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APPROXIMATIONS TO RANDOM FLIGHT FUNCTIONS IN ONE,
TWO, AND THREE DIMENSIONS

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The integral form of the random-walk function in ν dimensions has been derived by Watson ¹⁾. The special cases of $\nu=2$ and $\nu=3$ have been treated by Rayleigh. Details may be found in Watson ¹⁾ or Chandrasekhar ²⁾. For n , the number of steps, very large the function approaches a Gaussian (see also Watson ¹⁾). For moderate values of n , however, this is not a good approximation, because - as will be shown - one neglects then terms of order $1/n$. We propose here to compute those terms. A simple method for doing this has been given by Tricomi ³⁾, for the case of steps of equal length. We shall apply his ideas to the general case, for $\nu=3, 2, 1$.

I. The random-walk function in three dimensions

The probability that n vectors of given lengths $a_1, a_2, \dots, a_n$ but random directions add up to a vector sum of length r is given by

$$p_n(r) = \frac{2^{\frac{n-1}{2}} r^{\frac{3}{2}} (\Gamma(\frac{3}{2}))^{n-1}}{(a_1 a_2 \dots a_n)^{\frac{1}{2}}} \int_0^\infty t^{-\frac{n-1}{2}} J_{\frac{1}{2}}(rt) \prod_{i=1}^n J_{\frac{1}{2}}(a_i t) t dt \quad (1)$$

1) G.N. Watson, Bessel Functions, Cambridge 1945

2) S. Chandrasekhar, Revs. Mod. Phys. 15, 1 (1943)

3) F.G. Tricomi, Funzioni Ipergeometriche Confluenti, Roma 1954

The integral can, by the way, be solved in a rather elementary way and leads to an algorithm to construct $J_n(r)/r^2$ which is very suited for machine computation¹⁾.

Here, however, we are after a simple approximation formula, good up to terms of order $1/n^2$. We shall keep the argument fairly general, so that it may also be applied to two dimensions (or any other number, larger than 1).

We remark first that the Bessel function $J_\nu(z)$ is - apart from a factor - an entire function of z^2 . Let us therefore define the function

$$E_\nu(z) = z^{-\frac{\nu}{2}} J_\nu(2\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\nu+n+1)} \frac{z^n}{n!} \quad (2)$$

The next essential step is to approximate this function by an exponential, with certain corrections. Let us write the first terms of the development of each of those functions; for (1) we need $2\sqrt{z} = at$, and for later convenience let us put

$$at \equiv 2a\sqrt{\frac{\tau}{n}}$$

Then

$$\Gamma(\nu+1) E_\nu(a^2 \frac{\tau}{n}) = 1 - \frac{1}{\nu+1} \frac{\tau}{n} a^2 + \frac{1}{2!(\nu+1)(\nu+2)} \left(\frac{\tau}{n}\right)^2 a^4 - \dots \quad (3)$$

and

$$e^{\frac{\tau}{(\nu+1)n} a^2} = 1 + \frac{1}{\nu+1} \frac{\tau}{n} a^2 + \frac{1}{2!(\nu+1)^2} \left(\frac{\tau}{n}\right)^2 a^4 + \dots \quad (4)$$

1) F. Cerulus and R. Hagedörn, Suppl. Nuovo Cim. 9, 646 (1958) (Appendix)

Multiplying both series one gets

$$\Gamma(v+1) E_v \left(a^2 \frac{\tau}{n} \right) = e^{-\frac{\tau}{n} \frac{a^2}{v+1}} \sum_{k=0}^{\infty} C_k \left(\frac{a^2 \tau}{n} \right)^k \quad (5)$$

with

$$C_0 = 1$$

$$C_1 = 0$$

$$C_2 = - \frac{1}{2(v+1)^2 (v+2)} \quad (6)$$

$$C_3 = \frac{6v+4}{3! (v+1)^3 (v+2)(v+3)}$$

etc.

