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SOME SIMPLE INEQUALITIES IN THE SCATTERING BY COMPLEX POTENTIALS

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ABSTRACT

It is shown that if the range of the imaginary part of a complex potential is not too small as compared to the range of the real part of the potential a number of inequalities between total cross-sections, absorption cross-sections and elastic cross-sections can be derived by extensive use of the Schwarz inequality. This not only holds for the whole process but also for a given partial wave. Upper and lower bounds of the various cross-sections are obtained. An important result is that the elastic cross-section decreases faster with angular momentum than the absorption cross-section. We investigate whether this result still holds when the above conditions are released. It turns out that this depends on the specific shape of the potential. For a strictly finite range potential this result is still true, but for Yukawa shaped potentials at sufficiently high energy the absorption cross-section decreases faster with increasing angular momentum than the elastic cross-section if the range of the imaginary potential is less than half the range of the real potential.

I. Introduction

In many scattering processes the disappearance of particles from the initial channel can be described by a complex potential. This has been done not only in low energy nuclear physics ¹⁾ but also in elementary particle physics, for instance in the case of nucleon-antinucleon scattering ²⁾. Furthermore an unambiguous procedure has been given by Gy. Targonski and the author ³⁾ to associate to a given scattering process at a given energy a potential, provided the scattering amplitude under consideration satisfies the Mandelstam representation. We wish to present here a few simple results on the scattering by complex potentials in general. Under certain conditions on the ranges of the real and imaginary potentials, which are unfortunately rather restrictive, one can find a number of inequalities between scattering cross-sections, absorption cross-sections, and total cross-sections. These inequalities hold for the over-all process and for a given partial wave. In the high partial wave case one can establish the validity of the Born approximation and investigate the case where the conditions on the ranges of the potentials are not fulfilled.

II. Inequalities for the over-all process

The particles interact through a central complex potential

$$V_1(x) - i V_2(x) \quad (1)$$

where $V_2(x)$ is purely positive, i.e. purely absorptive. We denote $\psi_{\underline{k}}(\underline{x})$ as the wave function including a plane wave $e^{i\underline{k}\underline{x}}$ plus an outgoing wave. The scattering amplitude is given by

$$f(\underline{k}_f, \underline{k}_i) = \frac{2m}{4\pi} \int e^{-i\underline{k}_f \underline{x}} \left[V_1(\underline{x}) - i V_2(\underline{x}) \right] \psi_{\underline{k}_i}(\underline{x}) d^3x \quad (2)$$

where m is the reduced mass, ($\hbar = c = 1$).

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From (2) the total cross-section can be obtained by making use of the optical theorem :

$$\sigma_t = \frac{4\pi}{k} \operatorname{Im} f(\mathbf{k}_i, \mathbf{k}_i) \quad (3)$$

The absorption cross-section may be shown to be given by

$$\sigma_a = \frac{2m}{k} \int |\psi_{\mathbf{k}_i}(\mathbf{x})|^2 V_2(\mathbf{x}) d^3x \quad (4)$$

We can majorate (3) by replacing $\operatorname{Im} f$ by $|f|$. Using (2) and making use of the Schwarz inequality we get :

$$\frac{\sigma_t^2}{\sigma_a} < \frac{2m}{k} \int \left[\frac{V_1^2(\mathbf{x})}{V_2(\mathbf{x})} + V_2(\mathbf{x}) \right] d^3x = \frac{2m}{k} I \quad (5)$$

$$I = \int W(\mathbf{x}) d^3x, \quad \text{where } W(\mathbf{x}) = \frac{V_1^2(\mathbf{x})}{V_2(\mathbf{x})} + V_2(\mathbf{x})$$

An immediate consequence of (5) is

$$\sigma_a < \sigma_t < \frac{2m}{k} I \quad (6)$$

Another consequence, obtained by imposing that σ_t must be larger than the elastic cross-section σ_e is

$$\sigma_e < \frac{1}{4} \frac{2m}{k} I \quad (7)$$

Of course (6) and (7) are much weaker than (5). At this point let us make a few remarks. First the integral I does not necessarily make sense. In the case of strictly finite potentials we see that the range of the imaginary potential must be at least as large as the range of the real potential. In the case of a potential with an exponential or Yukawa tail the characteristic range

of the imaginary potential must be larger than half the characteristic range of the real potential; this is an unfortunate limitation which restricts very seriously the field of application of our inequalities; on the other hand singularities as strong as $r^{\epsilon-3}$ at the origin are permitted if they are both present in the real and imaginary parts of the potential. The second thing we notice is that inequality (5) blows up as V_2 goes to zero; this is natural since then, the absorption cross-section goes to zero. Finally, we see that for fixed $V_1(x)$ we get the lowest bound for $V_2(x) = V_1(x)$.

