CERN LIBRARIES, GENEVA

CM-P00056761

SELECTED TOPICS ON ANALYTICITY IN POTENTIAL SCATTERING

André Martin

CERN, Geneva

Summer School of Theoretical Physics, 1961, Yugoslavia

TABLE OF CONTENTS

	<u>page</u>
I. Introduction -----	1
II. Analytic properties of partial waves from Yukawa-Like potentials	2
a) S wave case -----	2
b) Higher angular momenta -----	10
c) Case where tensor forces are present -----	14
III. Application of the partial wave properties for generalised Yukawa potentials -----	21
a) Numerical calculations -----	21
b) The deuteron wave function and the anomalous threshold problem -----	23
c) The connection between the unphysical cut of a partial wave amplitude, the S matrix, and the potential -----	29
IV. Analytic properties for fixed energy -----	38
References -----	51

* * *

I. Introduction

In recent years a large amount of work has been done on the analytic properties of scattering and production amplitudes in field theory. However, at the time where these studies were started, very little had been done about the analytic properties in potential scattering except in the case where the interaction took place in a strictly finite region ¹⁾. The first really important contribution in this field was that of Khuri ²⁾ who showed that the scattering amplitude derived from a Yukawa-like potential obeys a fixed transfer dispersion relation.

More recently, assumptions have been made about the analytic properties of the scattering amplitude in field theory, both in energy and momentum transfer ³⁾. There does not exist yet any complete proof that these assumptions (Mandelstam representation) are valid. However, in this case, the potential model is a useful guide which shows that at least for generalised Yukawa potential the Mandelstam representation holds ⁴⁾.

We shall not speak here about the complete derivation of this result which has been given by Prof. T. Regge here. We shall first consider the analytic properties of a given partial wave. These analytic properties would be obtained by projecting the Mandelstam representation on a given angular momentum state. However, a direct method of obtaining the analytic properties of a partial wave proves very useful. Indeed, it turns out that the method we shall follow for Yukawa-like potentials furnishes many other results. In particular it furnishes a very nice integral representation of the wave functions (especially of the deuteron wave function) which permits to study certain matrix elements such as deuteron electromagnetic form factors and deuteron photodisintegration and to show that the so-called anomalous thresholds (at least the "normal" anomalous thresholds) are connected with the extension of the deuteron wave function.

In addition, the discontinuities of the ℓ th-wave scattering amplitude across the unphysical cut can be expressed explicitly from the knowledge of the potential. It turns out that the reverse is true: given the discontinuity one can construct explicitly the potential. Since the discontinuities are at

present the quantities which the dispersionists calculate first one sees that for a given partial wave one can associate a potential and, therefore a wave function. In this way one can connect the information about the discontinuities for nucleon-nucleon scattering and the external part of the deuteron wave function.

A similar connection between potential and discontinuity can be established in the case of fixed energy dispersion relations. We shall have to sketch first a method to establish the analytic properties for fixed t which is completely different from what you heard from Dr. Regge. Then we shall be able to show how one can construct from it the potential. As an application we shall show how one can settle the problem of the range of the imaginary potential in nucleon-antinucleon scattering.

II. Analytic properties of partial waves from Yukawa-like potentials

a) S wave case ⁵⁾

In the preceding lectures Prof. T. Regge has shown to you the analytic properties of finite range potentials and, also, established the existence of an analyticity strip for potentials decreasing faster than an exponential. We shall consider here the analytic properties for potentials of the type

$$V(r) = \int_0^{\infty} C(\alpha) e^{-\alpha r} d\alpha \quad (1)$$

which include the Yukawa potential ($C(\alpha) = \text{const.}$).

It proves useful to consider the following two solutions of the Schrödinger equation

$$\left[\frac{d^2}{dr^2} + k^2 - V(r) \right] u^{\pm}(k, r) = 0 \quad (2)$$

$$\begin{cases} u^+(k, r) = e^{ikr} f(-k, r) \\ u^-(k, r) = e^{-ikr} f(k, r) \end{cases} \quad \text{with} \quad \begin{cases} f(-k, \infty) = 1 \\ f(k, \infty) = 1 \end{cases} \quad (3)$$

whether or not such solutions exist for complex values of k will be established by trying to construct f . Notice that $f(k, 0)$ is just the Jost function about which you have already heard.

If the function $f(k, r)$ is known, one can easily construct the S matrix (I should say S functions), defined by

$$u_{\text{regular}} \sim e^{-ikr} - S(k) e^{ikr}$$

(for large r) at least for real k .

Since

$$u_{\text{regular}} = \frac{u^+(k, r)}{u^+(k, 0)} - \frac{u^-(k, r)}{u^-(k, 0)}$$

we see that

$$S(k) = \frac{f(k, 0)}{f(-k, 0)} \quad (4)$$

For complex values of k , the latter expression will be maintained as a definition of $S(k)$.

Let us now speak about the construction of f .

In the especially simple case of a pure exponential potential ⁶⁾
 $(c(\alpha) = \delta(\alpha - \mu))$ the solution of the S wave Schrödinger equation is known and one can easily recognise that $f(k, r)$ can be expanded as a series of exponentials of the type

4.

$$f(k, r) = 1 + \sum_{n=1}^{\infty} C_n(k) e^{-n\mu r}$$

for a potential $v = v_0 e^{-\mu r}$.

The same can be done for more complicated potentials

$$V = \sum_{N=1}^M V_N e^{-\mu_N r}$$

for which no exact solution is known. Here one can write

$$f(k, r) = 1 + \sum_{n_1, n_2, \dots, n_M} C_{n_1, n_2, \dots, n_M} \exp \left[- \left(n_1 \mu_1 + n_2 \mu_2 + \dots + n_M \mu_M \right) r \right]$$

This strongly suggests that when $v(r)$ is a continuous superposition of exponentials in which the largest range is $1/\mu$, $f(k, r)$ can be written as

$$f(k, r) = 1 + \int_{\mu}^{\infty} \rho_k(\alpha) e^{-\alpha r} d\alpha \quad (5)$$

We shall take this formula as a starting point and check later on that our assumption was correct.

Inserting $u^- = f(k, r) e^{-ikr}$ in the Schrödinger equation we get

$$f'' - 2ik f' - V(r) f = 0 \quad (6)$$

Replacing v and f by their integral representations (1) and (5) respectively, and using the well-known theorem about the product of two Laplace transforms (which appear in vf), we end up with an equation for $\rho_k(\alpha)$:

$$\alpha(\alpha + 2ik) \rho_k(\alpha) = C(\alpha) + \int C(\alpha - \beta) \rho_k(\beta) d\beta \quad (7)$$

Now we have to fix the upper and lower limits in the integral. Obviously the lower limit is μ because $c(\alpha) = 0$ for $\alpha < \mu$. From this fact we easily see that $\rho_k(\alpha) = 0$ in the range $\alpha = 0$ to $\alpha = \mu$, which is consistent with the assumption we made. Finally we can write

$$\alpha(\alpha + 2ik) \rho_k(\alpha) = C(\alpha) + \theta(\alpha - 2\mu) \int_{\mu}^{\alpha - \mu} C(\alpha - \beta) \rho_k(\beta) d\beta \quad (8)$$

This integral equation has the great advantage that it can be solved very easily by iteration for any finite value of α :

$$\begin{aligned} \text{for } 0 < \alpha < \mu & \quad \rho = 0 \\ \mu < \alpha < 2\mu & \quad \rho = \frac{C(\alpha)}{\alpha(\alpha + 2ik)} \\ 2\mu < \alpha < 3\mu & \quad \rho = \frac{1}{\alpha(\alpha + 2ik)} \left[C(\alpha) + \int_{\mu}^{\alpha - \mu} \frac{C(\alpha - \beta) C(\beta)}{\beta(\beta + 2ik)} d\beta \right] \end{aligned}$$

and, more generally, if $\rho_k(\alpha)$ is known for $\alpha < n\mu$ one can insert it in the right hand side and obtain ρ for $\alpha < (n+1)\mu$.

It is clear that if $(\alpha + 2ik)$ cannot vanish, i.e. if k lies outside a line on the imaginary axis for $k = i\frac{\mu}{2}$ to $i\infty$, $\rho_k(\alpha)$, for fixed α is a well defined function of k , is analytic in k . We have now to see what kind of condition should be imposed on $C(\alpha)$ to ensure that these properties hold for $f(k, r)$. We shall look for sufficient but not necessary conditions.

Let us assume that there exist a monotonous increasing function $B(\alpha)$ such that

$$|C(\alpha)| < B(\alpha) \quad (9)$$

6.