In (1) occur essentially products of expressions like (5); written with the new integration variable (1) is

$$\begin{aligned} \rho_n(x) = & \frac{4}{\sqrt{\pi}} x^2 n^{-3/2} \int_0^{\infty} \tau^{1/2} e^{-\frac{2}{3n} \left[\sum_{i=0}^n a_i^2 \right] \tau} E_{1/2} \left(\frac{x^2}{n} \tau \right) \\ & \times \prod_{i=1}^n \left[\sum_{k=0}^{\infty} C_k \left(\frac{a_i^2 \tau}{n} \right)^k \right] d\tau \end{aligned} \quad (7)$$

We expand the product of n series

$$\prod_{i=1}^n \left[\sum_{k=0}^{\infty} C_k (a_i^2 z)^k \right] = \sum_{k=0}^{\infty} C_k^{(n)} z^k \quad (8)$$

The new coefficients are derived from the c_k

$$C_0^{(n)} = 1 \quad C_1^{(n)} = 0 \quad C_2^{(n)} = c_2 \left(\sum_{i=1}^n a_i^4 \right)$$

$$C_3^{(n)} = c_3 \left(\sum_{i=1}^n a_i^6 \right)$$

$$C_4^{(n)} = c_4 \left(\sum_{i=1}^n a_i^8 \right) + c_2^2 \left(\sum_{i=1}^n \sum_{\substack{j=1 \\ j < i}}^n a_i^4 a_j^4 \right) \quad (9)$$

etc.

One can write a recurrence relation in n

$$C_k^{(n)} = \sum_{h=0}^k C_{k-h}^{(n-1)} c_k a_n^{2h} \quad (10)$$

where $C^{(n-1)}$ depends only on $a_1, \dots, a_{n-1}$, and $C_0^{(n)} = C_0^{(n-1)} c_0$. One sees immediately that $C_k^{(n)}$ is a homogeneous multinomial of order $2k$ in the variables $a_1, a_2, \dots, a_n$

$$C_k^{(n)} = \sum_{v_1 + v_2 + \dots + v_n = k} a_1^{2v_1} a_2^{2v_2} \dots a_n^{2v_n} c_{v_1} c_{v_2} \dots c_{v_n} \quad (10')$$

We can now compute the first terms of (7). The integrals are Laplace transforms of a standard type; one uses the following identities

$$E_{1/2}(k\tau) = (\pi k\tau)^{-1/2} \sin(2\sqrt{k\tau})$$

$$(\pi k)^{-1/2} \int_0^\infty e^{-s\tau} \sin(2\sqrt{k\tau}) d\tau = s^{-3/2} e^{-k/s}$$

$$(\pi k)^{-1/2} \int_0^\infty e^{-s\tau} \sin(2\sqrt{k\tau}) \tau^2 d\tau = 2^{-5/2} k^{-1/2} s^{-3} e^{-k/s} \text{He}_5\left(\sqrt{\frac{2k}{s}}\right)$$

etc.

The result is, putting $\sum_{i=1}^n a_i^2 = A^2$ and $\sum_{i=1}^n a_i^4 = B^4$

$$p(r) = \frac{4}{\sqrt{\pi}} r^2 \left(\frac{2A^2}{3}\right)^{-3/2} e^{-\frac{3}{2} \frac{r^2}{A^2}} \left[1 - \frac{1}{45} B^4 \left(\frac{2A^2}{3}\right)^{-2} \left(15 - 20 \frac{3r^2}{2A^2} + 4 \left(\frac{3r^2}{2A^2}\right)^2 \right) + \dots \right] \quad (11)$$

We are mainly interested, in the application of the random-walk function to phase space calculations, to sets of $a_1, \dots, a_n$ which are of the same order of magnitude. If we put for a moment $a_1 = a_2 = \dots = a_n \equiv a$, we get

$$p(r) = \frac{4}{\sqrt{\pi}} r^2 \left(\frac{2na^2}{3}\right)^{-3/2} e^{-\frac{3r^2}{2na^2}} \left[1 - \frac{1}{20} \frac{1}{n} \left(15 - 20 \frac{3r^2}{2na^2} + 4 \left(\frac{3r^2}{2na^2}\right)^2 \right) + \dots \right] \quad (12)$$

and it is easily proved in this case that the terms which are neglected are at most of order $1/n^2$, (they have the structure $C_k^{(n)} n^{-k} \text{He}_{2k+1}(\sqrt{\frac{3r^2}{2na^2}})$ and it can be proved that $C_k^{(n)}$ is of order $k-2$ for $k > 2$; the proof is given by Tricomi).

