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QUASI-ELASTIC NUCLEON-NUCLEUS SCATTERING AT HIGH ENERGY *)

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A B S T R A C T

The angular distribution for elastic nucleon-nucleon scattering at high energy is sharply peaked in the forward direction. As a consequence, there is an angular interval in nucleon-nucleus scattering to which successive small-angle elastic nucleon-bound nucleon scatterings are more likely to contribute than a single nucleon-bound nucleon scattering through the angle of observation. Such "plural" collisions result in outgoing nucleons which have suffered a smaller energy loss than nucleons that are scattered from hydrogen at the same angle. The Fermi motion of the bound-nucleons produces a broadening in the energy distribution of the scattered nucleons, but displaces the centre of the distribution only negligibly. The results of calculations with a simple model are in satisfactory agreement with the observations on proton-carbon scattering.

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I. Introduction

In the scattering of a nucleon by a nucleus, above an incident energy of about 300 MeV, an easily identifiable group of the scattered nucleons satisfies the kinematic requirements of nucleon-nucleon collisions. The expression "quasi-elastic scattering" is used to describe this process, which is attributed to the elastic scattering of the incident particles by single nucleons within the target nucleus ¹⁾; the energy transfer to the struck nucleon, of course, is somewhat larger than the energy transfer to the carbon nucleus in genuine elastic scattering from the latter, but it is not as large as it is in processes involving meson production ²⁾.

Experimental work with very high-energy (15-26 GeV) incident protons being scattered at small angles by carbon, tantalum, and beryllium has revealed a structure near the upper limit of the energy spectrum of the scattered protons that appeared to confirm the usefulness of the model in which a single bound nucleon acts as the target ³⁾. More precise measurements of the secondary energy, however, showed that the group of most energetic scattered protons had suffered an average energy loss only two-thirds that calculated for elastic scattering from single target nucleons ⁴⁾. Accordingly, it appeared likely that the effective target consisted of more than one single nucleon. Because the effective target acquires a large momentum near 1 GeV/c, it seemed implausible to postulate that it consists of a bound fragment such as a deuteron; instead, we investigated the possibility that the scattering took place successively on two uncorrelated nucleons in the nucleus. It is our purpose to show that the observations can be accounted for in this way. As a matter of fact, such plural interaction had been considered as a possible mechanism for producing many mesons in high-energy nuclear stars ⁵⁾, but they have never been studied in connection with quasi-elastic scattering in the absence of particle production.

At first thought it seems implausible that the probability for two elastic collisions should compete favorably with the probability for one collision in a small, strongly absorbing object such as a carbon nucleus (or a deuteron⁶). While the total probabilities unquestionably favor the single collision, the comparison becomes more difficult when we focus our attention on a fixed scattering angle, for then one must have information about the angular distribution in single scattering. In particular, if the angular distribution is sharply peaked in the forward direction, the relative probabilities may be inverted: one scattering at a substantial angle may be less probable than two successive scatterings, each at an angle only one-half as great.

In this article, we shall calculate the importance of the plural scattering for a simple model of the nucleus and the elementary cross-sections. The nucleus of mass number A is treated as a spherical region of radius ($\hbar = c = 1$)

$$R(A) = r_0 A^{1/3}, \quad (r_0 \simeq 1.17 \text{ f}) \quad (1)$$

filled with uncorrelated nucleons with a momentum distribution

$$P(\vec{p}) = \left(\frac{2M\varepsilon}{\pi} \right)^{3/2} e^{-\frac{\vec{p}^2}{2M\varepsilon}} \quad (2)$$

($M = \text{nucleon mass}$, $\varepsilon/M = 1/70$).