Let us take k outside the cut on the imaginary axis; more precisely we exclude the domain indicated on Fig. 1, where ε is an arbitrarily small angle

Fig. 1

then, for $\alpha > \mu$

$$|\alpha + 2ik| > \alpha \sin \varepsilon$$

$$|\alpha + 2ik| > 2|k| \sin \varepsilon$$

then we get

$$|p_k(\alpha)| < \frac{B(\alpha)}{\alpha^2 \sin \varepsilon} \left[1 + \int_{\mu}^{\alpha} |p_k(\beta)| d\beta \right] \quad (10)$$

Let us call

$$1 + \int_{\mu}^{\alpha} |p_k(\beta)| d\beta = y_k(\alpha)$$

then

$$\frac{y'}{y} = \frac{B(\alpha)}{\alpha^2 \sin \varepsilon}$$

Hence

$$y_k(\alpha) < \exp \int_{\mu}^{\alpha} \frac{d\beta B(\beta)}{\beta^2 \sin \varepsilon}$$

It is clear now that if $\int_{\mu}^{\infty} \frac{d\beta B(\beta)}{\beta^2}$ is bounded $1 + \int_{\mu}^{\infty} |\rho_k(\alpha)| d\alpha$ is uniformly bounded with respect to k when k is outside the neighbourhood of the cut. Therefore a sufficient condition for the existence and holomorphy of $f(k,r) = 1 + \int_{\mu}^{\infty} e^{-\alpha r} \rho_k(\alpha) d\alpha$, for fixed arbitrary r with respect to k is

$$|C(\alpha)| < M \alpha^{1-\eta} \quad (11)$$

η arbitrarily small.

another sufficient condition would be

$$|C(\alpha)| < \frac{M \alpha}{\lambda + \ln^{1+\eta}\left(\frac{\alpha}{\mu}\right)} \quad (12)$$

This ensures, in particular that $f(k,0)$ is regular outside the cut $i \frac{\mu}{2} - i\infty$. Further, making use of the uniform convergence property we see easily that since $\rho_k(\alpha) \rightarrow 0$ as $k \rightarrow \infty$ in any direction except the positive imaginary axis

$$\lim_{|k| \rightarrow \infty} f(k,r) = 1 \quad (13)$$

Further, in the case when condition (11) is fulfilled a more careful examination shows that

$$\begin{aligned} |f(k,r) - 1| &\sim |k|^{-\eta} && \text{for } \eta < 1 \\ \text{as } |k| \rightarrow \infty &\sim \frac{\ln |k|}{|k|} && \text{for } \eta \geq 1 \end{aligned} \quad (14)$$

The conditions imposed on $C(\alpha)$ essentially mean that $V(r)$ has to be less singular than $1/r^2$ at the origin of coordinates. We know that if this is not so difficulties arise, and in particular for sufficiently strong $1/r^2$ attractive potentials the Schrödinger equation has no regular solution.

8.

The question whether one can deal with very singular repulsive potentials using our method is not yet settled, but we would like to show that when $C(\alpha)$ is bounded by an arbitrary polynomial $f(k, r)$ retains all the desired properties for $r \neq 0$. In this case

$$|C(\alpha) e^{-\alpha r}| \quad (15)$$

is bounded by, say, B ; then

$$|p_k(\alpha) e^{-\alpha r}| < \frac{B}{\alpha^2 \sin \varepsilon} \left[1 + \int_{\mu}^{\alpha} |p_k(\beta) e^{-\beta r}| d\beta \right] \quad (16)$$

hence one shows easily that

$$1 + \int_{\mu}^{\infty} |p_k(\alpha) e^{-\alpha r}| d\alpha$$

is uniformly convergent and all the above mentioned properties hold.

Up to now we have established the analytic properties of the out-going and in-going wave functions for fixed r . In case condition (11) or (12) are satisfied it is an almost trivial matter to get the analytic properties of

$$S(k) = \frac{f(k, 0)}{f(-k, 0)}$$

It has two cuts from $k = i\mu/2$ to $k = i\infty$ and from $k = -i\infty$ to $k = -i\mu/2$.

To show that these are real cuts we should establish explicitly the existence of a discontinuity of $S(k)$ across these lines. This will be postponed to Section III. In addition to these cuts $S(k)$ has poles which come from the zeros of $f(-k, 0)$. You have probably already heard from Regge that the zeros of $f(-k, 0)$ in the upper half k plane necessarily lie on the imaginary axis ^{*)} and correspond to bound states whose wave function is just

$$e^{-\kappa r} \left[1 + \int_{\mu}^{\infty} p_{i\kappa}(\alpha) e^{\alpha r} d\alpha \right] \quad \kappa \text{ real} > 0 \quad (17)$$

*) This is due to the reality of the potential.

From $f(k,0) \rightarrow 1$ for $k \rightarrow \infty$ we get $S(k) \rightarrow 1$. Using the asymptotic behaviour of $f(k,0) - 1$ we can show that, provided condition (11) is fulfilled, one can write unsubtracted Cauchy integral representations (or dispersion relations) for

$$\frac{S(k) - 1}{2i} = e^{i\delta} \sin \delta$$

Finally, taking into account the symmetries of f :

$$f(k,0) = f^*(-k^*,0) \quad (18)$$

we get the following symmetries for S

$$S(k)S(-k) = 1 \quad S(k) = S^*(-k^*) \quad (19)$$

and the results are summarized in the following picture

As a last comment of this section we would like to say that when $C(\infty)$ increases like an arbitrary polynomial $f(k,0)$ and $f(-k,0)$ do not exist. However, we believe (without being able to prove it at present) that

$$S(k) = \lim_{r \rightarrow 0} \frac{f(k,r)}{f(-k,r)}$$

might still exist if $C(\alpha)$ is positive (repulsive potential at short distances) for large α . The analytic properties would remain unchanged except perhaps the behaviour at infinity. Our belief is based on the remark that for a hard core potential :

$$V(r) = +\infty \quad \text{for } r < r_0$$

$$V(r) = \int_{\mu}^{\infty} C(\alpha) e^{-\alpha r} d\alpha \quad \text{for } r > r_0 \quad (20)$$

where $C(\alpha)$ is bound by a polynomial the analytic properties still hold, but, at infinity, $S(k) \sim e^{2ikr_0}$. Then, we believe that letting r_0 go to zero will not seriously change the picture, or, in other words, since here we have replaced the singular region by a worse singularity in the potential, nothing worse than what happened in the hard core case can happen.

b) Higher angular momenta 7)

Here all the trouble comes from the centrifugal terms in the Schrödinger equation

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2 - V(r) \right] u^{\pm}(k, r) = 0 \quad (21)$$

In this case we have two possibilities :

- either to keep that same integral representation

$$u^-(k, r) = e^{-ikr} \left[1 + \int_{\mu}^{\infty} \rho_k(\alpha) e^{-\alpha r} d\alpha \right] \quad (22)$$

but then, since the range of the centrifugal potential is infinite we expect that the lower limit in the integral representation will be zero rather than μ and the iterative property of the integral equation for ρ is lost at least in appearance.

- or to try to generalise the S wave case by using functions which satisfy the free equations (with centrifugal term included). In the S wave case we had

$$u^-(k, r) = e^{-ikr} + \int_0^\infty \rho_k(\alpha) e^{-(k+\alpha)r} d\alpha$$

in the $l \neq 0$ case one can take

$$u^-(k, r) = i^{l+1} kr \left[h_l^{(2)}(kr) + \int_0^\infty \rho_k(\alpha) h_l^{(2)}[(k+i\alpha)r] d\alpha \right] \quad (23)$$

where $h_l^{(2)}$ is defined as in Schiff, Quantum Mechanics, p. 79.

The latter method which was used by De Alfaro and Rossetti⁸⁾ avoids the difficulties associated with the centrifugal term itself but, in order to be able to write the product $u^-(k, r) v(r)$ as a superposition of Hankel functions, which is possible but rather tedious and difficult.

Here we shall prefer to use the first method and show how one can get rid of the centrifugal term. Let us first merely include the centrifugal term in the potential. Since

$$\frac{1}{r^2} = \int_0^\infty \alpha e^{-\alpha r} d\alpha$$

it is not difficult to see that starting from

$$f(k, r) = 1 + \int_0^\infty \rho_k(\alpha) e^{-\alpha r} d\alpha$$

one gets

$$\begin{aligned} \alpha(\alpha + 2ik) \rho_k(\alpha) &= C(\alpha) + l(l+1)\alpha \\ &+ \int_0^\alpha [C(\alpha-\beta) + l(l+1)(\alpha-\beta)] \rho_k(\beta) d\beta \end{aligned} \quad (24)$$

What one would like to do is to transform this equation into an iterative type integral equation. One can notice that when $C(\alpha) = 0$ the solution of this equation is known, because it can be obtained by means of Laplace transformation from the Hankel functions. We shall prefer to show it directly. Let us call $\rho_k^0(\alpha)$ the solution for $C(\alpha) \equiv 0$

$$\alpha(\alpha + 2ik) \rho_k^0(\alpha) = l(l+1)\alpha + \int_0^\alpha (\alpha - \beta) \rho_k^0(\beta) d\beta \quad (25)$$

by differentiation of both sides of this equation we readily get a differential equation for the quantity

$$y_k^0(\alpha) = 1 + \int_0^\alpha \rho_k^0(\beta) d\beta$$

$$[\alpha(\alpha + 2ik) y_k^0(\alpha)]' = l(l+1) y_k^0(\alpha) \quad (26)$$

if we compare this with the Legendre equation :

$$- [(1-x^2)u']' = l(l+1)u \quad (27)$$

We readily see that $P_l(1 + \frac{\alpha}{ik})$ and $Q_l(1 + \frac{\alpha}{ik})$ are solutions of the equation for $y_k^0(\alpha)$.

Actually, since $\rho_k^0(\alpha)$ is regular at $\alpha = 0$, and since $y_k^0(0) = 1$, we conclude that

$$\rho_k^0(\alpha) = \frac{d P_l(1 + \frac{\alpha}{ik})}{d\alpha} \quad (28)$$

Let us notice that $P_l(1 + \frac{\alpha}{ik})$ is regular in the whole complex k plane, with a pole at $k = 0$, which means that even for $k = ix$ the equation for ρ_k^0 has a well behaved solution. It will be useful to notice that the other solution $Q_l(1 + \frac{\alpha}{ik})$, for fixed α , has - and it can be shown from

$$Q_n(z) = \frac{1}{2} \int_{-1}^{+1} \frac{P_n(t) dt}{z - t}$$

a cut along $k = i \ X$ where S is a real number going from $\frac{1}{2}$ to $+\infty$.