In the special case of a purely imaginary potential (5) and (6) become

$$\frac{\sigma_t^2}{\sigma_a} < \sigma_{a \text{ Born}}; \quad \sigma_t < \sigma_{a \text{ Born}}. \quad (8)$$

So, in this case, as soon as the Born approximation becomes valid for the absorption cross-section there is no room left for elastic scattering.

The weakness of inequality (5) lies in the fact that it does not take advantage of the sign of the real potential. This is probably the reason why a comparison with actual calculations with Yukawa wells ²⁾ shows that (5) is almost an equality for purely imaginary potentials but that it is far from being the case when a real potential is present.

One may as well derive an inequality of the elastic cross-section. From (2) :

$$\frac{d\sigma_e}{d\Omega} < \left(\frac{2m}{4\pi}\right)^2 \left[\int \left| V_1 - i V_2 \right| \left| \psi_{k_i} \right|^2 d^3x \right]^2,$$

at any angle. Applying once more the Schwarz inequality, we get

$$\frac{\sigma_e}{\sigma_a} < \frac{mk}{2\pi} I \quad (9)$$

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One can overcome the difficulty that this inequality becomes meaningless as k goes to infinity by a more refined treatment. Using the integral form of the Schrödinger equation :

$$e^{ik_1 x} = \psi_{k_1}(x) - \frac{2m}{4\pi} \int \frac{e^{ik|x-x'|}}{|x-x'|} [V_1(x') - iV_2(x')] \psi_{k_1}(x') d^3x' \quad (10)$$

we can obtain, by substitution in (2) and (3), making use of (4) :

$$\sigma_e = \text{Im} \frac{(2m)^2}{4\pi k} \int \left[(V_1 - iV_2) \psi \right]_x^* \frac{e^{ik|x-x'|}}{|x-x'|} \left[(V_1 - iV_2) \psi \right]_{x'} d^3x d^3x'$$

Then using the Schwarz inequality we get

$$\frac{\sigma_e}{\sigma_a} < \frac{mk}{2\pi} J \quad (11)$$

where

$$J = \frac{1}{k} \int \int W(x) \frac{\sin^2 k|x-x'|}{|x-x'|^2} W(x') d^3x d^3x' < I$$

for $k=0$ $J=I$, but as k goes to infinity kJ has a finite limit.

Multiplication of (7) by (9) shows that $\sigma_e < \frac{m^2}{\pi} I^2$, which proves that if $V_2(x)$ does not vanish at zero energy (the potential might be energy dependent) the elastic cross-section is finite at zero energy.

More sophisticated inequalities might be obtained by making use of the integral equation for ψ they are of the type:

$$\begin{aligned} [\sigma_t - \sigma_{t \text{ nth Born}}]^2 / \sigma_a &< C_n \\ [\sigma_e - \sigma_{e \text{ nth Born}}] / \sigma_a &< D_n \end{aligned}$$

So far we have obtained upper limits of the various cross-sections. In fact it turns out that under the same conditions on the ranges of the potential, a lower limit of the absorption cross-section can be obtained. Starting from Eq. (10) we can show, by Schwarz inequality :

$$1 < |\psi(x)| + \frac{2m}{4\pi} \sqrt{\frac{d^3x W(x')}{|x-x'|}} \sqrt{|\psi|^2 V_2(x'') d^3x''}$$

multiplying by $V_2(x)$, integrating, and applying the Schwarz inequality to the two terms in the right hand side we eventually get

$$\sigma_a > \sigma_{a \text{ Born}} \left[1 + \frac{2m}{4\pi} \sqrt{\frac{d^3x d^3x' V_2(x) W(x')}{|x-x'|^2}} \right]^{-2} \quad (12)$$

