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A RELATIVISTIC SCATTERING MODEL SATISFYING UNITARITY,
CROSSING SYMMETRY AND ANALYTICITY

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A B S T R A C T

A model is constructed which can generate a class of scattering amplitudes that have the same analytic properties as conventional perturbation theory, are crossing symmetric, and satisfy the unitarity condition in the elastic region in all three channels. The model consists of a non-linear off-shell integral equation whose solution gives a different amplitude for each different first order term or "generalized potential" that one starts with. The main new condition on these "potentials" is that they cannot have elastic cuts in any channel. The relation of this model to the problem of the uniqueness of the Chew-Mandelstam program is discussed.

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I. Introduction

Recently, in an effort to by-pass the difficulties of conventional field theory, attempts have been made to construct scattering amplitudes from the unitarity conditions, the condition of crossing symmetry, and the analytic properties of the usual Feynman diagrams. Such a program was first initiated by Chew and Mandelstam ¹⁾ and has been applied by several authors ²⁾ to problems of strong interactions.

This general approach is based on the assumption that for moderate energies the nearest singularities dominate the behaviour of the amplitude. This means that in the unitarity conditions one takes only the contributions from one- and two-particle terms. Indeed, in their treatment of the pion-pion problem, Chew and Mandelstam only make use of the two-particle elastic unitarity relation and neglect all inelastic contributions.

The other starting point of the Chew-Mandelstam program, namely that of analyticity, is usually expressed by the Mandelstam representation. However, the validity of the Mandelstam representation is not necessary for the Chew-Mandelstam approach. The only requirement, in view of the assumption concerning the nearby singularities, is that any extra singularities not exhibited by the Mandelstam representation should not lie near the low energy region and could be neglected with the inelastic singularities.

It would be interesting to investigate the uniqueness of amplitudes determined by the three starting points of Chew and Mandelstam: elastic unitarity, crossing symmetry, and the analytic properties of perturbation theory. If these conditions do not uniquely determine a scattering amplitude, one would like to see explicitly the nature of the non-uniqueness in the hope of finding out what other physical information is needed to determine a scattering amplitude.

In this paper we shall obtain a model for the scattering of two neutral bosons of equal mass. This model can generate an infinite set of scattering amplitudes which:

- i) are relativistically invariant,
- ii) satisfy unitarity in the elastic region,
- iii) are crossing symmetric
- iv) have the same analytic properties as the conventional Feynman graphs.

Our model consists of an off-shell non-linear integral equation for the scattering matrix element, which can generate by iteration a different scattering amplitude for each different first order term, i.e., "potential", that one starts with.

Even though one obtains an infinite number of amplitudes, it is still true that the four conditions listed above are very restrictive, and they impose strong conditions on the "potentials" one can start with. It turns out that none of these potentials can have a cut in the elastic region (in our model below $16 m^2$). One might argue that in view of the assumption about the nearest singularities, all these "potentials" are equivalent since they somehow represent inelastic effects. However, one can easily show that these potentials contribute to the Mandelstam spectral functions in the region of the elastic strip. Thus by taking very different potentials one can get very different contributions to the Mandelstam weight functions in the strip. Even with the assumption about nearest singularities this non-uniqueness is real and has to be dealt with in any calculation.

In Section II we start with a Bethe-Salpeter type equation and then construct an equivalent non-linear off-shell equation whose structure is similar to that of unitarity. This non-linear equation is obtained by first

constructing the total Green's function of the Bethe-Salpeter differential equation from a complete set of off-shell solutions of a certain off-shell generalization of the Bethe-Salpeter differential equation. The off-shell generalization needed turns out to be quite different from the obvious one and gives a novel off-shell two-particle propagator. The relation between this non-linear equation and unitarity is discussed.

In Section III a crossing symmetric amplitude is defined by a symmetrized generalization of the non-linear equation of Section II. The form of the equation and the nature of the symmetrization process are such that unitarity in the elastic region is still valid if our generalized "potentials" have no elastic cuts.

II. A non-linear Bethe-Salpeter equation

We start by considering a relativistically invariant differential equation for two interacting, spinless and neutral particles of mass m .

We write

$$(\square_1 - m^2)(\square_2 - m^2) \Psi(x_1, x_2) = -\frac{i}{2} V(x_1, x_2) \quad (1)$$

Here $V(x_1, x_2)$ is a relativistically invariant function of the difference of the two four vectors, $(x_1 - x_2)^2$, which will be defined in more detail below.

We consider solutions of (1) that satisfy the following integral equation

$$\Psi_{p_1 p_2}(x_1, x_2) = \Phi_{p_1 p_2}(x_1, x_2) - \frac{i}{2} \iint d^4 x'_1 d^4 x'_2 \Delta_c(x_1 - x'_1) \Delta_c(x_2 - x'_2) V(x'_1, x'_2) \Psi(x'_1, x'_2), \quad (2)$$

where Δ_c is the usual causal Green's function ³⁾

$$\Delta_c(x) = -\frac{1}{(2\pi)^4} \int d^4 k \frac{e^{ik \cdot x}}{k^2 - m^2 + i\epsilon}, \quad (3)$$

and

$$\Phi_{p_1 p_2}(x_1, x_2) = \frac{1}{(2\pi)^4} e^{-ip_1 \cdot x_1} e^{-ip_2 \cdot x_2}, \quad (4)$$

$$p_1^2 = p_2^2 = m^2.$$

To define a scattering matrix element we set

$$T(p'_1 p'_2 | p_1 p_2) = \left(\Phi_{p'_1 p'_2}, V \Psi_{p_1 p_2} \right). \quad (5)$$