Let us now return to the equation for $\rho(\alpha)$ in the case $C(\alpha) \neq 0$. It may be written

$$\begin{aligned} \alpha(\alpha + 2ik) \rho_k(\alpha) - \ell(\ell+1) \left[\alpha + \int_0^\alpha (\alpha - \beta) \rho_k(\beta) d\beta \right] \\ = C(\alpha) + \int_0^\alpha C(\alpha - \beta) \rho_k(\beta) d\beta \equiv \phi(\alpha) \end{aligned} \quad (29)$$

Notice that $\phi(\alpha) = 0$ for $\alpha < \mu$; so for $\alpha < \mu$ $\rho_k(\alpha) \equiv \rho_k^0(\alpha)$. We can as before use $y_k(\alpha) = 1 + \int_0^\alpha \rho_k(\beta) d\beta$ and get

$$\left[\alpha(\alpha + 2ik) y_k'(\alpha) \right]' - \ell(\ell+1) y_k(\alpha) = \frac{d\phi}{d\alpha} \quad (30)$$

ϕ will be treated as a known right hand side according to standard methods.

Then one can formally solve the equation and obtain

$$\begin{aligned} y_k(\alpha) = P_\ell \left(1 + \frac{\alpha}{ik} \right) \\ + \int_0^\alpha \left[P_\ell \left(1 + \frac{\alpha}{ik} \right) Q_\ell \left(1 + \frac{\beta}{ik} \right) - P_\ell \left(1 + \frac{\beta}{ik} \right) Q_\ell \left(1 + \frac{\alpha}{ik} \right) \right] \frac{d\phi}{d\beta} d\beta \end{aligned} \quad (31)$$

In this solution a choice of the homogeneous solution has been made: it is such that for $\alpha < \mu$ $y_k(\alpha) \equiv y_k^0(\alpha)$ since for $\alpha < \mu$ $\frac{d\phi}{d\alpha} = 0$.

At this point we could still transform a little bit the equation, but the main point is already exhibited here :

$$\phi(\alpha) = C(\alpha) + \int_0^{\alpha-\mu} C(\alpha-\beta) y'_k(\beta) d\beta$$

(because $C(\alpha) = 0$ for $\alpha < \mu$).

Therefore, if we know $y(\beta)$ for $\beta < n\mu$ we know $\phi(\beta)$ for $\beta < (n+1)\mu$ and inserting this in the r.h.s. of equation (31) we obtain $y_k(\beta)$ for $\beta < (n+1)\mu$. In this new form the iterative property is re-established.

One might think that a troublesome underlying assumption of our method of transformation of the equation is the derivability of $\phi(\alpha)$ especially at the point $\alpha = \mu$ (which is not valid for a pure Yukawa potential). Actually, by partial integration and derivation one can transform the equation into an equation for $\rho_k(\alpha)$ where $\phi(\beta)$ rather than $\frac{d\phi}{d\beta}$ appears. It is not difficult to believe that the lack of derivability of $\phi(\beta)$ will not alter the final result. We shall not, however, enter into these details. From what we know about the analytic properties of Q and P it is clear that $\rho_k(\alpha)$, again, is analytic provided k is outside the origin and outside the cut $k = i\mu/2 \rightarrow i\infty$. To prove that this is also true for $f(k,r)$ is tedious, because one needs to introduce upper bounds of the various functions appearing in the equation, but nevertheless possible. All the results which have been established in the S wave case can be proved in the present case. To establish the properties of $S(k) = \lim_{r \rightarrow 0} \frac{f(+k,r)}{f(-k,r)}$ one can notice that $S(k)$ is also $\lim_{\alpha \rightarrow \infty} \frac{\rho_k(\alpha)}{\rho_{-k}(\alpha)}$.

c) Case where tensor forces are present ⁹⁾

Since the actual nucleon-nucleon forces are not central but contain spin orbit and tensor forces, it is desirable to extend our method to the case where $L = J-1$ and $L = J+1$ states are coupled through a non-central potential. Then the starting equations are

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{J(J-1)}{r^2} - V_{J-1}(r) \right] u_{J-1}(r) = V_T(r) u_{J+1}(r)$$

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{(J+1)(J+2)}{r^2} - V_{J+1}(r) \right] u_{J+1}(r) = V_T(r) u_{J-1}(r) \quad (32)$$

Let us remind first a few facts about scattering by tensor forces in general because, though these results are well-known by specialists, they are rather unfamiliar to the non-specialist.

When in equations (32) the boundary condition $u_{J-1}(0) = u_{J+1}(r) = 0$ is disregarded, the system admits 4 independent solutions. Here we shall select the independent solutions by their asymptotic behaviour

$$\begin{pmatrix} u_{J-1} \\ u_{J+1} \end{pmatrix} = \psi_1^+ \sim \begin{pmatrix} e^{ikr} \\ e^{ikr} \end{pmatrix}, \psi_2^+ \sim \begin{pmatrix} e^{ikr} \\ e^{-ikr} \end{pmatrix}, \psi_1^- \sim \begin{pmatrix} e^{-ikr} \\ e^{-ikr} \end{pmatrix}, \psi_2^- \sim \begin{pmatrix} e^{-ikr} \\ -e^{-ikr} \end{pmatrix} \quad (33)$$

(Of course other linearly independent sets of solutions could be used). If we impose the boundary conditions at $r = 0$ we still have two linearly independent solutions.

The scattering matrix can be written as a two by two matrix :

$$\begin{pmatrix} S_{J-1 J-1} & S_{J-1 J+1} \\ S_{J+1 J-1} & S_{J+1 J+1} \end{pmatrix} \quad (34)$$

$S_{J-1 J-1}$ measures the amplitude of the out-going $J-1$ wave when the initial in-going wave is a pure $J-1$ wave; $S_{J-1 J+1}$ measures the amplitude of the $J-1$ out-going wave when one starts from a pure $J+1$ in-going wave, and so on. So to construct the scattering matrix elements one has to fulfil at the same time certain boundary conditions at the origin and at infinity. For instance to determine $S_{J-1 J-1}$ and $S_{J+1 J-1}$ we should start from a combination

$$a_1 \psi_1^+ + a_2 \psi_2^+ + c (\psi_1^- + \psi_2^-) \quad (35)$$

where no $L = J+1$ in-coming wave is present. Then the $L = J-1$ asymptotic wave is

$$(a_1 + a_2) e^{ikr} + 2c e^{-ikr} \quad (36)$$

hence, by comparison with $-S_{J-1, J-1} e^{ikr} + e^{-ikr}$ we get

$$S_{J-1, J-1} = -\frac{a_1 + a_2}{2c}$$

and also, clearly

$$S_{J+1, J-1} = \frac{a_2 - a_1}{2c}$$

once $\psi_1^+, \psi_2^+, \psi_1^-, \psi_2^-$ are known, a_1/c and a_2/c are determined by the condition

$$a_1 \psi_1^+(0) + a_2 \psi_2^+(0) + c [\psi_1^-(0) + \psi_2^-(0)] = 0 \quad (37)$$

which furnishes two equations *).

The S matrix constructed in this way has to be unitarity and symmetric : for real energies this means :

$$\begin{aligned} S_{J-1, J+1} &= S_{J+1, J-1} \\ |S_{J+1, J-1}|^2 + |S_{J-1, J-1}|^2 &= 1 \\ |S_{J+1, J+1}|^2 + |S_{J-1, J+1}|^2 &= 1 \\ S_{J-1, J-1} S_{J-1, J+1}^* + S_{J+1, J-1} S_{J+1, J+1}^* &= 0 \end{aligned} \quad (38)$$

*) The effect of the centrifugal term will be considered more seriously later.

The latter equation is already satisfied in modulus if the three first equations are fulfilled. Therefore the 8 real variables appearing in the four matrix elements are connected by 5 independent linear equations and the scattering matrix can be expressed in terms of 3 real quantities.

This is probably the reason why many people prefer to express the information on the scattering in an other way where no superabundant quantities appear. In the representation above considered the S matrix connects the in-going states

$$\begin{pmatrix} e^{-ikr} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ e^{-ikr} \end{pmatrix}$$

with the out-going states

$$\begin{pmatrix} e^{ikr} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ e^{ikr} \end{pmatrix}$$

One could try to choose another orthogonal set of in-going and out-going states such that the S matrix becomes diagonal, i.e. make an orthogonal transformation on S. We take the in-going states

$$U_{\alpha}^{-} \sim \frac{1}{\sqrt{1+\eta^2}} \begin{pmatrix} e^{-ikr} \\ \eta e^{-ikr} \end{pmatrix} \quad U_{\beta}^{-} \sim \frac{1}{\sqrt{1+\eta^2}} \begin{pmatrix} -\eta e^{-ikr} \\ e^{-ikr} \end{pmatrix}$$

and out-going states

$$U_{\alpha}^{+} \sim \frac{1}{\sqrt{1+\eta^2}} \begin{pmatrix} e^{ikr} \\ \eta e^{ikr} \end{pmatrix} \quad U_{\beta}^{+} \sim \frac{1}{\sqrt{1+\eta^2}} \begin{pmatrix} -\eta e^{ikr} \\ e^{ikr} \end{pmatrix} \quad (39)$$

then the unitarity condition remains valid after the transformation and if we choose η such that $S_{\alpha\beta} = S_{\beta\alpha} = 0$ then $|S_{\alpha\alpha}| = |S_{\beta\beta}| = 1$. η is determined by the equation

$$S_{J-1 J+1} (1-\eta^2) + \eta (S_{J+1 J+1} - S_{J-1 J-1}) = 0 \quad (40)$$

which expressed the fact that η can take two values η_α, η_β such that $\eta_\alpha \eta_\beta = -1$, which just corresponds to the two in-going solutions we have considered. One can check, using the conditions from unitarity and symmetry that η is real.