This establishes the behaviour in $\frac{1}{k}$ of the absorption cross-section as k goes to zero [for fixed $V_2(x)$]. As a consequence we see that

$$\frac{(\sigma_t)^2}{\sigma_e} > 4 \sigma_{a \text{ minimum}} \quad (13)$$

The quantity $\frac{(\sigma_t)^2}{\sigma_e}$ is considered as a measure of the "range" R of the interaction ⁴⁾ through the formula

$$\frac{(\sigma_t)^2}{\sigma_e} \ll \frac{4\pi}{k^2} (kR+1)^2.$$

Unfortunately (12) does not provide a very severe limitation for strong potentials.

From (12) one could get a lower limit of the elastic cross-section if one knows the number of partial waves contributing to the process. If not, it is impossible. So we postpone this question to the last section.

III. Inequalities for the partial wave cross-sections

Let us call u_ℓ the radial, reduced, wave function. Its normalisation is fixed by the asymptotic behaviour

$$u_\ell(r) \simeq e^{-i(kr - \ell\pi/2)} - \eta_\ell e^{i(kr - \ell\pi/2)}$$

It satisfies the integral equation

$$u_\ell(r) = 2i kr j_\ell(kr) + \frac{2m}{k} \int j_\ell(kr <) h_\ell^{(1)}(kr >) k^2 r r' [V_1(r') - iV_2(r')] u_\ell(r') dr' \quad (14)$$

where the j 's and the h 's are defined as in Schiff's book ⁵⁾.

We write down the standard formulae for the various cross-sections ⁶⁾:

$$\begin{aligned} \sigma_{e,\ell} &= \frac{\pi}{k^2} (2\ell+1) |1 - \eta_\ell|^2 \\ \sigma_{a,\ell} &= \frac{\pi}{k^2} (2\ell+1) (1 - |\eta_\ell|^2) \\ \sigma_{t,\ell} &= \sigma_{a,\ell} + \sigma_{e,\ell} = \frac{2\pi}{k^2} (2\ell+1) [1 - \operatorname{Re} \eta_\ell] \end{aligned} \quad (15)$$

Using standard techniques we get the following expressions for these quantities:

$$1 - \eta_\ell = -\frac{2m}{k} \int_0^\infty kr j_\ell(kr) [V_1(r) - iV_2(r)] u_\ell(r) dr \quad (16)$$

$$1 - |\eta_\ell|^2 = \frac{2m}{k} \int_0^\infty V_2(r) |u_\ell(r)|^2 dr \quad (17)$$

$$\operatorname{Re}(1 - \eta_\ell) = -\frac{2m}{k} \operatorname{Re} \int_0^\infty kr j_\ell(kr) [V_1(r) - iV_2(r)] u_\ell(r) dr \quad (18)$$

Application of the same technique as in the preceding section yields :

$$\frac{|1-\eta_\ell|^2}{1-|\eta_\ell|^2} \leq \frac{2m}{k} \int_0^\infty [kr j_\ell(kr)]^2 W(r) dr = 2mk K_\ell \quad (19)$$

$$\frac{[\operatorname{Re}(1-\eta_\ell)]^2}{1-|\eta_\ell|^2} \leq 2mk K_\ell \quad (20)$$

and, hence

$$\frac{\sigma_{e,\ell}}{\sigma_{a,\ell}} \leq 2mk K_\ell \quad (21)$$

$$\sigma_{t,\ell} < \frac{(\sigma_{t,\ell})^2}{\sigma_{a,\ell}} \ll \frac{4\pi m}{k} (2\ell+1) K_\ell \quad (22)$$

A very crude bound for K_ℓ may be obtained by noticing that $j_\ell^2(kr) < \frac{1}{2\ell+1}$ (see Appendix) so $K_\ell < \frac{1}{2\ell+1} \frac{1}{4\pi}$. Hence whenever the inequalities for the over-all cross-sections are valid the inequalities for the partial wave cross-sections are valid too. We immediately see that under the above stated conditions on the respective ranges of V_1 and V_2 the ratio elastic/absorption goes to zero as ℓ increases, and the total cross-section in the ℓ^{th} wave is bounded by a fixed number. For specific shapes of V_1 and V_2 more accurate estimations of K_ℓ can be done. For instance in the case of a square well of radius R , since $|j_\ell(X)| < \frac{X^\ell}{(2\ell+1)!!}$,

$$K_\ell < \left(\frac{V_1^2}{V_2} + V_2 \right) \frac{k^2 R^{2\ell+3}}{(2\ell+1)!!(2\ell+3)!!} \quad (23)$$

so, K_ℓ decreases very rapidly with ℓ .