The elementary nucleon-nucleon cross-section $[d\sigma(p, \theta)/d\Omega]_{\text{lab.}}$ at incident momentum p and scattering angle θ (in the "lab. system", where the target nucleon is at rest) we take as

$$\left[\frac{1}{p^2} \frac{d\sigma(p, \theta)}{d\Omega} \right]_{\text{lab}} \simeq \left(\frac{\sigma_{\text{tot}}}{4\pi} \right)^2 e^{-\alpha p^2 \theta^2}, \quad (3)$$

$$\sigma_{tot} = 40 \text{ mb}, \quad \alpha \simeq 7/M^2, \quad \sigma_e = \sigma_{tot}^2 / 16\pi\alpha \simeq 10 \text{ mb}, \quad (4)$$

independent of spin and charge. There is very little experimental information about these quantities except σ_{tot} . It seems reasonable in the light of present theoretical views to assume that the angular variation Eq. (3) depends on the square of the invariant momentum transfer

$$t \simeq p^2 \theta^2 \quad (5)$$

at small angles ; then a combination of measurements at different energies yields the form we have chosen for $t \lesssim M^2$ 4). For larger t the cross-section drops less steeply. The integrated value obtained for the elastic cross-section also is reasonable in the light of data at lower energies. We neglect the effect of nuclear binding on the cross-sections except to the extent to which the Pauli principle (and coherent effects) reduce the quasi-elastic scattering at very small angles and give $\sigma_{qe} \simeq 7 \text{ mb}$ for the quasi-elastic cross-section and 37 mb for the "total" cross-section.

The calculation now proceeds in three steps. First, we evaluate the cross-section $\sigma_n(A)$ for a high-energy nucleon to undergo n elastic collisions within a nucleus of mass number A . Second, we calculate the angular distribution for the outgoing nucleon in the hypothetical case in which the bound nucleons are stationary. Third, we take into account the motion of the bound nucleons and evaluate the energy spectrum of the nucleons scattered at the angle of observation. The effects of the exclusion principle are estimated before, in conclusion, we apply the results to the scattering from nuclei of mass numbers 12 and 189 to obtain a possible comparison with experiment.

II. The plural scattering cross-section

For the purposes of this section, we shall treat the nucleus as an absorbing medium rather than as a collection of discrete scatterers. The complications that the latter more consistent description would entail are not justified. As a matter of fact, the high-energy cross-section of nucleons is such that the nucleon may be thought of as a gray region with radius 1.2 f. The nucleus is, therefore, filled about uniformly with this nuclear medium ⁷⁾.

In terms of the mean-free-paths λ_e and λ_t for elastic and total scattering in the nuclear medium, the probability $P_n(z)$ that a nucleon suffers n elastic collisions after traversing a thickness of material z is

$$P_n(z) = \frac{1}{n!} \left(\frac{z}{\lambda_e} \right)^n e^{-z/\lambda_t} \quad (6)$$

Since the particle wave-length is small and the deflection and momentum loss in a single scattering are negligible, we calculate the cross-section $\sigma_n(A)$ for n elastic collisions in a sphere of radius $R(A)$ by using geometrical optics and straight rays in the sphere,

$$\begin{aligned} \sigma_n(A) &= \int_0^R 2\pi b db P_n(2\sqrt{R^2 - b^2}) \\ &= \frac{\pi \lambda_t^2}{2 n!} \left(\frac{\lambda_t}{\lambda_e} \right)^n \int_0^{2R/\lambda_t} x^{n+1} e^{-x} dx \quad (7) \end{aligned}$$

$$= \frac{\pi}{2} (n+1) \lambda_t^2 \left(\frac{\sigma_{ge}}{\sigma_{tot}} \right)^n \left[\Gamma_{2R/\lambda_t}(n+2) / \Gamma(n+2) \right]$$

The quantity in brackets is obtainable from tables of the incomplete Γ -function³⁾, where it is denoted by the symbol $I\left(\frac{2R}{\lambda_t}, n+1\right)$. When numerical values are inserted into Eq. (7), we obtain

$$\sigma_n(A) = 54 \left(\frac{\sigma_{qe}}{\sigma_{tot}} \right)^n (n+1) I\left(\frac{2R}{\lambda_t}, n+1\right) \text{ mb.} \quad (8)$$