Obviously once one has

$$S_{\alpha\alpha} = e^{2i\delta_\alpha} \quad \text{and} \quad S_{\beta\beta} = e^{2i\delta_\beta} \quad *) \quad \text{and, with, } \eta_\alpha \quad (41)$$

one can reconstruct the S matrix in its original form if one wishes.

For complex energies (if the S matrix makes sense), the symmetry property of the S matrix still holds, but the unitarity condition becomes $S^*(k^*)S(k) = 1$. It is still possible to diagonalise the S matrix; the equation for η still holds, with $\eta_\alpha \eta_\beta = -1$ but η_α becomes complex, and $|S_{\alpha\alpha}| \neq 1$, $|S_{\beta\beta}| \neq 1$.

To be complete we should add that in order to compute a cross-section, one has to adjust the amount of in-going wave to what is present in an incident plane wave, and using the S matrix obtain the out-going scattered wave.

Let us now return to our specific problem where $\psi_1^+, \psi_2^+, \psi_1^-, \psi_2^-$ are obtained from potentials of the type

$$\begin{aligned} V_{J-1} &= \int_{\mu}^{\infty} C_{J-1}(\alpha) e^{-\alpha r} d\alpha \\ V_{J+1} &= \int_{\mu}^{\infty} C_{J+1}(\alpha) e^{-\alpha r} d\alpha \\ V_T &= \int_{\mu}^{\infty} C_T(\alpha) e^{-\alpha r} d\alpha \end{aligned} \quad (42)$$

*) $\delta_\alpha, \delta_\beta$ are called "eigenphases"; η_α, η_β are the "admixture".

Then we try to represent the wave functions as

$$\psi_1^- = e^{ikr} \begin{pmatrix} f_1(k,r) \\ g_1(k,r) \end{pmatrix} \quad \psi_2^- = e^{-ikr} \begin{pmatrix} f_2(k,r) \\ g_2(k,r) \end{pmatrix} \quad (43)$$

ψ_1^+ and ψ_2^+ are obtained by changing k to $-k$.

We assume (as will be a posteriori checked to be right)

$$\begin{aligned} f_1(k,r) &= 1 + \int_0^{\infty} \rho_{1k}(\alpha) e^{-\alpha r} d\alpha \\ f_2(k,r) &= 1 + \int_0^{\infty} \rho_{2k}(\alpha) e^{-\alpha r} d\alpha \\ g_1(k,r) &= 1 + \int_0^{\infty} \sigma_{1k}(\alpha) e^{-\alpha r} d\alpha \\ g_2(k,r) &= -1 + \int_0^{\infty} \sigma_{2k}(\alpha) e^{-\alpha r} d\alpha \end{aligned} \quad (44)$$

then we insert in the coupled Schrödinger equations

$$\begin{aligned} f'' - 2ikf' - \frac{(J-1)J}{r^2} f - V_{J-1} f &= V_T g \\ g'' - 2ikg' - \frac{(J+1)(J+2)}{r^2} g - V_{J+1} g &= V_T f \end{aligned} \quad (45)$$

The equations can be treated exactly in the same way as in the $l \neq 0$ case.

It turns out then that they can be transformed to be of the form

$$\rho_{1k}(\alpha) = \rho_{1k}^0(\alpha) + \psi(\alpha) + \int_0^{\alpha} K_{J-1}(\alpha, \beta) \phi(\beta) d\beta \quad (46)$$

and so on. Where $\rho_k^0(\alpha) = \frac{d}{d\alpha} P_{J-1} \left(1 + \frac{\alpha}{ik}\right)$. $\psi(\alpha)$ is known and vanishes

for $\alpha < \mu$ and $\rho(\beta)$, vanishing for $\beta < \mu$, contains $\rho_{1k}(\alpha)$ and $\sigma_{1k}(\gamma)$ for arguments γ smaller than $\beta - \mu$. The iterative procedure works provided one solves at the same time the equation for $\rho_{1k}(\alpha)$ and $\sigma_{1k}(\alpha)$. Eventually, one finds for $f_1^{(k,r)} g_1 f_2 g_2$ the same analytic properties with respect to k as in the S wave case.

A slight difficulty occurs when one wants to construct for $f_1 g_1 f_2 g_2$ the S matrix. As it was sketched in the general considerations about tensor forces one has to take suitable combinations of $\psi_1^+ \psi_2^+ \psi_1^- \psi_2^-$ and to impose regularity conditions at the origin on the combination. Unfortunately, due to the centrifugal terms, $f_{1,2}$ and $g_{1,2}$ are singular at the origin. One would naïvely think that

$$\lim_{r \rightarrow 0} r^{J-1} f_1(k, r) \quad \text{and} \quad \lim_{r \rightarrow 0} r^{J-1} f_2(k, r)$$

are finite, and in the same way

$$\lim_{r \rightarrow 0} r^{J+1} g_1(k, r) \quad \text{and} \quad \lim_{r \rightarrow 0} r^{J+1} g_2(k, r).$$

If this were true it would not be difficult to apply the regularity condition at the origin. This is just what is done in the $\ell \neq 0$ uncoupled case. However, it was shown by H. Cornille¹⁰⁾ that this behaviour at the origin is only correct in one case, which is when the coupling potential $V_T(r)$ vanishes at the origin. This means that when $V_T(0) = 0$ there is no difficulty to construct $S_{J-1, J-1}$, $S_{J-1, J+1}$ and $S_{J+1, J+1}$. If $V_T(0) \neq 0$ a serious trouble arises because one can find a set of four solutions, equivalent to the set $\psi_1^+ \psi_2^+ \psi_1^- \psi_2^-$, two of which are regular solutions, two of which are irregular solutions with different degrees of singularity at the origin. Any kind of limiting process to deal with finite quantities will project out the most singular part of $f_1 f_2 g_1 g_2$ and therefore the less singular of the irregular solutions will not appear. So the number of equations to determine the matrix elements will not be sufficient. A more elaborate procedure must be used: one must first take into account the worse singularity and then subtract out from $f_1 f_2 g_1 g_2$ the contribution of the most singular solution. This will

give just the complementary equation needed. Let us notice that certain field theoretical potentials such as the Gartenhaus potential satisfy the condition $V_T(0) = 0$ so that the method can be applied without complication.

In any case the matrix elements $S_{J-1 J-1}$, $S_{J+1 J-1}$, $S_{J+1 J+1}$ have the same analytic properties as the $S(k)$ function in the uncoupled case. They have the same cuts and further one can show that the poles in the upper half k plane lie on the imaginary axis and correspond to bound states.

The analytic properties of the diagonalised S matrix and of the admixture parameter are not simple. This comes from the fact that the diagonal matrix element, as well as the admixture are functions with two determinations. However, one can construct directly from $\psi_1^+ \psi_2^+ \psi_1^- \psi_2^-$ the eigensolutions: one starts from solutions

$$\left(\psi_1^- + A \psi_2^- \right) - S' \left(\psi_1^+ + A \psi_2^+ \right)$$

these are automatically eigensolutions. The two regularity conditions determine A (related to the admixture) and S . This may be useful in calculations for real energies.

III. Applications of the partial wave properties for generalised Yukawa potentials

a) Numerical calculations

The method we have developed in order to obtain the analytic properties of partial waves may be also considered as a practical method of calculation. Let us consider the S wave case, an approximate $S(k)$ will be obtained by taking

$$S_{\text{Approx.}}(k) = \frac{1 + \int_0^{N\mu} P_k(\alpha) d\alpha}{1 + \int_0^{N\mu} P_{-k}(\alpha) d\alpha} \quad (47)$$

and for real energies

$$\delta_{\text{Approx}}(k) = \text{Arg} \left[1 + \int_{\mu}^{N\mu} \rho_k(\alpha) d\alpha \right] \quad (48)$$

The question is to know what is the error committed.

When $C(\alpha) < M \alpha^{1-\eta}$, for real k for instance,

$$1 + \int_{\mu}^{\infty} |\rho_k(\alpha)| d\alpha < \exp \left(\frac{M}{\eta} \frac{1}{\mu^{\eta}} \right) \quad (\text{case } \eta < 1)$$

and

$$\left| \int_{N\mu}^{\infty} \rho_k(\alpha) d\alpha \right| < \int_{N\mu}^{\infty} M \frac{d\alpha}{\alpha^{1+\eta}} \exp \left(\frac{M}{\eta} \frac{1}{\mu^{\eta}} \right)$$

$$\left| \int_{N\mu}^{\infty} \rho_k(\alpha) d\alpha \right| < \frac{M}{\eta} \frac{1}{(N\mu)^{\eta}} \exp \left(\frac{M}{\eta} \frac{1}{\mu^{\eta}} \right) = r \quad (49)$$

r goes to zero as N goes to infinity.

Therefore

$$\delta(k) = \text{Arg} \left[1 + \int_{\mu}^{N\mu} \rho_k(\alpha) d\alpha \right] \pm \sin^{-1} \frac{r}{\left| 1 + \int_{\mu}^{N\mu} \rho_k(\alpha) d\alpha \right|} \quad (50)$$

So an explicit bound can be put on the error committed.

As an application let us mention this small result if we take the Yukawa potential : the condition to get a bound state at zero energy is

$$1 + \int_{\mu}^{\infty} \rho_{k=0}(\alpha) d\alpha = 0$$

replacing this by

$$1 + \int_{\mu}^{\infty} \rho(\alpha) d\alpha = 1 - \frac{V_0}{2\mu} - \frac{V_0}{6\mu} + \left| \frac{V_0}{\mu} \right|^2 \left[\frac{1}{3} - \log \frac{4}{3} \right] = 0$$

we get a well depth parameter

$$\frac{\mu}{|V_0|} = 0.595$$

to be compared with the exact result 0.5953.