Here we also take $p_1'^2 = p_2'^2 = m^2$. We use the notation

$$T(p'_1 p'_2 | p_1 p_2) = T(s, t) \delta^4(p'_1 + p'_2 - p_1 - p_2), \quad (6)$$

with

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p'_1)^2, \quad u = (p_1 - p'_2)^2. \quad (7)$$

From (5) and (2) it is very easy to obtain a linear integral equation of the Bethe-Salpeter type for T . One gets

$$T(p'_1 p'_2 | p_1 p_2) = U(p'_1 p'_2 | p_1 p_2) - \frac{i}{2} \int d^4 k_1 d^4 k_2 \frac{U(p'_1 p'_2 | k_1 k_2) T(k_1 k_2 | p_1 p_2)}{(k_1^2 - m^2 + i\epsilon)(k_2^2 - m^2 + i\epsilon)}, \quad (8)$$

where

$$U(p'_1 p'_2 | p_1 p_2) = U(t) \delta^4(p'_1 + p'_2 - p_1 - p_2), \quad (9)$$

$$U(t) = \frac{1}{(2\pi)^4} \int e^{-i(p_1 - p'_1) \cdot x} V(x) d^4 x.$$

The equation (9) is not completely defined until we specify how we are going to define the off-shell $U(p_1' p_2' / k_1 k_2)$. In order to satisfy condition iV) and end up with a T with essentially the same analytic properties as in the usual perturbation theory, we restrict ourselves to "potentials" which are of the form

$$V(x_1, x_2) = \frac{(2\pi)^4}{\pi} \int_{4m^2}^{\infty} \sigma(\mu^2) \Delta_c(\mu^2 | x_1, x_2) d\mu^2, \quad (10)$$

where σ is real. Using (3) and (9) we get

$$\begin{aligned} U(q_1, q_2 / k_1, k_2) &= U((q_1 - k_1)^2) \delta^4(q_1 + q_2 - k_1 - k_2) \\ U([q_1 - k_1]^2) &= -\frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\sigma(\mu^2)}{(q_1 - k_1)^2 - \mu^2 + i\epsilon} d\mu^2 \end{aligned} \quad (11)$$

In this last equation any or all of the vectors q_1, q_2, k_1, k_2 could be off the mass shell. In (10) and (11) we have neglected to put pole terms just to simplify the writing. They can be included without changing the essence of the arguments that follow in this section.

The iteration of (8) now leads to a series of ladder type diagrams of the form shown in Fig. 1 where the straight internal lines correspond to the usual Feynman propagators $(k^2 - m^2 + i\epsilon)^{-1}$ and the curved lines correspond to $(k^2 - \mu^2 + i\epsilon)^{-1} \sigma(\mu^2)$. The only difference between our diagrams and the usual Feynman diagrams is the final integration over the weight functions σ .

The amplitude T defined by (8) satisfies the elastic unitarity condition for $4m^2 < s < 16m^2$ as it is the usual Bethe-Salpeter amplitude.

$$\begin{aligned} \text{Im } T(p_1' p_2' | p_1 p_2) \\ = \frac{(2\pi)^2}{4} \int d^4 q_1 d^4 q_2 T(p_1' p_2' | q_1 q_2) T^*(p_1 p_2 | q_1 q_2) \delta(q_1^2 - m^2) \delta(q_2^2 - m^2) \\ \times \theta(q_1) \theta(q_2). \end{aligned} \quad (12)$$

Here $\text{Im} T$ is taken at fixed t . If (11) has a pole term at $(q_1 - k_1)^2 = m^2$ then (12) is only valid for $4m^2 < s < 9m^2$, otherwise if U is represented as in (11) then (12) is valid at least for $4m^2 < s < 16m^2$.

To define a crossing symmetric scattering amplitude one can easily crossing symmetrize (8) and obtain a linear equation that will define a new T which is crossing symmetric. However, this new amplitude will not, in general, satisfy unitarity any more. To avoid this difficulty and still be able to construct a unitary crossing symmetric amplitude, we choose to work with a non-linear equation which is equivalent to (8) but whose structure is more similar to that of unitarity. We can obtain such an equation by first constructing the total Green's function of equation (1) from a complete set of off-shell solutions of (1).

We consider the following off-shell generalization of (1),

$$\begin{aligned} (\square_1 - m^2)(\square_2 - m^2) \Psi_{k_1 k_2}(x_1, x_2) - (m^2 - k_1^2)(m^2 - k_2^2) \bar{\Psi}_{k_1 k_2}(x_1, x_2) \\ = -\frac{i}{2} V(x_1, x_2) \Psi_{k_1 k_2}(x_1, x_2). \end{aligned} \quad (13)$$

When $k_1^2 = k_2^2 = m^2$, this equation obviously reduces to (1). The choice of this off-shell generalization instead of the obvious one, i.e.

$$(\square_1 - k_1^2)(\square_2 - k_2^2) \bar{\Psi}_{k_1 k_2} + \frac{i}{2} V \bar{\Psi}_{k_1 k_2} = 0 ,$$

is necessary if one wants to construct the total Green's function of (1) from the off-shell $\bar{\Psi}$'s in a way analogous to ordinary potential scattering.