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Another example is the case of two Yukawa potentials

$$V(r) = C_1 \frac{e^{-\mu_1 r}}{r} - iC_2 \frac{e^{-\mu_2 r}}{r},$$

with $2\mu_1 > \mu_2$, then

$$K = \frac{1}{2k^2} \left[\frac{C_1^2}{C_2} Q e^{\left(\frac{2k^2 + (2\mu_1 - \mu_2)^2}{2k^2} \right)} + C_2 Q e^{\left(\frac{2k^2 + \mu_2^2}{2k^2} \right)} \right] \quad (24)$$

(see Appendix).

It is shown in the Appendix that the Q 's decrease exponentially with ℓ , more precisely

$$Q_\ell(x) < \frac{1}{2} \log \frac{x+1}{x-1} \left(\frac{1}{x + \sqrt{x^2 - 1}} \right)^\ell \quad \text{for } x > 1.$$

So, in specific cases, we can get an estimate of the number of partial waves contributing to the total cross-section and an estimate of the number of partial waves contributing to the elastic cross-section. With increasing ℓ , if the conditions on the ranges are fulfilled, the elastic cross-section vanishes before the total cross-section.

As was done in Section II a lower limit of the absorption cross-section in the ℓ^{th} wave can be obtained. One starts from the integral equation (14) and one uses the following upper bound (see Appendix)

$$kr < kr > \left| j_\ell(kr <) h_\ell(kr >) \right| < k \sqrt{r < r >} \sqrt{\frac{\pi}{2\ell + 1}} \quad (25)$$

then repeated use of the Schwarz inequality yields :

$$\sigma_{a,\ell} > \sigma_{a,\ell}^{\text{Born}} \left[1 + 2m \sqrt{\frac{\pi}{2\ell+1}} \sqrt{\int V_2(r)r \, dr} \sqrt{\int W(r)r \, dr} \right]^{-2} \quad (26)$$

So, for large ℓ , the lower limit coincides with the Born approximation. In particular, comparing with the upper bound of $\sigma_{a,\ell}$, we see that for a purely imaginary potential the Born approximation is valid for large ℓ .

A question now arises; what happens if the conditions on the ranges of the potentials are not fulfilled? In the preceding section we did not make any attempt to answer this question. In the partial wave case we have, however, the chance that the Born approximation holds for large ℓ . We shall repeat the proof here. Combining the integral equation (14) and the bound (25) we get

$$|u_\ell(r)| < 2kr |j_\ell(kr)| + 2m \sqrt{r} \int \sqrt{r'} |V_1(r') - iV_2(r')| u_\ell(r') dr' \times \sqrt{\frac{\pi}{2\ell+1}}$$

this yields

$$\int \sqrt{r} |V_1 - iV_2| |u_\ell(r)| dr < \frac{\int \sqrt{r} 2kr |j_\ell(kr)| |V_1 - iV_2| dr}{1 - 2m \sqrt{\frac{\pi}{2\ell+1}} \int r |V_1 - iV_2| dr}$$

provided $\int r |V_1 - iV_2| dr$ exists, which is a much weaker requirement than the previous ones, and ℓ is chosen large enough so that the denominator is positive. Now

$$\begin{aligned} & \left| \int 2iu_\ell (V_1 - iV_2) kr j_\ell(kr) dr - \int (2i)^2 (V_1 - iV_2) [kr j_\ell(kr)]^2 dr \right| \\ & < \sqrt{\frac{\pi}{2\ell+1}} \frac{2m \left[\int \sqrt{r} 2kr |j_\ell(kr)| |V_1 - iV_2| dr \right]^2}{1 - 2m \sqrt{\frac{\pi}{2\ell+1}} \int r |V_1 - iV_2| dr} \end{aligned} \quad (27)$$

Let us now make the assumption (probably unnecessary) that V_1 and V_2 have each one definite sign.