For $\ell \neq 0$ or for tensor forces the practical use of the method is not so obvious. However, the method we use seems to be very suitable for a computer in view of its iterative character.

b) The deuteron wave function and the anomalous threshold problem 11), 12), 8), 13), 14)

We have already said that our method furnishes an integral representation of the in-going and out-going wave functions. In the special case of a bound state the wave function $f(-ix, r)e^{-xr}$ is the regular wave function because $f(-ix, 0) = 0$. Hence the deuteron wave function may be written (not the reduced one) :

$$\psi_D(r) = N_D \frac{e^{-xr}}{r} \left[1 + \int_{\mu}^{\infty} \rho_D(\alpha) e^{-\alpha r} d\alpha \right] \quad (51)$$

This is the most general form of the deuteron wave function for Yukawa-like potentials. In a sense it is a generalisation of the Hulthén wave function

$$\psi_{\text{Hulthén}} = N \frac{e^{-xr} - e^{-\beta r}}{r}$$

The analytic properties of any non-relativistic matrix element involving a deuteron in its initial or final state can be investigated by inserting in it the deuteron wave function.

This has been applied to two cases :

- i) the non-relativistic deuteron electromagnetic form factor

$$F(\vec{q}^2) = \int \psi_D(\vec{r}) e^{i\vec{q} \cdot \vec{r}} \psi_D(\vec{r}) d^3r \quad (52)$$

Typical contributions to this will be obtained by replacing $|\psi_D(r)|^2$ by

$$\left(\frac{e^{-\mu r}}{r}\right)^2, \quad \frac{e^{-2\mu r} e^{-\alpha r}}{r^2} \quad (\alpha > \mu)$$

$$e^{-\alpha r} e^{-\beta r} / r^2 \quad (\alpha, \beta > \mu)$$

Let us call

$$\begin{aligned} \phi(\alpha, \beta, \vec{q}^2) &= \int e^{i\vec{q} \cdot \vec{r}} \frac{e^{-\alpha r} e^{-\beta r}}{r^2} d^3r \\ \frac{d\phi}{d\alpha} &= - \int e^{i\vec{q} \cdot \vec{r}} \frac{e^{-(\alpha+\beta)r}}{r} d^3r \\ &= - \frac{1}{(\alpha+\beta)^2 + \left(\frac{\vec{q}}{2}\right)^2} \times \text{cst.} \end{aligned}$$

Hence

$$\phi(\alpha, \beta, \vec{q}^2) = \text{const.} \int_{\alpha}^{\infty} \frac{d\gamma}{(\gamma+\beta)^2 + \left(\frac{\vec{q}}{2}\right)^2}$$

therefore in the (q^2) complex plane $\phi(\alpha, \beta, (q^2))$ has a branch point at $q^2 = -4(\alpha+\beta)^2$.

From this we conclude that the analytic properties of $F(q^2)$ are as follows : it has a branch point at $q^2 = -16x^2 = -4MB$ (where B is the deuteron binding energy) and then a cut from $q^2 = -4(2x+\mu)^2$ to $q^2 = -\infty$ (see Fig. 2).

Fig. 2

It can be noticed that the discontinuity across the cut starting from $q^2 = -16x^2$ (which is placed for convenience in the same direction as the next cut) can be calculated exactly in the region

$$-16x^2 \longrightarrow -4(2x+\mu)^2$$

from the knowledge of x and N alone, i.e. of the asymptotic form of the deuteron wave function.

If the N-P potential is partly known or, more precisely if $C(\alpha)$ is known in the range

$$\mu < \alpha < N\mu$$

(see next section) one can use our equations to calculate $p_{-ix}(\alpha)$ and then the discontinuity of $F(q^2)$ across the cut till $q^2 = -4(2x+(N+1)\mu)^2$.

It is interesting to compare the result of this analysis with the field theoretical treatment. In this treatment one considers the deuteron as an elementary particle coupled to two nucleons. In perturbation theory the simplest process of interaction of a photon with the deuteron is :

Fig. 3

Let me remind you that the standard procedure to look for singularities with respect to q^2 , the 4 momentum of the photon is to look at the lowest intermediate state in $\gamma \rightarrow D+\bar{D}$. The lowest intermediate state lies at $q^2 = -(4\mu)^2$; one would believe that between $q^2 = 0$ and $q^2 = -(4\mu)^2$ there is no singularity. However, if one calculates the singularities by explicit calculation of the Feynman denominator, as was done by Karplus, Sommerfield and Wichmann¹⁵⁾, it turns out that the first branch point is much closer to the physical region. It is called an anomalous threshold. With very good accuracy it lies at $q^2 = -16x^2$, which is just (setting $q_0^2 = 0$, which is the case if we consider the form factor as a part of e-D scattering amplitude in c.m. system) the position at which we found the first branch point in the non-relativistic treatment. Therefore the appearance of anomalous thresholds is directly related to the weak binding or the large extension of the deuteron. If the deuteron binding energy were much bigger, the relativistic treatment predicts that the anomalous thresholds should disappear. The non-relativistic treatment is unable to give this result, and this for the simple reason that in a tightly bounded deuteron the non-relativistic quantum mechanics would no longer be valid.

ii) a similar but much more complicated case is the deuteron photodisintegration matrix element. We can write this matrix element, neglecting spin complications as

$$\int \psi_{NP}^*(\vec{x}) e^{i\vec{q}\vec{x}} \psi_D(\vec{x}) d^3x \quad (53)$$

where $\psi_{NP}^*(\vec{x})$ is the NP wave function (in-coming wave + plane wave) with c.m. momentum $\vec{p}$, and $\vec{q}$ is the photon momentum. One has to study the analytic properties of

$$\int \psi_{NP}^*(\vec{x}) \frac{e^{-\alpha x}}{x} e^{i\vec{q}\vec{x}} d^3x$$

where $\alpha = x_D$ or $\alpha > x_D \mu$. This object is very similar to a scattering matrix element

$$\int \psi_{NP}^{\vec{p}_f}(\vec{x}) e^{i\vec{q} \cdot \vec{x}} \left[\int_{\mu}^{\infty} \frac{C(\alpha) e^{-\alpha x}}{x} d\alpha \right] d^3x$$

As far as the analytic properties with respect to the angle $\vec{p}_f \vec{q}$ for fixed $|\vec{p}_f|$ and $|\vec{q}|$ are concerned, both matrix elements can be treated in the same way. One can just copy the results for the analytic properties of an off-energy shell ($|\vec{p}_i| \neq |\vec{p}_f|$) scattering matrix element to get the analytic properties with respect to the angle.

We shall speak later about the fixed energy analytic properties. So we shall just give here the result for the deuteron photodisintegration matrix element :

it has a pole at $(\vec{p}_f - \vec{q}/2)^2 = -\kappa^2$

and a cut starting at $(\vec{p}_f - \vec{q}/2)^2 = -(\kappa + \mu)^2$

The fixed transfer dispersion relations for this matrix element are not very easily established because of the complications due to kinematics. This, however, can be done ¹⁴⁾ and by combination with the fixed energy dispersion relations for nearly real energies one gets a Mandelstam representation for this matrix element.

A much simpler thing to do is to look at the analytic properties of the partial waves for this matrix element ⁸⁾. Then one can also replace the final wave function by its integral representation derived in Section II. Needless to say that the analytic properties obtained in this way coincide with those obtained by projecting the Mandelstam representation.

The field theoretical analogue of this matrix element, described by the following graphs

28.

Fig. 4

The second order graphs correspond to the poles appearing in the energy or in the momentum transfer variables. The fourth order graph gives rise to anomalous thresholds in agreement with the non-relativistic treatment (cut at $(\vec{p}_f - \frac{\vec{q}}{2})^2 = - (x + \mu)^2$).

It is interesting to notice that in spite of the presence of anomalous thresholds the Mandelstam representation holds, at least in the non-relativistic approach.

Other applications can be made using this technique, for instance to study the electromagnetic form factor of the α particle ¹⁶⁾, which can be considered either as a D-D system or a N - He³ system. Then one gets cuts due to the extension of the alpha particle but one can show that the closest cut starts relatively far from the physical region ($q^2 = 11 \mu^2$) and that it is possible to observe the effect of light intermediate particles, for instance a T = 0 J = 1 particle of mass $2.2 \mu_\pi$, which gives rise to closer singularities

Fig. 5

- c) The connection between the unphysical cut of a partial wave amplitude, the S matrix, and the potential ¹⁷⁾

In the recent years many theoreticians have attempted to relate the nucleon-nucleon scattering matrix to pion-pion and pion-nucleon scattering, using Mandelstam representations and unitarity. In many approaches, the primary object which is calculated or estimated is the discontinuity of the S matrix of a given wave on the unphysical cut in the energy complex plane.

Once this discontinuity is known one has to construct a scattering amplitude such that it has the correct discontinuity on the left and fulfills the unitarity condition on the physical cut ¹⁸⁾.

In the simple case of potential scattering (for Yukawa-like potentials) one would like to be able to construct the S scattering amplitude (or any other partial wave amplitude) from the information on the discontinuity of the left hand cut in the k^2 (i.e. kinetic energy) plane. This might prove useful in the more general case, because, if we succeed in doing so, we may forget the intermediate assumption of the existence of a potential, and obtain an acceptable solution, as long as relativistic kinematical effects are unimportant.

In our study of analytic properties of partial waves we have used k rather than k^2 as a variable : the upper half k plane corresponds to the first k^2 Riemann sheet (with respect to the physical cut). The cut $k = \frac{i\mu}{2} \rightarrow i\infty$ is precisely the unphysical cut. The lower cut of the k plane is hidden in the second Riemann sheet of the energy complex plane.

So, what is supposed to be known is the discontinuity $S(ix+\varepsilon) - S(ix-\varepsilon)$ across the upper cut in the k plane :

$$S(ix + \varepsilon) - S(ix - \varepsilon) = D(x) \times 2\pi i \quad (54)$$

where D is a real function.