As in (2), we take solutions of (13) which satisfy the following integral equation

$$\begin{aligned} & \bar{\Psi}_{k_1 k_2}(x_1, x_2) \\ &= \bar{\Phi}_{k_1 k_2}(x_1, x_2) - \frac{i}{2} \iint d^4 x'_1 d^4 x'_2 K(k_1^2, k_2^2 | x_1 - x'_1, x_2 - x'_2) V(x'_1, x'_2) \bar{\Psi}_{k'_1 k'_2}(x'_1, x'_2) \end{aligned} \quad (14)$$

where the off-shell propagator K is given by

$$\begin{aligned} & K(k_1^2, k_2^2 | x_1 - x'_1, x_2 - x'_2) \\ &= \frac{1}{(2\pi)^8} \int d^4 q_1 d^4 q_2 \frac{e^{i q_1 \cdot (x_1 - x'_1)} e^{i q_2 \cdot (x_2 - x'_2)}}{[(q_1^2 - m^2 + i\epsilon_1)(q_2^2 - m^2 + i\epsilon_2) - (k_1^2 - m^2 - i\epsilon_2)(k_2^2 - m^2 - i\epsilon_2)]} \end{aligned} \quad (15)$$

The choice of sign of the $i\epsilon_2$ in the second term in the denominator will be explained later on. Nevertheless, it is still obvious that

$$K(m^2, m^2 | x_1 - x'_1, x_2 - x'_2) = \Delta_c(x_1 - x'_1) \Delta_c(x_2 - x'_2) .$$

Thus equation (14) reduces to (2) when $k_1^2 = k_2^2 = m^2$ and $\epsilon_2 \rightarrow 0$. The function Φ_{k_1, k_2} in (14) is just a product of two plane waves as in (4) and for any k_1 and k_2 is a solution of the homogeneous off-shell equation

$$[(\square_1 - m^2)(\square_2 - m^2) - (m^2 - k_1^2)(m^2 - k_2^2)] \Phi_{k_1, k_2}(x_1, x_2) = 0. \quad (16)$$

In order to express the Green's function of (1) in terms of an expansion of products of eigenfunctions the usual procedure has to be slightly modified since the differential operator we are dealing with, $D = (\square_1 - m^2)(\square_2 - m^2) + \frac{i}{2}V$, is not hermitian. We rewrite (13) in an abstract form

$$D \Psi_\alpha = \alpha \Psi_\alpha, \quad (13')$$

where $\alpha = (m^2 - k_1^2)(m^2 - k_2^2)$. The operator D^* has the following set of eigenstates

$$D^* \tilde{\Psi}_\beta = \beta \tilde{\Psi}_\beta. \quad (17)$$

Here we choose solutions of (17) that satisfy the following integral equation

$$\tilde{\Psi}_{k_1, k_2}(x_1, x_2) = \Phi_{k_1, k_2}(x_1, x_2) + \frac{i}{2} \iint K(k_1^2, k_2^2 | x_1 - x'_1, x_2 - x'_2) V^*(x'_1, x'_2) \tilde{\Psi}_{k_1, k_2}(x'_1, x'_2) dx'_1 dx'_2. \quad (18)$$

Now it is an elementary matter to show that $\widetilde{\Psi}_\beta$ and $\widetilde{\Psi}_\alpha$ are orthogonal when $\alpha \neq \beta$, i.e. $\int \widetilde{\Psi}_\beta^* \widetilde{\Psi}_\alpha d^4x_1 d^4x_2 = 0$, and $\widetilde{\Psi}$ and Ψ form a bi-orthogonal set of eigenfunctions ⁴⁾, (one should notice from (18) that $\widetilde{\Psi}$ is not Ψ^*). The total Green's function for (13) can now be written as

$$G_{k_1, k_2}(x_1, x_2 / y_1, y_2) = \int d^4k'_1 d^4k'_2 \frac{\Psi_{k'_1, k'_2}(x_1, x_2) \widetilde{\Psi}_{k'_1, k'_2}^*(y_1, y_2)}{[(k_1'^2 - m^2 + i\epsilon_1)(k_2'^2 - m^2 + i\epsilon_1) - (k_1'^2 - m^2 - i\epsilon_2)(k_2'^2 - m^2 - i\epsilon_2)]} \quad (19)$$

The total Green's function for (1) is obtained from (19) by setting $k_1^2 = k_2^2 = m^2$ and letting $\epsilon_2 \rightarrow 0$. The Green's function defined in (19) is a solution of the following equation

$$[(\square_1 - m^2)(\square_2 - m^2) + \frac{i}{2}V] G_{k_1, k_2} - (m^2 - k_1^2)(m^2 - k_2^2) G_{k_1, k_2} = \delta^4(x_1 - y_1) \delta^4(x_2 - y_2). \quad (20)$$