Then, since

$$\int |A(x) + iB(x)| dx < \int |A(x)| dx + \int |B(x)| dx < \sqrt{2} \left| \int A(x) dx + i \int B(x) dx \right|$$

if A and B are one sign functions we get, using once more the Schwarz inequality :

$$\left| \frac{1 - \eta_\ell}{(1 - \eta_\ell)_{\text{Born}}} - 1 \right| < 2m \sqrt{\frac{2\pi}{2\ell+1}} \int r |V_1 - iV_2| dr \left[1 - 2m \sqrt{\frac{\pi}{2\ell+1}} \int r |V_1 - iV_2| dr \right]^{-1} \quad (28)$$

Therefore the convergence towards the Born result for large ℓ is established provided V_1 , as well as V_2 , have definite sign. This probably still holds if V_1 changes sign a finite number of times, because as ℓ increases, larger and larger distances in the potential are selected by $j_\ell(kr)$, and so this brings us back to the one sign case.

Starting from the Born approximation we easily get the ratio $\sigma_{e,\ell} / \sigma_{a,\ell}$

$$\left(\frac{\sigma_{e,\ell}}{\sigma_{a,\ell}} \right)_{\text{Born}} = \frac{2m}{k} \frac{\left[\int [kr j_\ell(kr)]^2 V_1(r) dr \right]^2 + \left[\int [kr j_\ell(kr)]^2 V_2(r) dr \right]^2}{\int [kr j_\ell(kr)]^2 V_2(r) dr}$$

Let us apply this to the two specific cases previously considered.

If V consists of the sum of two square wells the replacement of $j_\ell(x)$ by $x^\ell [2\ell+1]^{-1}$ is justified for large ℓ and one easily sees that $\sigma_{e,\ell} / \sigma_{a,\ell}$ still goes to zero as ℓ goes to infinity.

If V is the sum of two Yukawa potentials

$$V(r) = C_1 \frac{e^{-\mu_1 r}}{r} - iC_2 \frac{e^{-\mu_2 r}}{r},$$

then,

$$\left(\frac{\sigma_{e,\ell}}{\sigma_{a,\ell}} \right)_{\text{Born}} = \frac{\frac{m}{k} \left[C_1^2 Q_\ell^2 \left(1 + \frac{\mu_1^2}{2k^2} \right) + C_2^2 Q_\ell^2 \left(1 + \frac{\mu_2^2}{2k^2} \right) \right]}{C_2 Q_\ell \left(1 + \frac{\mu_2^2}{2k^2} \right)}$$

The problem is to study the asymptotic behaviour of

$$Q_\ell^2 \left(1 + \frac{\mu_1^2}{2k^2} \right) / Q_\ell \left(1 + \frac{\mu_2^2}{2k^2} \right).$$

It turns out (see Appendix) that

$$\lim_{\ell \rightarrow \infty} \left[Q_\ell^2(X)/Q_\ell(Y) \right]^{\frac{1}{\ell}} = \frac{Y + \sqrt{Y^2 - 1}}{(X + \sqrt{X^2 - 1})^2}$$

This enables us :

i) to confirm our previous result that if

$$\mu_2 < 2\mu_1$$

is satisfied,

$$\sigma_{e,\ell} / \sigma_{a,\ell} \rightarrow 0 \quad \text{as} \quad \ell \rightarrow \infty.$$

ii) to treat the complementary case. Then one easily finds

$$\sigma_{e,\ell} / \sigma_{a,\ell} \rightarrow 0 \quad \text{as} \quad \ell \rightarrow \infty.$$

if

$$k^2 < \frac{\mu_1^4}{\mu_2^2 - 4\mu_1^2}$$

$$\sigma_{e,\ell} / \sigma_{a,\ell} \rightarrow \infty$$

if

$$k^2 > \frac{\mu_1^4}{\mu_2^2 - 4\mu_1^2}$$

This means for instance that if one admits in nucleon-antinucleon scattering a range of the potential as small as $1/2M$ ⁷⁾ where M is the nucleon mass, the large ℓ waves contribute to scattering and not to absorption, except at extremely low energies, where the high waves are negligible.