We shall write

$$S(k) = \frac{f(k, 0)}{f(-k, 0)}$$

as was previously done and remember that for potentials such that $C(\infty) < \infty^{1-\eta}$

$$\lim_{|k| \rightarrow \infty} \frac{|f(k, 0) - 1|}{|k|^\eta} < \text{Cst.}$$

This enables us to write an unsubtracted integral representation for f :

$$f(k, 0) = 1 - i \int_{\mu/2}^{\infty} \frac{R(x) dx}{k - ix} \quad (55)$$

$R(x)$ is a real function because $f^*(-k^*, 0) = f(k, 0)$. We get

$$S(k) = \frac{1 - i \int_{\mu/2}^{\infty} \frac{R(x) dx}{k - ix}}{1 + i \int_{\mu/2}^{\infty} \frac{R(x) dx}{k + ix}} \quad (56)$$

it is a trivial matter to evaluate the discontinuity of S across the upper cut.

$$D(x) = - \frac{R(x)}{1 + \int_{\mu/2}^{\infty} \frac{R(x') dx'}{x' + x}} \quad *) \quad (57)$$

*) In ref. 17) the sign is incorrect

or

$$R(x) = D(x) \left[1 + \int_{\mu/2}^{\infty} \frac{R(x') dx'}{x' + x} \right] \quad (58)$$

This equation can be considered as a Fredholm (or almost Fredholm) type integral equation determining $R(x)$, and hence $S(k)$ from the information on the discontinuity. This equation was obtained by Noyes and Wong¹⁸⁾ for the nucleon-nucleon case, at least for the non-relativistic kinematics, in a rather involved way, as the result of the combination of two coupled integral equations, one expressing the fact that the left hand cut has the correct discontinuity, the other expressing that the physical cut discontinuity (in the energy variable) satisfies unitarity.

Our method can be actually generalised to the general case where no potential is assumed. This has been done by Omnès¹⁹⁾. One still writes

$$S(k) = \frac{\hat{f}(k)}{\hat{f}(-k)}$$

$\hat{f}(k)$, as in the potential scattering case has no singularity in the lower half k plane, and goes to unity as $|k| \rightarrow \infty$. A natural definition for $\hat{f}(k)$ will be

$$\hat{f}(k) = \exp - \frac{1}{\pi} \int_0^{\infty} \frac{\delta(\nu') d\nu'}{\nu' - \nu}$$

(for k in the lower half plane), where ν is the kinetic energy. When the Levinson theorem is fulfilled and when no bound state is present (this imposes $\delta(0) - \delta(\infty) = 0$) $\hat{f}(k)$ coincides with our $f(k,0)$. $\hat{f}(k)$ can be prolonged in the upper half plane by

$$\hat{f}(k) = S'(\nu) \hat{f}(-k)$$

Then one shows that $\hat{f}(k)$ satisfies the same kind of integral representation as the one we proposed. The equation one ends up with is similar to ours but a little bit more complicated because :

- i) of relativistic kinematics
- ii) the bound states no longer appear as zeros of $f(-k)$ for k in the upper half plane but as poles of $\hat{f}(k)$
- iii) if $\partial(0) - \partial(\infty) \neq 0$, but rather $n\pi$, $\hat{f}(k)$ has a multiple pole at $k = 0$.

Let us come back to our original equation. Its solutions have been studied by Regge and De Alfaro ¹⁾ who have shown that if

$$\frac{1}{2} \int_{\mu/2}^{\infty} \frac{|D(x)| dx}{x} = A < \ln 2$$

the solution is unique; also, if $D(x)$ is positive and A finite the solution exists. Further they show that the number of resonances + antibound states is less than two plus the number of zeros of $D(x)$, if the above condition is fulfilled.

The next question we shall consider now is the relation between the potential and the discontinuity $D(x)$. One might question about the usefulness of constructing the potential since we have seen that from $D(x)$ one can directly construct the S matrix. However, three points are worth noticing :

- i) when the discontinuity is not completely known S cannot be constructed, but it will turn out that one can construct explicitly the external part of the potential
- ii) information on the external part of the potential permits to determine the external (however, inside the interaction region) part of the deuteron wave function which as it appeared in b) is so important for the deuteron form factor
- iii) nuclear physicists have to deal with potentials in the many-body problem.

We shall first calculate the discontinuity from the potential. In order to do this we have to look more carefully at what happens for $k = ix \pm \epsilon$ (x real $> \mu/2$).

$f(k, r)$ has no longer a unique meaning but we have some hope to define

$$f^+(ix, r) = \lim_{\varepsilon \rightarrow 0} f(ix + \varepsilon, r) \quad (59)$$

and, similarly $f^-(ix, r)$. Correspondingly we define $\rho_{ix}^+(\alpha)$ and $\rho_{ix}^-(\alpha)$. Now we notice that from the equation

$$\alpha(\alpha - 2x \pm i\varepsilon) \rho_{ix}^{\pm}(\alpha) = C(\alpha) + \int_{\mu}^{\alpha - \mu} C(\alpha - \beta) \rho_{ix}^{\pm}(\beta) d\beta \quad (60)$$

i) $\rho_{ix}^+(\alpha) - \rho_{ix}^-(\alpha) = 0$ for $\alpha < 2x$

ii) near $\alpha = 2x$

$$\rho_{ix}^+(\alpha) - \rho_{ix}^-(\alpha) = -2\pi i \delta(\alpha - 2x) \frac{C(\alpha) + \int_{\mu}^{\alpha - \mu} C(\alpha - \beta) \rho_{ix}(\beta) d\beta}{\alpha} \quad (61)$$

There is no ambiguity in the right hand side because of i).

iii) in the range $2x + \varepsilon < \alpha < 2x + \mu - \varepsilon$

Now there are several possible ways of calculating the discontinuity of $S(k)$

$$S^+(ix) - S^-(ix) = 2\pi i D(x) = \frac{\int [\rho_{ix}^+(\alpha) - \rho_{ix}^-(\alpha)] d\alpha}{1 + \int_{\mu}^{\infty} \rho_{ix}(\alpha) d\alpha} \quad (62)$$

The most straightforward may consist in noticing that $f^+(ix, r)e^{xr}$, $f^-(ix, r)e^{xr}$, $f(-ix, r)e^{-xr}$ are not independent solutions of the Schrödinger equation. The

only possible relation between these quantities, if we take into account the fact that

$$f^+(ix, \infty) = f^-(ix, \infty) = f(-ix, \infty) = 1$$

is

$$f^+(ix, r) - f^-(ix, r) = 2i\pi A(x) e^{-2xr} f(-ix, r)$$

If we look at this relation for $r \rightarrow 0$ we see that $A(x)$ is identical with $D(x)$. So we can evaluate $D(x)$ by taking the most suitable value of r ; in particular

$$\begin{aligned} 2\pi i D(x) &= \lim_{r \rightarrow \infty} e^{2xr} [f^+(ix, r) - f^-(ix, r)] \\ &= \lim_{r \rightarrow \infty} e^{-2xr} \int_{2x-\varepsilon}^{2x+\varepsilon} [\rho_{ix}^+(\alpha) - \rho_{ix}^-(\alpha)] e^{-\alpha r} d\alpha \\ &\quad + \lim_{r \rightarrow \infty} e^{-2xr} \int_{2x+\mu-\varepsilon}^{\infty} [\rho_{ix}^+(\alpha) - \rho_{ix}^-(\alpha)] e^{-\alpha r} d\alpha \end{aligned} \quad (63)$$

The last term does not contribute to $D(x)$ because it contains a factor $\exp[-(\mu-\varepsilon)r]$. The contribution of the first term is trivially evaluated. It gives just

$$D(x) = - \frac{C(2x) + \int_{\mu}^{2x-\mu} C(2x-\alpha) \rho_{ix}(\alpha) d\alpha}{2x} \quad (64)$$

In this formula $\rho_{ix}(\alpha)$ is a well defined quantity because the range of integration is such that the condition $\alpha < 2x$ is always fulfilled. At this point we have really proved that the lines

$$k = i\frac{\mu}{2} \rightarrow i\infty \quad \text{and} \quad k = -i\frac{\mu}{2} \rightarrow -i\infty$$

are cuts of the function $e^{2i\delta(x)}$. We have an explicit expression of the potential. One can notice that if the potential is written

$$V(r) = \lambda \int_{\mu}^{\infty} C(\alpha) e^{-\alpha r} d\alpha$$

the discontinuity in the range $x = n \frac{\mu}{2}$ $x = (n+1) \frac{\mu}{2}$ is given by an n^{th} order polynomial in λ . Therefore it is just $S(ix+\epsilon) - S(ix-\epsilon)$ calculated till the n^{th} Born approximation. However, the expression we have derived is exact, even when the Born series fails to converge.