Here one uses the completeness of the off-shell $\widetilde{\Psi}$'s which can be written as

$$\iint d^4k'_1 d^4k'_2 \Psi_{k'_1, k'_2}(x_1, x_2) \widetilde{\Psi}_{k'_1, k'_2}^*(y_1, y_2) = \delta^4(x_1 - y_1) \delta^4(x_2 - y_2). \quad (21)$$

With the help of (19) one can formally write down the solution of (14)

$$\begin{aligned} \Psi_{k,k_2}(x_1, x_2) \\ = \Phi_{k,k_2}(x_1, x_2) - \frac{i}{2} \iint d^4 y_1 d^4 y_2 G_{k,k_2}(x_1, x_2 | y_1, y_2) V(y_1, y_2) \Phi_{k,k_2}(y_1, y_2). \end{aligned} \quad (22)$$

Similarly, one can construct the Green's function $\tilde{G}$ for (17) which will be given by the same formula as (19) with Ψ and $\tilde{\Psi}$ interchanged. One obtains for the solutions of (18) the following

$$\begin{aligned} \tilde{\Psi}_{k,k_2}(x_1, x_2) \\ = \Phi_{k,k_2}(x_1, x_2) + \frac{i}{2} \iint d^4 y_1 d^4 y_2 \tilde{G}_{k,k_2}(x_1, x_2 | y_1, y_2) V^*(y_1, y_2) \Phi_{k,k_2}(y_1, y_2). \end{aligned} \quad (23)$$

One can now define the following two off-shell matrix elements

$$\begin{aligned} T(q, q_2 | k, k_2) &= (\Phi_{q, q_2}, V \Psi_{k, k_2}), \\ \tilde{T}(q, q_2 | k, k_2) &= (\tilde{\Psi}_{k, k_2}, V \Phi_{q, q_2}). \end{aligned} \quad (24)$$

From (14) and (18) and the definitions (24) we find that T and $\tilde{T}$ satisfy the following two linear integral equations, which are off-shell generalizations of (8),

$$\begin{aligned}
& T(q, q_2 / k, k_2) \\
&= U(q, q_2 / k, k_2) - \frac{i}{2} \iint d^4 k' d^4 k'_2 K(k'^2, k'^2_2, \epsilon_1 / k^2, k^2_2, \epsilon_2) U(q, q_2 / k', k'_2) T(k', k'_2 / k, k_2),
\end{aligned}
\tag{25}$$

$$\begin{aligned}
& \tilde{T}(q, q_2 / k, k_2) = U(k, k_2 / q, q_2) \\
& - \frac{i}{2} \iint d^4 k' d^4 k'_2 K^*(k'^2, k'^2_2, \epsilon_2 / k^2, k^2_2, \epsilon_1) U(k', k'_2 / q, q_2) \tilde{T}(k', k'_2 / k, k_2).
\end{aligned}
\tag{26}$$

The propagator $K(k_1'^2, k_2'^2, \epsilon_1 / k_1^2, k_2^2, \epsilon_2)$ is just the Fourier transform of that given in (15) and is given by

$$\begin{aligned}
& K(k_1'^2, k_2'^2, \epsilon_1 / k_1^2, k_2^2, \epsilon_2) = \\
& \left[(k_1'^2 - m^2 + i\epsilon_1)(k_2'^2 - m^2 + i\epsilon_2) - (k_1^2 - m^2 - i\epsilon_2)(k_2^2 - m^2 - i\epsilon_2) \right]^{-1}
\end{aligned}
\tag{27}$$

Clearly, $K(k_1'^2, k_2'^2, \epsilon_1 / m^2, m^2)$ reduces to the product of two Feynman propagators as $\epsilon_2 \rightarrow 0$. The relabelling of ϵ_1 and ϵ_2 in the kernel of equation (26) is just for later convenience and has no significance at the present stage.

The non-linear equations we are seeking follow immediately from the definitions (24) if one substitutes in them the expressions (22) and (23) for $\underline{\Psi}$ and $\widetilde{\underline{\Psi}}$ making use of (19). One obtains

$$T(q'_1 q'_2 | q_1 q_2) = U(q'_1 q'_2 | q_1 q_2) \quad (28)$$

$$- \frac{i}{2} \iint d^4 k_1 d^4 k_2 T(q'_1 q'_2 | k_1 k_2) \widetilde{T}(q_1 q_2 | k_1 k_2) K(k_1^2 k_2^2, \epsilon_1 | q_1^2 q_2^2, \epsilon_2),$$

and

$$\widetilde{T}(q'_1 q'_2 | q_1 q_2) = U(q_1 q_2 | q'_1 q'_2)$$

$$- \frac{i}{2} \iint d^4 k_1 d^4 k_2 \widetilde{T}(q'_1 q'_2 | k_1 k_2) T(q_1 q_2 | k_1 k_2) K^*(k_1^2 k_2^2, \epsilon_2 | q_1^2 q_2^2, \epsilon_1). \quad (29)$$