III. The differential cross-section

For single quasi-elastic scattering, we write the relative differential cross-section $r_1(p, \theta)$ at the incident momentum p and angle θ

$$r_1(p, \theta) = \frac{1}{\sigma_{qe}} \cdot \frac{d\sigma_{qe}}{d\Omega} = \left(\frac{\sigma_e}{\sigma_{qe}} \right) \cdot \frac{\alpha p^2}{\pi} e^{-\alpha p^2 \theta^2} \quad (9)$$

where we neglect the effects of the exclusion principle and coherence that become important at momentum transfers $p\theta \lesssim p_F = 250 \text{ MeV/c}$. The angular distribution of double scattering is then given by the integral

$$r_2(p, \theta) = \frac{1}{2\pi} \int r_1(p, \theta_1) r_1(p, \theta_2) d\cos\theta_1 d\cos\theta_2 \times d\phi_1 d\phi_2 \delta\left(\cos\theta - \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)\right), \quad (10)$$

since the energy loss in each collision is so small that the decrease in incident momentum may be neglected. After the azimuthal integrations are carried out, it is helpful to approximate the trigonometric functions by

their small-angle expansions to second order and to extend the range of integration to infinity. The angular distribution function now becomes

$$r_2(p, \theta) = \left(\frac{\sigma_e}{\sigma_{qe}} \right) \int_0^\infty d(\theta_1^2) \int_0^\infty d(\theta_2^2) \left(\frac{\alpha p^2}{\pi} \right)^2 e^{-\alpha p^2 (\theta_1^2 + \theta_2^2)} \times \frac{\Theta(4\theta_1^2\theta_2^2 - (\theta - \theta_1^2 - \theta_2^2)^2)}{\sqrt{4\theta_1^2\theta_2^2 - (\theta - \theta_1^2 - \theta_2^2)^2}} \quad (11)$$

The main contribution comes from the smallest values of $\theta_1^2 + \theta_2^2 \simeq \frac{1}{2}\theta^2$ permitted by the step function Θ . An exact evaluation of the integral is possible because of the form we have chosen for the angular dependence. All the integrals can be found with the help of the integral formula

$$\int_0^\infty dx_1 \int_0^\infty dx_2 f\left(\frac{x_1}{a_1} + \frac{x_2}{a_2}\right) \left\{ \frac{\Theta(4x_1x_2 - (y - x_1 - x_2)^2)}{\sqrt{4x_1x_2 - (y - x_1 - x_2)^2}} \right\} = \frac{\pi a_1 a_2}{a_1 + a_2} \int_{y/(a_1+a_2)}^\infty dz f(z) \quad (12)$$

By the use of this formula we obtain from Eq. (11) the result

$$r_2(p, \theta) = \left(\frac{\sigma_e}{\sigma_{qe}} \right)^2 \frac{\alpha p^2}{2\pi} e^{-\alpha p^2 \theta^2/2} \quad (13)$$

The obvious iteration of this expression according to Eq. (10) leads to the further values

$$r_m(p, \theta) = \left(\frac{\sigma_e}{\sigma_{qe}} \right)^n \frac{\alpha p^2}{n\pi} e^{-\alpha p^2 \theta^2/n} \quad (14)$$

for n fold scattering. We must remember, however, that the effects of the exclusion principle become more severe, since the higher order scattering depends on contributions from smaller and smaller angles. Hence n should not exceed $\sim p\theta/p_F$.

The differential cross-section for plural scattering is

$$\frac{d\sigma_n(A)}{d\Omega} = \sigma_n(A) r_n(p, \theta). \quad (15)$$

By comparing Eqs. (9) and (13), we see that double scattering becomes comparable to single scattering when

$$\frac{1}{2} \left(\frac{\sigma_{qe}}{\sigma_{tot}} \right) e^{\alpha p^2 \theta^2 / 2} \sim 1 \quad \text{or} \quad p\theta \sim 0.8M, \quad (16)$$

a value of momentum transfer within the range of the recent experiments. We see that the quasi-elastic cross-section σ_{qe} cancels in Eq. (15); obviously, when the exclusion principle can be ignored, the free elastic cross-section determines the relative contributions. Further numerical conclusions are postponed until the final section of the paper.

IV. The energy spectrum - single scattering

The spectrum of quasi-elastically scattered nucleons is not mono-energetic because of the motion of the bound nucleons. This effect gives to the secondary nucleons at the angle θ a momentum distribution of a width

$$\sim \frac{p\theta}{M} \cdot p_F$$

as we shall show below. The centre of this spectrum is displaced by a negligible amount from the secondary momentum p'_0 resulting in free nucleon-nucleon collisions at small angles

$$p'_0 \simeq p - \frac{p^2 \theta^2}{2M}. \quad (17)$$

The plural scattering we have described also affects the spectrum of the nucleons scattered at an angle θ , because the momentum loss $\Delta p(\theta)$ depends on the scattering angle in each of the quasi-elastic processes that took place. By using Eq. (17) several times, we obtain

$$\Delta p(\theta) = \frac{p^2}{2M} \sum_{i=1}^n \theta_i^2 \sim \frac{p^2}{2M} \frac{\theta^2}{n}, \quad (18)$$

according to the properties of the plural scattering noted after Eq. (11). As expected the momentum loss for plural scattering is smaller than for single quasi-elastic scattering, and corresponds approximately to that evaluated for scattering from a more massive target with mass $\sim nM$.