It is interesting to notice that an attractive potential with $C(\alpha) < 0$ gives rise to a purely positive $D(x)$. Indeed looking at

$$\lambda(\alpha - 2\kappa) \rho_{ix}(\alpha) = C(\alpha) + \int_{\mu}^{\alpha-\mu} C(\alpha-\beta) \rho_{ix}(\beta) d\beta$$

we easily see that $C < 0$ makes $\rho_{ix}(\alpha) > 0$ for $\alpha < 2x$. Inserting this in equation (64) we see that $D(x)$ has to be positive. Returning to equation (58) we see that if there is no bound state with binding energy larger than $B = \frac{\mu^2/4}{M}$ (which is the case of the NP system)

$$1 + \int_{\mu/2}^{\infty} \frac{R(x') dx'}{\kappa' + \kappa}$$

is positive for $x > \frac{\mu}{2}$, and hence $R(x)$, if $D(x)$ is positive, is purely negative. Then

$$1 + \int_{\mu/2}^{\infty} \frac{R(x') dx'}{\kappa' + \kappa}$$

can only vanish once in the range $x = 0$ $x = \frac{\mu}{2}$ and there is at most one bound state. Hence, if a system interacting by a purely attractive potential has two bound states one of these bound states must have a binding energy larger than $\frac{\mu^2/4}{M}$. Let us also notice that if $R(x)$ is strictly negative there is no zero of $f(k,0)$ outside the imaginary axis because then $f(k,0)$ is a Herglotz function ($\text{Im } f$ has the sign of $\text{Re } k$). This means that there is no resonance since there is no pole close to the real axis in the lower complex k half plane. There may be, however, a pole on the imaginary axis between $k = 0$ and $k = -i\frac{\mu}{2}$ which would correspond to a "virtual state" (as in P-P system), but then the true bound state cannot exist.

Let us now exploit formula (64) in order to construct the potential from the information on the discontinuity.

If we know $D(x)$ in the range $x = \frac{\mu}{2} \rightarrow x = 2\frac{\mu}{2}$ we get

$$C(2x) = -2x D(x)$$

Let us go now to the range $x = \mu \rightarrow x = \frac{3\mu}{2}$ we know from the preceding result that for $\alpha < 2\mu$

$$\rho_{ix}(\alpha) = \frac{C(\alpha)}{\alpha(\alpha-2x)} = \frac{D(\alpha/2)}{2x-\alpha}$$

Hence

$$\begin{aligned} -2x D(x) &= C(2x) + \int_{\mu}^{2x-\mu} \frac{C(2x-\beta) D(\beta/2) d\beta}{2x-\beta} \\ &= C(2x) - \int_{\mu}^{2x-\mu} D(x-\beta/2) D(\beta/2) d\beta \end{aligned}$$

or

$$C(2x) = -2x D(x) + \int_{\mu}^{2x-\mu} D(x-\beta/2) D(\beta/2) d\beta$$

So, here again we are able to construct $C(\alpha)$ from D .

The procedure can go on : if we know $C(\alpha)$ in the range $\alpha = \mu$ to $\alpha = n\mu$ and $D(x)$ in a larger range, say

$$x = \mu/2 \quad \text{to} \quad x = \frac{n+1}{2} \mu$$

We know $\rho_{ix}(\alpha)$ for $\alpha < n\mu$ and we can calculate the integral appearing in the r.h.s. of (64); the l.h.s. is known by hypothesis; hence $C(2x)$ can be determined for $n\mu < 2x < (n+1)\mu$.

In general, when the discontinuity is known for $2x < N\mu$ the contribution to the potential

$$\int_{\mu}^{N\mu} C(\alpha) e^{-\alpha r} d\alpha$$

is known.

We see that the further the discontinuity is known, the better the inner part of the potential is known. If the discontinuity is completely known and if the conditions imposed by Regge and De Alfaro are fulfilled, it is that the complete potential

$$\int_{\mu}^{\infty} C(\alpha) e^{-\alpha r} d\alpha$$

is acceptable because we know anyway from previous work by Gel'fand, Levitan and others that one can fit the phase shift by a potential.

The condition that the maximum range of the potential is $1/\mu$ is a very important one. It permits, as long as there are no bound states with $x > \mu/2$ to distinguish between the zeros of $f(-k, 0)$ and the singularities of $f(k, 0)$. This is the reason why in our treatment we do not get various phase equivalent potentials.

In case a hard core is present, the treatment as given here breaks down because the scattering amplitude has an essential singularity at infinity. However, if the range of the hard core is known the procedure can be adapted to this case.

One starts from

$$V(r) = +\infty \quad \text{for } r_0 > r$$

$$V(r) = \int_{\mu}^{\infty} C(\alpha) \exp[-\alpha(r-r_0)] d\alpha \quad \text{for } r_0 < r$$

one defines

$$S'(k) = S(k) e^{2ikr_0}$$

and

$$D'(x) = D(x) e^{-2\pi r_0}$$

from $D'(x)$ one can construct $C(\alpha)$ in the ordinary way.

The connection between the discontinuity and the potential can be extended to an arbitrary partial wave. Formula (64) can be written as

$$D(x) = - \lim_{\alpha \rightarrow 2x} \rho_{ix}(\alpha) (\alpha - 2x) \quad (65)$$

Then the expression of $D(x)$ can be obtained in terms of $C(\alpha)$ and $\rho(\alpha)$ which itself can be calculated from $C(\alpha)$. When one tries to go the reverse way, one finds that at each step of the calculation one must solve a Volterra equation to construct C from D . In this latter case it is clear that D must satisfy stringent conditions to ensure that the phase shift has a correct low energy behaviour. These conditions are not apparent in our treatment because they can always be satisfied by a convenient choice of $D(x)$ for large x .

The treatment can as well be extended to the construction of potentials in presence of tensor forces. Starting from the discontinuities D_{00} , D_{02} , D_{22} of $S_{00}(k)$, $S_{02}(k)$, $S_{22}(k)$ one can construct the inverse Laplace transform of V_0 , V_2 , V_{tensor} .

IV. Analytic properties for fixed energy ⁴⁾

The complete scattering amplitude is a function of two variables $f(E, t)$, where $t = -2k^2 (1 - \cos \Theta)$, k is the c.m. momentum and Θ the scattering angle, and $E = k^2$ is the energy.

The most straightforward approach to the analytic properties for fixed physical E with respect to t is to make use of the Born expansion. However, it will turn out eventually that the analytic properties of the exact amplitude can be obtained from those derived by using the Born expansion in a very easy way. The only question this approach cannot solve is the behaviour of $f(E, t)$ for $t \rightarrow \infty$. Among all the approaches to this problem only the one by Regge furnishes the solution. However, this does not mean that the present approach is useless, because it gives another insight of the structure of the discontinuities.

For convenience we assume for the potential

$$V(r) = \int_{\mu}^{\infty} \Gamma(\alpha) \frac{e^{-\alpha r}}{r} d\alpha \quad (66)$$

i.e. we take it as a superposition of Yukawa potentials rather than exponential potentials.

The n^{th} Born term can be written, dropping numerical factors :

$$\langle \vec{k}' | V G_0 V G_0 \dots V | \vec{k} \rangle \quad (67)$$

with n V 's.

and

$$G_0 = \frac{e^{ik|r-r'|}}{|r-r'|} \quad (68)$$

in coordinate space ,

$$= \frac{1}{k'^2 - k^2 - i\varepsilon}$$

in momentum space.

Inserting the representation for $V(r)$ we get a typical contribution to the n^{th} Born term :

$$\int k_1^2 dk_1 d\Omega_1 \dots \frac{1}{\alpha_1^2 + (\vec{k}' - \vec{k}_1)^2} \frac{1}{k_1^2 - k^2 - i\varepsilon} \frac{1}{\alpha_2^2 + (\vec{k}_1 - \vec{k}_2)^2} \dots \frac{1}{\alpha_n^2 + (\vec{k}_{n-1} - \vec{k})^2} \quad (69)$$

with $\alpha_1, \alpha_2, \dots, \alpha_n > \mu$ this formula should be integrated again on $\alpha_1, \dots, \alpha_n$, with the weight function $\Gamma(\alpha)$.

To study the analytic properties with respect to the angle $(\vec{k}', \vec{k})$ which is equivalent to the study of the analyticity in t we just need to make the angular integrals. Let us consider first the integration over k_1 .

$$\int d\Omega_1 \frac{1}{\alpha_1^2 + (\vec{k}' - \vec{k}_1)^2} \frac{1}{\alpha_2^2 + (\vec{k}_1 - \vec{k}_2)^2}$$

$$= \int \frac{d\Omega_1}{4k'k_2k_1^2} \frac{1}{U_1 - (\hat{k}', \hat{k}_1)} \frac{1}{U_2 - (\hat{k}_1, \hat{k}_2)}$$

with

$$U_1 = \frac{\alpha_1^2 + k_1'^2 + k_1^2}{2k'k_1} \quad U_2 = \frac{\alpha_2^2 + k_1^2 + k_2^2}{2k_1k_2}$$

and the $\hat{}$ mean unit vectors.

This integral by means of Feynman parametrisation can be transformed into

$$-\frac{2\pi}{4k'k_2k_1^2} \int_{-1}^{+1} \frac{dy}{\left[U_1 \left(\frac{1+y}{2} \right) + U_2 \left(\frac{1-y}{2} \right) \right]^2 - \frac{1+y^2}{2} - \frac{1-y^2}{2} (\hat{k}', \hat{k}_2)}$$

or

$$-\frac{2\pi}{4k'k_2k_1^2} \int_{-1}^{+1} \frac{dy}{\left(\frac{1-y^2}{2} \right) \left[\frac{\left[U_1 \left(\frac{1+y}{2} \right) + U_2 \left(\frac{1-y}{2} \right) \right]^2 - \frac{1+y^2}{2}}{1-y^2} - (\hat{k}', \hat{k}_2) \right]}$$

Let us rewrite the bracket as $U_{12} - \hat{k}'\hat{k}_2$ and minimize it first with respect to $|\vec{k}_1|$ (since k_1 goes from 0 to ∞)

$$U_{12} = \frac{\left[\frac{1}{4k_1} \left[(1+y) \frac{\alpha_1^2 + k_1'^2}{k_1'} + (1-y) \frac{\alpha_2^2 + k_2^2}{k_2} \right] + \frac{k_1}{4} \left[\frac{(1+y)}{k_1'} + \frac{(1-y)}{k_2} \right] \right]^2 - \frac{1+y^2}{2}}{\frac{1-y^2}{2}}$$

Now since $\frac{A}{X} + BX \gg 2\sqrt{AB}$

$$U_{12} > \frac{\frac{1}{2} \left[(1+y) \frac{\alpha_1^2 + k_1'^2}{k_1'} + (1-y) \frac{\alpha_2^2 + k_2^2}{k_2} \right] \left[\frac{1+y}{k_1'} + \frac{1-y}{k_2} \right] - (1+y^2)}{1-y^2}$$

Letting $t = \frac{1+y}{1-y}$ this becomes

$$U_{12} > \frac{1}{2} \left\{ \frac{\alpha_1^2 + k_1'^2}{k_1} t + \frac{\alpha_2^2 + k_2'^2}{k_2} \right\} \left\{ \frac{1}{k_1} + \frac{1}{t k_2} \right\} - \frac{1}{2} \left(t + \frac{1}{t} \right)$$

which, again, can be minimized with respect to t . Eventually

$$U_{12} > \frac{(\alpha_1 + \alpha_2)^2 + k_1^2 + k_2^2}{2 k_1 k_2}$$

Therefore

$$U_{12} = \frac{(\lambda_{12} + \alpha_1 + \alpha_2)^2 + k_1^2 + k_2^2}{2 k_1 k_2}$$

with $\lambda_{12} > 0$. λ_{12} is a function of y and k_1 . U_{12} has the same form as U_1 and U_2 .