The system of equations given above is completely equivalent to (25) and (26), and when the q 's are all on the mass shell (28) and (29) are equivalent to the Bethe-Salpeter equation, (8), that we started with. Superficially, however, it seems that an iteration of (28) and (29), even in the case when all the q 's are on the mass shell, will involve the use of the off-shell propagators K , while the iteration of (8) will have nothing but the usual Feynman propagators. It will be instructive for what follows for us to check explicitly at this stage that the two iterations are equivalent in lowest orders. That they are equivalent in first and second order is

obvious, and the first non-trivial case will be third order. The third order term in the iteration of (28) and (29) is

$$\begin{aligned}
 & T^{(3)}(q'_1 q'_2 | q_1 q_2) \\
 &= -\frac{1}{4} \int U(q'_1 q'_2 | k'_1 k'_2) K(k'_1 k'_2, \epsilon_1 | k_1^2 k_2^2, \epsilon_2) U(k'_1 k'_2 | k_1 k_2) K(k_1^2 k_2^2, \epsilon_1 | q_1^2 q_2^2, \epsilon_2) \\
 &\quad \times U(k_1 k_2 | q_1 q_2) d^4 k_1 d^4 k_2 d^4 k'_1 d^4 k'_2 \\
 &\quad - \frac{1}{4} \int U(q'_1 q'_2 | k_1 k_2) K(k_1^2 k_2^2, \epsilon_1 | q_1^2 q_2^2, \epsilon_2) U(k'_1 k'_2 | q_1 q_2) K^*(k'_1 k'_2, \epsilon_2 | k_1^2 k_2^2, \epsilon_1) \\
 &\quad \times U(k_1 k_2 | k'_1 k'_2) d^4 k_1 d^4 k_2 d^4 k'_1 d^4 k'_2
 \end{aligned} \tag{30}$$

In order to simplify this expression we need the following properties of the propagator K

$$K^*(k_1^2 k_2^2, \epsilon_1 | k'_1 k'_2, \epsilon_2) = -K(k'_1 k'_2, \epsilon_2 | k_1^2 k_2^2, \epsilon_1) \tag{31}$$

$$\begin{aligned}
 & K(k'_1 k'_2, \epsilon_1 | k_1^2 k_2^2, \epsilon_2) [K(k_1^2 k_2^2, \epsilon_1 | q_1^2 q_2^2, \epsilon_2) - K(k'_1 k'_2, \epsilon_1 | q_1^2 q_2^2, \epsilon_2)] \\
 &= K(k_1^2 k_2^2, \epsilon_1 | q_1^2 q_2^2, \epsilon_2) K(k'_1 k'_2, \epsilon_1 | q_1^2 q_2^2, \epsilon_2)
 \end{aligned} \tag{32}$$

These equations follow immediately from the definition, (27), of K . We now interchange the integration variables k and k' in the second term of (30) and then use (31) and (32) to get

$$\begin{aligned}
 & T^{(3)}(q'_1, q'_2 | q_1, q_2) \\
 &= -\frac{1}{4} \int U(q'_1, q'_2 | k', k'_2) K(k', k'_2 | q_1, q_2, \epsilon_2) U(k', k'_2 | k, k_2) K(k, k_2 | q_1, q_2, \epsilon_1) \\
 & \quad \times U(k, k_2 | q_1, q_2) d^4 k, d^4 k_2 d^4 k', d^4 k'_2. \quad (33)
 \end{aligned}$$

This is exactly the same form as the iteration of the linear equation (25) and when $q_1^2 = q_2^2 = m^2$ and $\epsilon_2 \rightarrow 0$ the two propagators K in (33) reduce to products of ordinary Feynman propagators as in the iteration of (8). A similar reduction takes place in higher orders. At this stage it becomes clear why we chose the signs of the $i\epsilon_2$ in (15) as we did, namely we wanted to guarantee that this reduction will take place. In doing this we were guided very much by the case of non-relativistic potential scattering where a similar non-linear off-shell equation exists which is equivalent to the linear Lippman-Schwinger equation. Essentially, the same kind of reduction takes place in potential scattering as that between (30) and (33).

In potential scattering the potentials are always real and the non-linear equation analogous to (28) and (29) gives immediately the unitarity condition. In the present case the connection between unitarity and (28) and (29) is not immediately apparent. We shall turn to a discussion of this point.

When all four external momenta are on the mass shell (28) can be rewritten as

$$T(p_1' p_2' / p_1 p_2) = U(p_1' p_2' / p_1 p_2)$$

$$= -\frac{i}{2} \int d^4 k_1 d^4 k_2 T(p_1' p_2' / k_1 k_2) \tilde{T}(p_1 p_2 / k_1 k_2) (k_1^2 - m^2 + i\epsilon)^{-1} (k_2^2 - m^2 + i\epsilon)^{-1} \quad (28')$$

Here we have relabelled ϵ_1 as ϵ . A similar equation holds for $\tilde{T}(p_1' p_2' / p_1 p_2)$. Now if we use the fact that $T(p_1' p_2' / p_1 p_2) = T(p_1 p_2 / p_1' p_2')$ when all the p 's are on the mass shell we obtain

$$\begin{aligned} T(p_1' p_2' / p_1 p_2) - \tilde{T}(p_1' p_2' / p_1 p_2) \\ = -\frac{i}{2} \int d^4 k_1 d^4 k_2 T(p_1' p_2' / k_1 k_2) \left[(k_1^2 - m^2 + i\epsilon)^{-1} (k_2^2 - m^2 + i\epsilon)^{-1} - (k_1^2 - m^2 - i\epsilon)^{-1} (k_2^2 - m^2 - i\epsilon)^{-1} \right] \times \\ \times \tilde{T}(p_1 p_2 / k_1 k_2). \end{aligned} \quad (34)$$