It is now our task to combine the two effects we have described. First, we obtain the energy spectrum of single quasi-elastic scattering. Accordingly, we consider the scattering of two nucleons with energy-momentum vectors $(\underline{p}_1, E_1)$ and $(\underline{p}_2, E_2)$ into a final state with $(\underline{p}'_1, E'_1)$ and $(\underline{p}'_2, E'_2)$. We use the invariant form for the differential scattering rate⁹⁾

$$B_0(s) d\sigma = |I(s, t)|^2 \frac{d^3 p'_1}{E'_1} \frac{d^3 p'_2}{E'_2} \delta(\underline{p}'_1 + \underline{p}'_2 - \underline{p}_1 - \underline{p}_2) \times \delta(E'_1 + E'_2 - E_1 - E_2), \quad (19)$$

where s is the mass invariant of the two-particle system

$$S = (E_1 + E_2)^2 - (\underline{p}_1 + \underline{p}_2)^2, \quad (20)$$

t is the invariant momentum transfer squared,

$$t = -(E_1' - E_1)^2 + (\underline{p}_1' - \underline{p}_1)^2, \quad (21)$$

$B_0(s)$ is the invariant flux,

$$B_0(s) = \frac{1}{2} \sqrt{s(s - 4M^2)} \quad (22)$$

and $I(s, t)$ is the invariant matrix element of the interaction. The assumed form Eq. (3) implies that

$$|I(s, t)|^2 = \frac{1}{4\pi} s(s - 4M^2) \alpha \sigma_e e^{-\alpha t} \quad (23)$$

at high energy and small angles. For the nucleus, the target momenta must be averaged over the distribution $P(\underline{p}_2)$ given in Eq. (2). We then obtain the differential cross-section for quasi-elastic scattering

$$\begin{aligned} \frac{d^2\sigma}{d\underline{p}_1' d\Omega_1'} &= \frac{1}{B_0} \frac{p_1'}{E_1'} \int \frac{d^3p_2}{E_2'} |I(s, t)|^2 P(\underline{p}_2) \\ &\quad \times \delta(E_1' + E_2' - E_1 - E_2), \end{aligned} \quad (24)$$

which neglects the effects of the exclusion principle. The momentum $\underline{p}_2'$ is to be expressed by means of the other three momenta, $\underline{p}_2' = \underline{p}_1 + \underline{p}_2 - \underline{p}_1'$. In carrying out the evaluation of Eq. (24), it is wise to make approximations based on the high-incident energy, the small scattering angles, and the non-relativistic nature of the target nucleons where possible. The

errors are characteristically of the order of $\frac{M}{E_1}$, or smaller. Thus we have ¹⁰⁾

$$\begin{aligned} E_1 &\simeq E_1' \simeq p_1 \simeq p_1', \quad S \simeq 2Mp_1, \quad t \simeq p_1^2 \theta^2, \\ B_0 &\simeq Mp_1, \\ |I(s, t)|^2 &\simeq \frac{1}{\pi} M^2 p_1^2 \alpha \sigma_e e^{-\alpha p_1^2 \theta^2}, \end{aligned} \quad (25)$$

$$|\vec{p}_1 - \vec{p}_1'| \simeq p_1^2 \theta^2 + (p_1 - p_1')^2.$$

It is useful to define the momentum loss

$$\Delta p = p_1 - p_1' \simeq p_1^2 \theta^2 / 2M, \quad (26)$$

which is an experimentally controlled variable.