That we have retained factors which are at first sight of order $1/n^2$ is because for larger r (the first term (in (12)) given a width of $r \approx \sqrt{2n/3} a$) this n -dependence vanishes.

One may, therefore, use this formula when computing spectra of particles, as the one particle momentum has to be at times much larger than the average momenta. If the $a_1 \dots a_n$ differ widely the convergence of the series is worse; take e.g. the extreme case that $a_1 = a$, $a_2 = a_3 = \dots = a_n = 0$. In that case $B^4/(A^2)^2 = 1$ and not $= 1/n$ and the next term contains $\sum a_i^6 / (\sum a_i^2)^3 = 1$ instead of $1/n^2$. In this extreme case one should, of course, have $p(r) = \delta(r-a)$, and the Gaussian is a bad first

approximation. In this extreme case also $c_k^{(n)} = a^{2k} c_k$ because the multinomial (10') reduces to one term. If not all a_i ($i > 1$) are zero, but small against a_1 , the other terms in the multinomial will also be small. One can, therefore, take as a measure of the convergence of (11) the convergence of the series

$$\frac{C_k^{(n)}}{n^{k-2}} \frac{\sum_{i=1}^n (a_i^{2k})}{\left(\sum_{i=1}^n a_i^2\right)^k} n^{k-3} H e_{2k+1} \left(\sqrt{\frac{3r^2}{2\sum a_i^2}} \right) 2^{-2k-1} \quad (13)$$

II. The random-walk function in two dimensions

We take again the general formula in integral form, from Watson

$$p(r) = r \int_0^\infty \bar{J}_0(rt) \prod_{m=1}^\infty \bar{J}_0(a_m t) t dt \quad (14)$$

As in the 3-dimensional case we put

$$E_0(z) = \bar{J}_0(2\sqrt{z})$$

$$t = 2\sqrt{\frac{\tau}{n}}$$

and we approximate $E_0(a^2 \frac{\tau}{n})$ by a Gaussian

$$E_0(a^2 \frac{\tau}{n}) = e^{-a^2 \frac{\tau}{n}} \left[1 + c_2 \left(\frac{\tau}{n} a^2 \right)^2 + c_3 \left(\frac{\tau}{n} a^2 \right)^3 + \dots \right] \quad (15)$$

with $c_2 = -\frac{1}{4}$.

Substituting from (15) in (14) one gets

$$n(r) = \frac{2r}{n} \int_0^\infty E_0(r^2 \frac{\tau}{n}) e^{-\frac{\tau}{n} A^2} \left[1 + c_2 \frac{\tau^2}{n^2} B^4 + \dots \right] d\tau \quad (16)$$

where again $A^2 = \sum_{i=1}^n a_i^2$ $B^4 = \sum_{i=1}^n a_i^4$. This Laplace transform is computed from the identities

$$\int_0^\infty e^{-s\tau} J_0(2\sqrt{k\tau}) d\tau = s^{-1} e^{-k/s} \quad (17)$$

$$\int_0^\infty e^{-s\tau} J_0(2\sqrt{k\tau}) \tau^2 d\tau = (2s^{-3} - 4ks^{-4} + k^2 s^{-5}) e^{-k/s} \quad (18)$$

and we obtain

$$n(r) = 2r \frac{1}{A^2} e^{-\frac{r^2}{A^2}} \left[1 - \frac{1}{4} \frac{B^4}{A^4} \left(2 - 4 \frac{r^2}{A^2} + \frac{r^4}{A^4} \right) + \dots \right] \quad (19)$$

For the special case of equal lengths this reduces to

$$p(r) = 2r \frac{1}{na^2} e^{-\frac{r^2}{na^2}} \left[1 - \frac{1}{4} \frac{1}{n} \left(2 - 4 \frac{1}{n} \frac{r^2}{a^2} + \frac{1}{n^2} \frac{r^4}{a^4} \right) + \dots \right] \quad (20)$$

The discussion of convergence, order of neglected terms etc. is almost the same as for the 3-dimensional case.