Some care must be taken when one tries to guess the range of the imaginary potential for a given process to apply the above considerations. Consider for instance nucleon-nucleon scattering at an energy above the meson production threshold. One would think a priori that the imaginary potential has a range $1/m_\pi$. Actually R.G.N. Phillips ⁸⁾ has shown that $\sigma_{e,\ell} / \sigma_{a,\ell} \rightarrow \infty$ on the basis of peripheral processes. In fact when a careful analysis is made according to the lines of ref. ³⁾ one finds indeed that the range of the imaginary potential is certainly less than $1/2m_\pi$.

IV. A lower limit of the elastic cross-section

We assume that the conditions under which integral I makes sense are satisfied. There are two possible ways of obtaining a lower bound of the elastic cross-section; either to consider the lower limit of the total absorption cross-section and estimate the maximum number of partial waves contributing or to start directly from the partial wave absorption cross-sections.

Let us sketch the former method first. Assume for instance a purely imaginary potential $V = -iV_0 e^{-\mu r}$. Then

$$\sum_{\ell=0}^L \sigma_{a,\ell} > \sigma_{a \text{ minimum}} - \frac{\sigma_{a \text{ Born}}}{4\pi} \sum_L^{\infty} \frac{(2\ell+1)^2}{\left[1 + \frac{\mu^2}{2k^2} + \sqrt{\left(1 + \frac{\mu^2}{2k^2}\right)^2 - 1} \right]^{\ell}}$$

by taking L large enough one can clearly make the right-hand side positive. Let us call its value $\sum_{a,L}$. Then a standard minimisation procedure ⁷⁾ yields

$$\sigma_e > \frac{\pi}{k^2} (L+1)^2 \left[1 - \sqrt{1 - \frac{k^2 \sum_{a,L}}{\pi (L+1)^2}} \right]^2$$

The second method is of course straightforward

$$\sigma_e > \frac{\pi}{k^2} \sum (2\ell+1) \left[1 - \sqrt{1 - \frac{k^2 \sigma_{a,\ell \text{ min}}}{\pi (2\ell+1)}} \right]^2 > \sum_0^{\infty} \frac{k^2}{4\pi (2\ell+1)} \left[\sigma_{a,\ell \text{ min}} \right]^2$$

In the case of a Yukawa imaginary potential this series can be summed if the denominator in $\sigma_{a,\ell \text{ min}}$ is overestimated by taking $\ell = 0$. Then

$$\sigma_{a,\ell \text{ min}} > \text{const} \times (2\ell+1) \frac{Q_{\ell}}{\ell} \left(1 + \frac{\mu^2}{2k^2} \right)$$

and since

$$\frac{1}{z^2 - 1} = \sum (2\ell+1) Q_{\ell}^2(z)$$

one very easily gets a lower limit for σ_e . This limit, however, turns out to be very small, as compared to the upper limit, partly because of the neglect of the ℓ dependence of the denominator in $\sigma_{a,\ell \text{ min}}$, but, clearly, a more refined evaluation is feasible.

V. Concluding remarks

Under the conditions that a complex potential $V = V_1(r) - iV_2(r)$ is such that

- i) $V_2(r)$ is purely positive,
- ii) $V_1^2(r)/V_2(r)$ is integrable when multiplied by r or r^2 ,

we have derived a series of inequalities between the cross-sections both for a given partial wave and for the over-all process. Further, upper and lower bounds have been obtained for the total cross-section, the elastic cross-section and the absorption cross-section, and this, again, both for a given partial wave and for the whole process. We are aware of the fact that the lower bounds we have obtained are very poor. The upper bounds are probably not so bad, especially in the case of a purely imaginary potential.