We now proceed to carry out the average over the Fermi momentum. The energy conserving δ function, of course, selects certain values of p_2 that make possible a particular outgoing momentum p_1' . If we introduce a co-ordinate system with the z axis along $\vec{p}_1 - \vec{p}_1'$, then we find p_{2z} is fixed,

$$p_{2z} \simeq \frac{-\frac{1}{2} p_1^2 \theta^2 + M \Delta p}{\sqrt{p_1^2 \theta^2 + (\Delta p)^2}}, \quad (27)$$

but the other two components of $\vec{p}_2$ are arbitrary. The integral over p_{2z} is

$$\begin{aligned} \frac{1}{E_1'} \int dp_{2z} \delta(E_1' + E_2' - E_1 - E_2) &= \frac{2}{\partial(E_2')^2 / \partial p_{2z}} \\ &= \frac{1}{|p_1 - p_1'|} \simeq \frac{1}{\sqrt{p_1^2 \theta^2 + (\Delta p)^2}} \simeq \frac{1}{p_1 \theta}. \end{aligned} \quad (28)$$

We shall make an approximation in Eqs. (27) and (28) by neglecting $(\Delta p)^2$ compared to $(p_1 \theta)^2$,

$$p_{2z} \approx -\frac{1}{2} p_1 \theta + \frac{M \Delta p}{p_1 \theta}. \quad (27')$$

This depends on the fact that $\Delta p/p_1 \theta \sim p_1 \theta/2M < \frac{1}{2}$ in the range of interest. Eq. (27') gives a much more manageable result than Eq. (27). The remaining integrations can be carried out with no difficulty to yield a shape function $G(\Delta p, p\theta)$

$$\begin{aligned} G(\Delta p, p_1 \theta) &= \frac{M}{p_1 \theta} \int [P(p_2)]_{p_{2z} = -\frac{1}{2} p_1 \theta + \frac{M \Delta p}{p_1 \theta}} d p_{2x} d p_{2y} \\ &\approx \frac{M}{p_1 \theta} \frac{\Theta(\Delta p)}{\sqrt{2\pi \epsilon M}} \exp \left[-\frac{M}{2 \epsilon p_1^2 \theta^2} \left(\Delta p - \frac{p_1^2 \theta^2}{2M} \right)^2 \right]. \end{aligned} \quad (29)$$

The differential cross-section for quasi-elastic scattering with momentum loss Δp is, therefore,

$$\frac{d^2 \sigma_{qe}}{d \Delta p d \Omega_i} \approx \frac{\alpha p_1^2 \sigma_e}{\pi} G(\Delta p, p_1 \theta) e^{-\alpha p_1^2 \theta^2}. \quad (30)$$

As expected, the most likely momentum loss is $\frac{p_1^2 \theta^2}{2M}$, the value for scattering on free nucleons. The half width of the secondary momentum distribution is $\frac{p_1 \theta}{M} \sqrt{2 \epsilon M}$.

V. The energy spectrum - plural scattering

We repeat now the discussion of Sect. II with the differential cross-section Eq. (30) rather than with the free cross-section Eq. (3)

for the elementary process. We again introduce a relative cross-section

$$r_1(\Delta p, p, \theta) = \frac{1}{\sigma_{qe}} \frac{d^2 \sigma_{qe}}{d\Delta p d\Omega}, \quad (31)$$

which is iterated according to the relation

$$\begin{aligned} r_n(\Delta p, p, \theta) &= \frac{1}{2\pi} \int r_1(\Delta p_1, p\theta_1) r_{n-1}(\Delta p - \Delta p_1, p\theta_2) \\ &\times d\cos\theta_1 d\cos\theta_2 d\phi_1 d\phi_2 d(\Delta p_1) \\ &\times \delta(\cos\theta - \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2)) \end{aligned} \quad (32)$$

analogous to Eq. (10). We extend the range of the Δp integration over the entire real axis. Then we can use the simple property of Gaussian distributions that

$$\begin{aligned} &\int_{-\infty}^{\infty} d(\Delta p_1) G(\Delta p_1, p\theta_1) G(\Delta p - \Delta p_1, p\theta_2) \\ &= G(\Delta p, p\sqrt{\theta_1^2 + \theta_2^2}). \end{aligned} \quad (33)$$