III. The random-walk problem in one dimension

In one dimension the random-walk function gives the probability that a point on a line has reached position r after n steps of length $a_1 \dots a_n$ when at each step it has equal probability to go in either direction. One obtains this function by summing over all possible "paths" of the particle, i.e.

$$p(r) = \frac{1}{2^n} \sum_{(\sigma)} \delta(\sigma_1 a_1 + \sigma_2 a_2 + \dots + \sigma_n a_n - r) \quad (21)$$

where the sum is over all combinations of the signs $\sigma_i = \pm 1$. Of course, this is not a continuous function, as not all points r can be reached in this manner, with prescribed $a_1 \dots a_n$. However, if the number of steps is large and the interval Δr is also sufficiently large (of order of the a_i), the probability that r will fall in Δr after n steps is a sufficiently smooth function of the position of Δr . What we want to know is then the density of the points r that can be reached in n steps of prescribed length.

Let us first write (21), using the Fourier transform of the δ function

$$\begin{aligned}
 \mu(r) &= \frac{1}{2\pi} \frac{1}{2^n} \sum_{(\sigma)} \int_{-\infty}^{+\infty} dt e^{-itr} e^{it(\sigma_1 a_1 + \dots + \sigma_n a_n)} \\
 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt e^{-itr} \prod_{i=1}^n \cos(ta_i) \\
 &= \frac{1}{\pi} \int_0^{\infty} dt \cos(tr) \prod_{i=1}^n \cos(ta_i) \quad (22)
 \end{aligned}$$

Here again we express $\cos(ta)$ as an exponential times a series

$$\cos(ta) = e^{-\frac{1}{2}t^2 a^2} \left[1 - \frac{1}{4!} t^4 a^4 + \dots \right] \quad (23)$$

Substituting in (22) one obtains

$$\mu(r) = \frac{1}{\pi} \int_0^{\infty} dt \cos(tr) e^{-\frac{1}{2}t^2 A^2} \left[1 - \frac{t^4}{4!} B^4 + \dots \right] \quad (24)$$

with $A^2 = \sum_{i=1}^n a_i^2$, $B^4 = \sum_{i=1}^n a_i^4$. This last integral goes over into a Laplace transform, if the substitution $\tau = n/2 t^2$ is made. One uses the identities

$$\begin{aligned}
 \int_0^{\infty} d\tau \cos(2\sqrt{a\tau}) \tau^{n-1/2} e^{-p\tau} \\
 = (-2)^{-n} \pi^{1/2} p^{-n-1/2} e^{-\frac{a}{p}} \text{He}_{2n}\left(\sqrt{2\frac{a}{p}}\right) \quad (25)
 \end{aligned}$$

and obtains

$$\mu(r) = \frac{1}{\sqrt{2\pi} A^2} e^{-\frac{r^2}{2A^2}} \left[1 - \frac{1}{4!} \frac{B^4}{A^4} \text{He}_4\left(\frac{r}{A}\right) + \dots \right] \quad (26)$$

The Hermite polynomial is equal to

$$He_4(x) = 3 - 6x^2 + x^4$$

For the special case of equal length steps this reduces to

$$p(r) = \frac{1}{\sqrt{2\pi} a} e^{-\frac{1}{2n} \frac{r^2}{a^2}} \left[1 - \frac{1}{4!} \frac{1}{n} \left(3 - \frac{6}{n} \frac{r^2}{a^2} + \frac{1}{n^2} \frac{r^4}{a^4} \right) + \dots \right] \quad (27)$$

IV. Conclusion

Formulae have been given for the random-walk functions in 1, 2 and 3 dimensions, for arbitrary lengths of the single steps. They give essentially correction terms to the usual Gaussian approximation. In case all steps are of the same order of magnitude one has retained terms of order $1/n$ and left out those of order $1/n^2$. The resultant need not be of the same order as the single steps. If the steps differ by an order of magnitude the series which corrects the Gaussian still converges, but the terms can no longer be distinguished by their order in $1/n$.