Among other results an interesting one is that under the above conditions the elastic cross-section decreases faster with increasing ℓ than the absorption cross-section. We have investigated what happens when the conditions are released. This is possible because we have shown (a well-known result) that the Born approximation holds for large ℓ if the integral $\int r |V_1(r) - iV_2(r)| dr$ converges. Then one sees that for square well potentials it is still true that the elastic cross-section decreases faster than the absorption cross-section. However, for Yukawa well potentials this is not so : if the range of the imaginary potential is less than half the range of the real potential, above a certain energy the absorption cross-section decreases faster than the elastic cross-section.

A P P E N D I X

We collect here the various auxiliary mathematical results which have been used in this paper

$$1) \quad j_\ell(x) < \frac{1}{\sqrt{2\ell+1}}$$

from the expansion

$$e^{ix \cos \theta} = \sum (2\ell+1) i^\ell j_\ell(x) P_\ell(\cos \theta)$$

we easily get by integration over $\cos \theta$ of

$$1 = e^{ix \cos \theta} e^{-ix \cos \theta}$$

$$1 = \sum (2\ell+1) j_\ell^2(x)$$

which proves the above inequality.

$$2) \quad \int \left[x j_\ell(x) \right]^2 \frac{e^{-\mu x}}{x} dx = \frac{1}{2} Q_\ell \left(1 + \frac{\mu^2}{2} \right)$$

this formula can be found in Magnus, Oberettringer and Tricomi, Laplace transforms ⁹⁾.

$$3) \quad Q_\ell(x) < \frac{1}{2} \log \left(\frac{x+1}{x-1} \right) \left(\frac{1}{x + \sqrt{x^2-1}} \right)^\ell$$

this is a consequence of the integral representation ¹⁰⁾

$$Q_\ell(x) = \int_0^\infty \frac{dt}{(x + \cosh t \sqrt{x^2-1})^{l+1}} < \left(\frac{1}{x + \sqrt{x^2-1}} \right)^\ell Q_0(x)$$

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in addition

$$Q_\ell(x) > \int_0^\infty \frac{dt}{(x + \sqrt{x^2 - 1})^{\ell+1} (Cht)^{\ell+1}} > \left(\frac{1}{x + \sqrt{x^2 - 1}} \right)^{\ell+1} \frac{1}{\ell+1}$$

so

$$\lim_{\ell \rightarrow \infty} \left[Q_\ell(x) (x + \sqrt{x^2 - 1}) \right]^{1/\ell} = 1$$

$$4) \quad \left| j_\ell(x) y h_\ell(y) \right| < \sqrt{xy} \sqrt{\frac{\pi}{2\ell+1}} \quad \text{for } y > x$$

first we notice that ¹⁰⁾

$$\left| y h_\ell(y) \right| \quad \text{is a decreasing function of } y.$$

So it is sufficient to investigate the case

$$x = y$$

From the expansion ¹⁰⁾

$$\frac{e^{i\sqrt{y^2 + x^2 - 2xy \cos \theta}}}{\sqrt{x^2 + y^2 - 2xy \cos \theta}} = i \sum_{\ell=0}^{\infty} (2\ell+1) j_\ell(x) h_\ell^{(1)}(y) P_\ell(\cos \theta)$$

We get, taking the limiting case $y \rightarrow x$

$$\left| j_\ell(x) h_\ell(x) \right| < \frac{1}{x} \int_{-1}^{+1} \frac{|P_\ell(t)| dt}{\sqrt{2} \sqrt{1-t}}$$

for $\ell = 0$ one can check directly the result. For $\ell \geq 1$ we can use

$$\left| P_\ell(\cos \theta) \right| < \frac{2}{\sqrt{\ell \pi \sin \theta}} \quad 10)$$

whenever the right hand side is less than unity and otherwise $|P_e| < 1$.
 Then we are left with a result for the right hand side integral

$$\sqrt{\frac{\pi}{2e}} - \frac{C}{e} + O\left(\frac{1}{e^2}\right)$$

which can be shown to be less than $\sqrt{\frac{\pi}{2e+1}}$. Hence for $Y > X$

$$\left| x j_e(x) y h_e(y) \right| < \left| x j_e(x) x h_e(x) \right| < x \sqrt{\frac{\pi}{2e+1}} \sqrt{xy} \sqrt{\frac{\pi}{2e+1}}$$

This form is especially suitable because it replaces the kernel of the integral equation by a separable kernel.

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