With the use of the formula (12), the double scattering relative cross-section r_2 can now be written as a single integral over the variable $u = \alpha p^2(\theta_1^2 + \theta_2^2)$,

$$r_2(\Delta p, p, \theta) = \frac{\alpha p^2}{2\pi} \left(\frac{\sigma_e}{\sigma_{qe}} \right)^2 \int_{\frac{\alpha p^2 \theta^2}{2}}^{\infty} du e^{-u} G(\Delta p, \sqrt{\frac{u}{\alpha}}) \quad (34a)$$

$$= \frac{\alpha p^2}{2\pi} \left(\frac{\sigma_e}{\sigma_{qe}} \right)^2 e^{-\alpha p^2 \theta^2 / 2} \int_0^{\infty} du' e^{-u'} G(\Delta p, \sqrt{\frac{u'}{\alpha} + \frac{p^2 \theta^2}{2}}) \quad (34b)$$

The integral in Eq. (34b) is shown in Fig. 1 as function of Δp for some values of momentum transfer $p\theta$. It is seen that the most likely momentum loss is about two-thirds of the value of the momentum loss in nucleon-nucleon scattering. By iteration one can again obtain the formula for n -fold scattering,

$$r_n(\Delta p, p\theta) = \frac{\alpha p^2}{n\pi} \left(\frac{\sigma_e}{\sigma_{ge}} \right)^n \frac{1}{(n-2)!} \int_{\frac{\alpha p^2 \theta^2}{n}}^{\infty} du e^{-u} G(\Delta p, \sqrt{\frac{u}{\alpha}}) \left(u - \frac{\alpha p^2 \theta^2}{n} \right)^{n-2} \quad (35)$$

($n \geq 2$),

but again it is not very meaningful for large n because the neglected exclusion principle severely affects the cross-section at the small momentum transfers and momentum losses that contribute heavily. After integration over Δp , Eq. (35) reduces to Eq. (14).

VI. The exclusion principle

We have referred already several times to the fact that the exclusion principle reduces the cross-section for quasi-elastic scattering at extremely small angles. In this section we shall attempt to reach an estimate of the effect without, however, pretending to make a complete, consistent theory of the "one-particle states" in a nucleus.

First, we may ask for the reduction in the elastic cross-section and the plural scattering angular distribution. To this end we make a Gaussian approximation to the average fraction f of states not available to a nucleon in a degenerate Fermi distribution (rather than the distribution (2)) when this nucleon receives an impulse $p\theta$,

$$f = \left(1 - \frac{p\theta}{2p_F} \right)^2 \left(1 + \frac{1}{4} \frac{p\theta}{p_F} \right) \approx e^{-1.25 \frac{p^2 \theta^2}{p_F^2}} \equiv e^{-\frac{5}{2} \alpha p^2 \theta^2} \quad (36)$$

The quasi-elastic differential cross-section then becomes

$$r_1' = \frac{\alpha p^2}{\pi} \left(\frac{\sigma_e}{\sigma_{qe}} \right) e^{-\alpha p^2 \theta^2} \left(1 - e^{-\frac{5}{2} \alpha p^2 \theta^2} \right) \quad (37)$$

whose integral determines σ_{qe} ,

$$\sigma_{qe} = \frac{5}{7} \sigma_e \simeq 7 \text{ mb}, \quad (38)$$

as stated earlier. We see also that scattering at angles such that $p\theta \gtrsim 2p_F$ is only slightly reduced. The Gaussian form of the correction makes it possible to calculate the modified $r_n'(p, \theta)$ by repeated use of Eq. (12). Thus we have

$$r_2'(p, \theta) = \frac{\alpha p^2}{2\pi} \left(\frac{\sigma_e}{\sigma_{qe}} \right)^2 e^{-\frac{1}{2} \alpha p^2 \theta^2} \cdot \left(1 - 0.89 e^{-0.278 \alpha p^2 \theta^2} + 0.28 e^{-1.25 \alpha p^2 \theta^2} \right), \quad (39)$$

$$r_3'(p, \theta) = \frac{\alpha p^2}{3\pi} \left(\frac{\sigma_e}{\sigma_{qe}} \right)^3 e^{-\frac{1}{3} \alpha p^2 \theta^2} \cdot \left(1 - 1.125 e^{-0.104 \alpha p^2 \theta^2} + 0.468 e^{-0.303 \alpha p^2 \theta^2} - 0.082 e^{-0.833 \alpha p^2 \theta^2} \right), \quad (40)$$

etc.