This last equation is not yet the usual unitarity condition but as we shall see below it is equivalent to it for $4m^2 < s < 16m^2$. Unitarity is concerned with the difference between $T^{(+)} \equiv T$ and $T^{(-)}$, where $T^{(-)}$ is obtained by changing all the $+i\epsilon$ in $T^{(+)}$ to $-i\epsilon$. However, $\tilde{T}$, as we have defined it, is obtained from $T^{(+)}$ by changing $+i\epsilon$ to $-i\epsilon$ only in Feynman propagators corresponding to the straight internal lines in Fig. 1 and not changing the sign of the $i\epsilon$ in the "potential" propagators corresponding to the curved lines of Fig. 1. Clearly then, for physical scattering and for "potentials" given by (11), $\tilde{T}$ is essentially the same as $T^{(-)}$ as long as $4m^2 < s < 16m^2$. Any singularities coming from cutting across a potential line will lie beyond $s = 16m^2$. One can now use the arguments of Cutkosky ⁵⁾ and write, from (34) for $4m^2 < s < 16m^2$

$$\begin{aligned} T^{(+)} - T^{(-)} \\ = -\frac{i}{2} \int d^4 k_1 d^4 k_2 T^{(+)}(p_1' p_2' / k_1 k_2) \left[(2\pi i)^2 \delta(k_1^2 - m^2) \delta(k_2^2 - m^2) \theta(k_1^0) \theta(k_2^0) \right] T^{(-)}(p_1 p_2 / k_1 k_2). \end{aligned} \quad (35)$$

This last equation is just the unitarity condition.

III. The crossing symmetric equation

In this section we shall symmetrize the set of equations (28) and (29) and obtain two equations which define a new amplitude that is both unitary and crossing symmetric and still has the same analytic properties as the usual perturbation theory.

The symmetrization is easier if we take all our external momenta as ingoing momenta and consider

$$M(q_1 q_2 | q_3 q_4) \equiv M(q_i q_j) \delta^4(q_1 + q_2 + q_3 + q_4). \quad (36)$$

In analogy to (28) we shall define M by the following non-linear equation

$$M(q_1 q_2 | q_3 q_4) = U(q_1 q_2 | q_3 q_4) + U(q_1 q_3 | q_2 q_4) + U(q_1 q_4 | q_2 q_3) \\ - \frac{i}{2} \int \left\{ \frac{1}{8} \sum_{\text{perm.}} M(q'_1 q'_2 | k_1 k_2) \tilde{M}(-q'_3 q'_4 | k_1 k_2) \right\} K(k_1^2 k_2^2, \epsilon_1 | q_3^2 q_4^2, \epsilon_2) d^4 k_1 d^4 k_2, \quad (37)$$

where q'_1, q'_2, q'_3, q'_4 is a permutation of q_1, q_2, q_3, q_4 and we sum over all permutations of the external momenta. Here U is given by a slightly different notation than in the previous section. We take

$$U(q_1 q_2 | q_3 q_4) = U([q_1 + q_2]^2) \delta^4(q_1 + q_2 + q_3 + q_4),$$

and $U([q_1 + q_2]^2)$ has the same form as in (11) except for the fact that, as we shall find out below, unitarity forces us in this case to have $\sigma(\mu^2) = 0$ for $\mu^2 < 16m^2$. Similarly, $\tilde{M}$ is defined by

$$\begin{aligned} \tilde{M}(q_1, q_2 | q_3, q_4) &= U(q_3, q_4 | q_1, q_2) + U(q_2, q_4 | q_1, q_3) + U(q_4, q_3 | q_1, q_2) \\ &- \frac{i}{2} \int \left\{ \frac{1}{8} \sum_{\text{perm.}} \tilde{M}(q'_1, q'_2 | k_1, k_2) M(-q'_3, -q'_4 | k_1, k_2) \right\} K^*(k_1^2, k_2^2, \epsilon_2 | q_1^2, q_2^2, \epsilon_1) \\ &\quad \times d^4 k_1 d^4 k_2. \end{aligned} \quad (38)$$

It is clear that M as defined in (37) is not completely crossing symmetric since the q_3^2 and q_4^2 appearing in the propagator K are not permuted and summed over with the permutations. However, when M is on the mass shell and $q_i^2 = m^2$, then M defined by (37) is manifestly crossing symmetric. We have in that case for $p_i^2 = m^2$

$$\begin{aligned} M(p_1, p_2 | p_3, p_4) &= U(p_1, p_2 | p_3, p_4) + U(p_1, p_3 | p_2, p_4) + U(p_1, p_2 | p_4, p_3) \\ &- \frac{i}{2} \int \left\{ \frac{1}{8} \sum_{\text{perm.}} M(p'_1, p'_2 | k_1, k_2) \tilde{M}(-p'_3, -p'_4 | k_1, k_2) \right\} (k_1^2 - m^2 + i\epsilon)^{-1} (k_2^2 - m^2 + i\epsilon)^{-1} d^4 k_1 d^4 k_2. \end{aligned} \quad (39)$$