We see here the transition from the larger angle plural scattering described by Eq. (15), which is determined by the free elastic cross-section, to the plural scattering at smaller angles, whose magnitude is

essentially determined by the quasi-elastic cross-section. Thus we have

$$\frac{r'_n}{r_n} \approx \left(\frac{\sigma_{qe}}{\sigma_e} \right)^n \quad \text{for } n = \frac{p\theta}{p_F}, \quad (41)$$

while the exclusion effects are small for $n < \frac{p\theta}{2p_F}$. Because of the crudeness of our one-parameter model, one must not push the use of Eqs. (39) and (40) to very small angles. It seems reasonable to go as far as is done in Eq. (41).

More difficult to determine is the modification in the momentum spectrum. We may proceed as above with a degenerate Fermi distribution and calculate a modified spectral function from Eq. (29) with a correction factor for the occupied states. Rather than trying to find a usable analytic form for this function, we shall take advantage of the fact that the width of the secondary spectrum becomes very small for small momentum transfer. The changes in the shape of this narrow spectrum are relatively less important than the reduction in its amplitude, a reduction that can be represented by the factor calculated in Eq. (36). We therefore introduce the new function

$$G'(\Delta p, p\theta) = G(\Delta p, p\theta) \left(1 - e^{-\frac{5}{2} \alpha p^2 \theta^2} \right) \quad (42)$$

which is to be used instead of G in the plural scattering calculation. The relative differential cross-sections then become

$$r'_i(\Delta p, p\theta) = \frac{\alpha p^2}{\pi} \left(\frac{\sigma_e}{\sigma_{qe}} \right) G(\Delta p, p\theta) e^{-\alpha p^2 \theta^2} \times \left(1 - e^{-\frac{5}{2} \alpha p^2 \theta^2} \right), \quad (43)$$

$$r'_2(\Delta p, p, \theta) = \frac{\alpha p^2}{2\pi} \left(\frac{\sigma_e}{\sigma_{qe}} \right)^2 \int_{\frac{1}{2}\alpha p^2 \theta^2}^{\infty} du G(\Delta p, \sqrt{\frac{u}{\alpha}}) \quad (44)$$

$$\times \left\{ e^{-u} + e^{-\frac{7}{2}u} - 2e^{-\frac{9}{4}u} I_0\left(\frac{5}{2}\sqrt{\frac{\alpha p^2 \theta^2}{2}\left(u - \frac{\alpha p^2 \theta^2}{2}\right)}\right) \right\},$$

and similar, more complicated expressions for the higher orders. The expression (44) has the reasonable properties that almost the entire spectrum is reduced in scale for severely affected scattering conditions ($\frac{p\theta}{p_F} \lesssim 2$), while for larger momentum transfer there is a suppression of the spectrum near $\Delta p \sim p^2 \theta^2 / 2M$, the momentum loss suffered in single collisions. Our naive model does not justify a more detailed investigation of the modifications in the spectrum.

VII. Conclusions

We begin this section by collecting the results of our calculation for the quasi-elastic plural scattering cross-sections. The angular distributions are Eqs. (7), (37), (39) and (40)

$$\frac{d\sigma_{qe}^{(n)}}{d\Omega} = \sigma_n(A) r'_n(p, \theta) \quad (45)$$

while the distribution in angle and momentum loss Δp is given by Eqs. (7), (43) and (44),

$$\frac{d^2\sigma_{qe}^{(n)}}{d\Delta p d\Omega} = \sigma_n(A) r'_n(\Delta p, p, \theta). \quad (46)$$