The fact that the K 's are kept outside the sum over the permutations allows us in the iteration of (37) and (38) to carry through the same reduction of the K propagators as was done in the previous section in equations (30)-(33). The algebra is still the same and the products of K and K^* propagators will appear in the same form in the iteration of (37) and (38) as in the iteration of (28) and (29). One can again check the reduction in lowest orders explicitly. Thus the iteration of (37) will reduce to terms that are composed of products of U 's and ordinary Feynman propagators when all four external momenta are on the mass shell. Some of the crossing symmetric sets of diagrams generated by (37) are shown in Fig. 2, where the straight lines

correspond to ordinary Feynman propagators $(k^2 - m^2 + i\epsilon)^{-1}$ and the circles represent the potentials. Each circle corresponds to a propagator of the form $(q^2 - \mu^2 + i\epsilon)^{-1} \sigma(\mu^2)$ joining two pairs of the four lines coming to the circle in one of three different ways. Except for the final integrations over the σ 's our iteration leads to ordinary Feynman graphs, and the analytic properties of our amplitude M will be the same as the analytic properties of ordinary Feynman diagrams. The graphs that we obtain are such that in at least one of the three s, t, u channels a graph has at least one two-particle intermediate state. If one cuts the graph perpendicular to the internal lines representing this intermediate state one obtains two graphs that still belong to the set generated by (37). In this section we are using the notation

$$s = (p_1 + p_2)^2; \quad t = (p_1 + p_3)^2; \quad u = (p_1 + p_4)^2.$$

We are left now with the task of demonstrating that the symmetrization effected in this section does not violate unitarity in the elastic region. It is immediately apparent from (39) that the potentials U cannot have a cut below $16m^2$ if we want to satisfy unitarity in the physical s -channel for $4m^2 < s < 16m^2$. For since our first order term consists of a sum of $[U(t) + U(u) + U(s)]$, (after we take out the over-all δ -function), we would then have a contribution to $\text{Im } M$ in first order in U which would be impossible to reconcile with unitarity. Hence we must restrict ourselves to U 's that can be represented as

$$U(t) = -\frac{1}{\pi} \int_{16m^2}^{\infty} \frac{\sigma(\mu^2)}{t - \mu^2 + i\epsilon} d\mu^2. \quad (11')$$

A limiting case of these potentials would occur when the onset of the cut moves out to infinity; i.e., $U(s)+U(t)+U(u) = \lambda$ where λ is just a constant. This would lead to ϕ^4 type theory and the circles in the diagrams of Fig. 2 will collapse to points. Another way to avoid having first order contributions to $\text{Im } M$ in the elastic region is to take U 's that consist of pole terms with the position of the poles lying below $4m^2$. We shall not discuss these cases here for they are uninteresting for the pion-pion problem and also lead to a more complicated situation in higher orders.

By restricting ourselves to U 's of the form given by (11') we not only avoid difficulties with the first order contribution but are also able to avoid further difficulties in higher orders. The argument now runs as in the previous section and one obtains, in analogy to (34), from (37) and (38)

$$\begin{aligned}
 & M(p_1 p_2 | p_3 p_4) - \tilde{M}(p_1 p_2 | p_3 p_4) \\
 &= -\frac{i}{2} \int \left\{ \frac{1}{8} \sum_{\text{perm.}} M(p'_1 p'_2 | k_1 k_2) \tilde{M}(-p'_3 - p'_4 | k_1 k_2) \right\} \left[\frac{(k_1^2 - m^2 + i\epsilon)^{-1} (k_2^2 - m^2 + i\epsilon)^{-1}}{-(k_1^2 - m^2 - i\epsilon)^{-1} (k_2^2 - m^2 - i\epsilon)^{-1}} \right] d^4 k_1 d^4 k_2, \quad (40)
 \end{aligned}$$

$$4m^2 < (p_1 + p_2)^2 < 16m^2,$$

where all $p_i^2 = m^2$; p'_1, p'_2, p'_3, p'_4 is a permutation of p_1, p_2, p_3, p_4 ; and the p 's are in the physical region of the s -channel. The term in the straight bracket is obviously zero when both k_1^2 and k_2^2 are off the mass shell. Now one again remarks that the only difference between $\tilde{M}$ and $M^{(-)}$ is the sign of the $i\epsilon$ in the potential propagators. Clearly, because of (11') and

the theorem of Cutkosky, the differences $M^{(+)} - \tilde{M}$ and $M^{(+)} - M^{(-)}$ are the same for $4m^2 < s < 16m^2$. In other words, in finding contributions to singularities in the elastic region one, because of (11'), does not have to cut the potential lines. If we now again use the Cutkosky theorem ⁵⁾ and replace the straight bracket term in (40) by a product of δ -functions we get the unitarity condition

$$M^{(+)}(p_1 p_2 | p_3 p_4) - M^{(-)}(p_1 p_2 | p_3 p_4) \\ = -\frac{i}{2} \int M^{(+)}(p_1 p_2 | k_1 k_2) \left[(2\pi i)^2 \delta(k_1^2 - m^2) \delta(k_2^2 - m^2) \theta(k_1^0) \theta(k_2^0) \right] M^{(-)}(p_3 p_4 | k_1 k_2) d^4 k_1 d^4 k_2, \quad (41)$$

$4m^2 < s < 16m^2.$

All the permutations in (40) which lead to δ -functions of the form $\delta(p_1 + p_3 + k_1 + k_2)$ and $\delta(p_1 + p_4 + k_1 + k_2)$ drop out when k_1^2 and k_2^2 are on the mass shell since we are considering physical scattering in the s-channel and both $t = (p_1 + p_3)^2$ and $u = (p_1 + p_4)^2$ are negative. Only eight of the 4! permutations contribute equally and the factor 1/8 gives us (41).

Similarly, by considering p_1, p_2, p_3, p_4 in the physical region of the t- and u-channels respectively one obtains equations analogous to (40) and (41) giving unitarity in the t-channel and the u-channel for $4m^2 < t < 16m^2$ and $4m^2 < u < 16m^2$ respectively.

The symmetrization exhibited in (37) and (38) does, of course, introduce diagrams which have intermediate states of four, six, eight or more particles. This is a necessary by-product of making the two-particle amplitudes crossing

symmetric, and it does not affect the unitarity condition below $16m^2$ in any channel. It is clear, however, that our amplitude M violates unitarity for $s, t, \text{ or } u$ greater than $16m^2$, even if $\sigma(\mu^2)$ is zero beyond $16m^2$. One should also add here that in the present case one has to be much more careful in applying the Cutkosky theorem and going from (40) to (41) than in the similar argument of the previous section. To check that in going from (40) to (41) we made no errors, we make the following remark. The class of Feynman diagrams generated by (37) has the property that whenever one diagram is cut perpendicular to any two-particle intermediate state, one obtains two lower order diagrams which are also in the class generated by (37). Similarly, it is evident from the structure of (37) that if any two diagrams generated by (37) are fused together by joining a pair of external lines from each, the resulting higher order diagram will also belong to the class generated by (37). These properties would be enough to guarantee unitarity below $16m^2$ if the number (or multiplicity) of our diagrams is the same as in the usual perturbation theory. We have checked the multiplicity of our diagrams in lowest orders and they give the correct result consistent with unitarity and perturbation theory. That the multiplicity is consistent with elastic unitarity is clear since equation (40) is similar enough to unitarity to guarantee at least that.

One final remark we can make here is that in writing down (37) and (38) one need not restrict himself to first order terms of the form $U(t)+U(s)+U(u)$. One can presumably start with any crossing symmetric function, $f(s,t,u)$, which satisfies the Mandelstam representation with a spectral function that vanishes whenever at least one of the variables is less than $16m^2$. This ensures that f has no elastic cuts. The off-shell "potential" f could then be obtained from the on-shell f by analytic continuation via the Mandelstam representation for f .

IV. Conclusions

In their present form equations (37) and (38) are not, to say the least, very useful for any direct calculation. In addition to their off-shell character they also generate diagrams which have the conventional divergences of ordinary perturbation theory. In principle, one can renormalize them by subtracting from (39) a similar equation for M evaluated at $s_0 = t_0 = u_0 = \frac{4m^2}{3}$. In the case where $U(s)+U(t)+U(u) = \lambda$, i.e., the ϕ^4 case, the potential will disappear in this subtraction and our new "coupling constant" will be $M(s_0, t_0, u_0)$ just as in the Chew-Mandelstam treatment of the pion-pion problem.

Nevertheless, the equations (37) and (38) answer the question posed in the introduction concerning the uniqueness of a scattering amplitude determined from the four conditions listed in Section I. As we mentioned earlier, even though $\sigma(\mu^2) = 0$ for $\mu^2 < 16m^2$ still our "potentials" will contribute to the Mandelstam spectral functions in second order in U in the elastic strip. In other words for large enough s there is a contribution to $\rho(s, t)$ in the region $4m^2 < t < 16m^2$, and similarly for large enough t there is a contribution to $\rho(s, t)$ in the region $4m^2 < s < 16m^2$. It is clear that we can choose very different σ 's that will generate quite different amplitudes. Thus, at least in the language of the present model, one must somehow determine σ by adding some extra conditions to the four listed in the introduction. Furthermore, we add here that if the Mandelstam representation is correct in perturbation theory (at least for the nearest singularities) then any set of equations derived from elastic unitarity and the Mandelstam representation such as those recently given by Ter-Martirosyan⁶⁾ and Zimmermann⁷⁾ and the original equations of Chew and Mandelstam¹⁾ do not have a unique solution. Any solution of (37) will be a solution of any of these equations.

In this connection, one can distinguish three different situations. The first would be that, as implied in the original approach to the pion-pion problem, the weight function $\sigma(\mu^2)$ is essentially zero even much beyond

$\mu^2 = 16m^2$ and its exact form is not needed. In this case one has essentially the ϕ^4 theory. However, in solving the Chew-Mandelstam equations one must make sure in this case that one picks the solution that corresponds to $\sigma(\mu^2) \approx 0$. The second situation would be that one would try to put more restrictions on $\sigma(\mu^2)$ by considering other physical conditions such as the idea of the saturation of the unitarity condition. The third situation would result if the short range effects are important. In that case one would have to put in more information and clearly one would not be able to construct the σ 's from any "bootstrap" operation. In fact, we would be fortunate if just the σ 's and the four conditions listed in the introduction are enough to describe the physical situation without the necessity of going into the details of four-particle intermediate states.

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Fig. 1.

Fig. 2.