For numerical evaluation, we tabulate $\sigma_n(A)$ for two mass numbers ($A = 12, 189$) in Table I and $\frac{M^2}{p^2} r'_n(p, \theta)$, which depends only on $p\theta$, for several values of this argument, in Table II. The predicted angular

distributions based on these values are shown in Figs. 2 and 3 for the two cases. As expected, there is a range of momentum transfer from 0,75 GeV/c to 1,5 GeV/c in which plural scattering dominates over single quasi-elastic scattering. At smaller momentum transfer the Pauli principle and the greater normalization of the single scattering make the plural scattering relatively less likely. At larger momentum transfers the nucleon-nucleon cross-section no longer decreases so rapidly and eventually remains larger than the still decreasing plural scattering cross-sections. The over-all effect on the angular distribution is to broaden it compared to the nucleon-nucleon cross-section, as though the bound nucleons were "smaller" scatterers than the free nucleons. A rough Gaussian fit of the form

$$e^{-\alpha' p^2 \theta^2} \quad (47)$$

is obtained with $\alpha' \sim 5/M^2$. From Fig. 2 we can also see that the carbon cross-section will be about of an order of magnitude greater than the hydrogen cross-section for $p\theta \sim 1$ GeV/c (plural scattering), but will exceed it only by a factor of 2 at small momentum transfer (single quasi-elastic scattering).

The momentum loss spectra are capable of giving more convincing evidence for the occurrence of plural scattering. As we have seen, the mean energy loss depends on the number of collisions. If we expect that the spectra from single and plural collisions are not resolved, we predict an observed mean momentum loss that is a weighted average of the momentum losses of the various processes. This quantity $\langle \Delta p \rangle_A$ for the nuclei with $A = 12$ and 189 is plotted in Fig. 4. The experimental points are also indicated. We consider the agreement satisfactory. Whether triple collisions also contribute to scattering from carbon is not clear. Blind acceptance of the results of our

calculation would suggest that they do. Still, carbon is such a small nucleus that treating it as a "medium", as we have done, is not a very accurate procedure. In larger nuclei, triple collisions are likely to play a role at momentum transfers from 1.3 to 1.7 GeV/c, and will broaden the momentum-loss spectrum. When our considerations are applied to the deuteron, then it turns out that single and double collisions with total momentum transfer of 1 GeV/c are about equally likely, if the density parameter is chosen so as to give the correct "shadowing" ⁶⁾. Because of the smaller momenta of the target nucleons, one expects to observe two well-resolved maxima at $\Delta p = \frac{p^2 \theta^2}{2M}$ and at $\Delta p = \frac{p^2 \theta^2}{4M}$ in the momentum loss spectrum.

Finally, we shall point out that the study of quasi-elastic collisions up to momentum transfers of about 2 GeV/c cannot give as direct information about the elementary nucleon-nucleon cross-section as one might have hoped.

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Table I

The plural scattering cross-sections

A	12	189
σ_1	11.2 mb	19.8 mb
σ_2	1.92	5.40
σ_3	0.248	1.24
σ_4	--	0.25

Table II

The relative angular distributions for plural scattering

$p\theta =$	.25	.50	.75	1.00	1.25	1.50 GeV/c
$\frac{M^2}{p^2} r_1'$	1.39	0.436	0.0347	0.00104	1.15×10^{-5}	-
$\frac{M^2}{p^2} r_2'$	-	0.436	0.177	3.79×10^{-2}	4.2×10^{-3}	2.8×10^{-4}
$\frac{M^2}{p^2} r_3'$	-	-	0.20	0.083	2.4×10^{-2}	4.5×10^{-3}

Figure Captions

- Fig. 1 The momentum-loss spectrum (normalized) in double scattering. The exclusion principle has not been taken into account.
- Fig. 2 The angular distribution for plural quasi-elastic scattering from carbon. The dashed part of the curve for $n = 1$ is based on the measured p-p cross-section ⁴⁾ rather than on our fit Eq.(3).
- Fig. 3 The angular distribution for plural quasi-elastic scattering from a heavy nucleus ($A = 189$). The dashed part of the curve for $n = 1$ is based on the measured p-p cross-section ⁴⁾ rather than on our fit Eq.(3).
- Fig. 4 The mean momentum loss for nuclear scattering compared with the momentum loss in p-p scattering at the same incident energy and scattering angle. For $A = 12$, single and double scattering have been used, for $A = 189$, single, double and triple scattering have been used. The circles represent the measurements ⁴⁾ but do not indicate probable errors.

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- 7) Of course, one must not make the mistake of using our procedure to calculate plural scattering from protons.
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- 10) In making these approximations, we neglect a small doppler shift in the momentum-loss spectrum. This small effect comes from the dependence of $|I(s,t)|^2$ on the momentum-loss and is of the order of ϵ .

FIG. 1

FIG. 2

FIG. 3

FIG. 4