We can now perform the integration on $d\Omega_2$. We have to integrate

$$\frac{1}{U_{12} - \hat{k}_1' \hat{k}_2} \frac{1}{U_3 - \hat{k}_2 \hat{k}_3}$$

obviously the same reasoning can be applied. Eventually the n^{th} Born term will be written as

$$\int \frac{d(\text{parameter variables}) d(\text{lengths of } k\text{'s}) \times \text{Green's function} \times 1}{\text{product of } \left(\frac{1-y^2}{2} \right)} \times \left[U_{123 \dots n} - \hat{k}_1' \hat{k}_n \right]$$

with

$$U_{12 \dots n} > \frac{(\alpha_1 + \alpha_2 + \dots + \alpha_n)^2 + k_1'^2 + k_n^2}{2 k_1 k_n} > \frac{n^2 \mu^2 + k^2 + k_1'^2}{2 k k'}$$

The remaining integrations give no trouble because they do not give any trouble in the physical range ^{*)}.

The structure of the denominator exhibits a cut in the $\cos\theta$ plane from

$$\cos\theta = \frac{n^2\mu^2 + k'^2 + k^2}{2kk'} \quad \text{to } +\infty \quad (70)$$

or in $t = -(\vec{k}' - \vec{k})^2$, from $t = n^2\mu^2$ to $+\infty$.

Notice that we do not assume $|\vec{k}| = |\vec{k}'|$. Notice also that the result holds if the k^2 in the free Green's function is different from k and k' . Let us now consider the full scattering amplitude

$$T = T_1 + T_2 + \dots + T_n + T_{n+1} + R_n \quad (71)$$

where

$$R_n = \int (T - T_1)(\vec{k}', \vec{k}'', k) G_{0k} T_n(\vec{k}'', \vec{k}, k) d^3k'' \quad (72)$$

in coordinate space

$$(T - T_1)(\vec{k}', \vec{k}'', k) = \int e^{-i\vec{k}'\vec{x}} V(x) G_k(x, x') V(x') e^{i\vec{k}''\vec{x}'} d^3x d^3x'$$

where G_k is the exact (not the free) Green's function. $G_k(x, x')$ is regular and bounded by $\frac{\text{const}}{|\vec{x}' - \vec{x}|}$ in the upper half complex plane in k . This still holds for real k . If $|V(x)| < \text{cte } e^{-\mu x}$ which is the case here, the convergence of

^{*)} The singularities at $y = \pm 1$ may be shown to be harmless.

this integral is dependent on the factor

$$\exp \left[-i \vec{k}' \vec{x}' - \mu |\vec{x}'| + i \vec{k}'' \vec{x} - \mu |\vec{x}| \right]$$

Let us take

$$\vec{k}'' = \vec{u} \cos \lambda \theta + \vec{v} \sin \lambda \theta$$

$$\vec{k}' = \vec{u} \cos(1-\lambda)\theta + \vec{v} \sin(1-\lambda)\theta$$

with $\vec{u}, \vec{v}$ are orthogonal unit vectors. Then the real part of the argument of the exponential is at most

$$|\vec{x}| \left[k'' \sqrt{(\sin \cos \lambda \theta)^2 + (\sin \sin \lambda \theta)^2} - \mu \right] \\ + |\vec{x}'| \left[k' \sqrt{[\sin \cos(1-\lambda)\theta]^2 + [\sin \sin(1-\lambda)\theta]^2} - \mu \right]$$

$\cos \lambda \theta = \cos \lambda (\theta_1 + i \theta_2) = \cos \lambda \theta_1 \operatorname{ch} \lambda \theta_2 - i \sin \lambda \theta_1 \operatorname{sh} \lambda \theta_2$; to make the first bracket negative we get

$$k'' \operatorname{sh} \lambda \theta_2 - \mu < 0$$

and the second

$$k'' \operatorname{sh}(1-\lambda)\theta_2 - \mu < 0$$

Therefore the integral exists in the smaller of the ellipses in the $\cos \theta$ plane with foci $-1, +1$ and semi major axis

$$\begin{cases} \operatorname{Ch} \theta_2 = \operatorname{Ch} \frac{1}{\lambda} \operatorname{sh}^{-1} \frac{\mu}{k''} \\ \operatorname{Ch} \theta_2 = \operatorname{Ch} \frac{1}{1-\lambda} \operatorname{sh}^{-1} \frac{\mu}{k'} \end{cases}$$

The maximum domain is obtained when these ellipses coincide, by convenient choice of λ .

Then

$$\theta_2 = sh^{-1} \frac{\mu}{k'} + sh^{-1} \frac{\mu}{k''}$$

and the semi major axis is

$$a = \sqrt{1 + \frac{\mu^2}{k'^2}} \sqrt{1 + \frac{\mu^2}{k''^2}} + \frac{\mu^2}{k'k''}$$

This means that $T(k'k'', k)$ can be expanded in Legendre polynomials

$$T(\vec{k}', \vec{k}'', k) = \sum A_\ell P_\ell(\hat{k}' \hat{k}'')$$

with

$$\limsup_{\ell \rightarrow \infty} (A_\ell)^{\frac{1}{\ell}} \leq \frac{1}{a + \sqrt{a^2 - 1}}$$

Now $T_n(\vec{k}'', \vec{k}, k)$ can also be expanded in Legendre polynomial because from its analytic structure this expansion is certainly convergent in an ellipse with foci $-1 +1$ and semi major axis

$$b = \frac{n^2 \mu^2 + k^2 + k''^2}{2kk''}$$

Therefore

$$T_n(\vec{k}'', \vec{k}, k) = \sum B_\ell P_\ell(\hat{k}'' \hat{k})$$

with

$$\limsup_{\ell \rightarrow \infty} (B_\ell)^{\frac{1}{\ell}} = \frac{1}{b + \sqrt{b^2 - 1}}$$

One can insert these expressions in the expression of R_n . The angular integrations can be performed because

$$\int P_\ell(\hat{k}' \hat{k}'') P_\ell(\hat{k}'' \hat{k}) d\Omega_{\hat{k}''} = \delta_{\ell\ell'} \frac{4\pi}{2\ell+1} P_\ell(\hat{k}' \hat{k})$$

Hence

$$R_n = \int \frac{k'^2 dk'}{k'^2 - k^2} \sum \frac{4\pi A_\ell B_\ell}{2\ell+1} P_\ell(\hat{k}' \hat{k})$$

The expansion under the integral is convergent inside an ellipse with foci -1 $+1$ whose semi major axis c is just given by

$$\frac{1}{c + \sqrt{c^2 - 1}} = \limsup (A_e B_e)^{1/e} = \frac{1}{a + \sqrt{a^2 - 1}} \frac{1}{b + \sqrt{b^2 - 1}}$$

or $c = ab + \sqrt{a^2 - 1} \sqrt{b^2 - 1}$.

Now

$$c = Ch \left[Sh^{-1} \frac{\mu}{k''} + Sh^{-1} \frac{\mu}{k'} + Ch^{-1} \frac{n^2 \mu^2 + k^2 + k''^2}{2 k k''} \right]$$

We have to minimize it with respect to k'' . Since Ch and Sh are monotonous we can minimize first

$$Sh \left[Sh^{-1} \frac{\mu}{k''} + Ch^{-1} \frac{n^2 \mu^2 + k^2 + k''^2}{2 k k''} \right]$$

if $X = \frac{\mu}{k''}$ $Y = \frac{n^2 \mu^2 + k^2 + k''^2}{2 k k''}$

This quantity is $XY + \sqrt{X^2 + 1} \sqrt{Y^2 - 1}$ but this is just the minimum with respect to $\lambda > 0$ of

$$\frac{1}{2} \left[\left(\lambda X + \frac{Y}{\lambda} \right) + \lambda^2 - \frac{1}{\lambda^2} \right]$$

We minimize first this with respect to k'' : it gives

$$\begin{aligned} & \frac{1}{2} \left[4 \left(\lambda \mu + \frac{k^2 + n^2 \mu^2}{2 \lambda k} \right) \left(\frac{1}{2 \lambda k} \right) + \lambda^2 - \frac{1}{\lambda^2} \right] \\ &= \frac{1}{2} \left[\frac{2 \mu}{\lambda} + \lambda^2 + \frac{n^2 \mu^2}{k^2 \lambda^2} \right] \geq \frac{(n+1) \mu}{k} \end{aligned}$$

Hence

$$c_{\text{minimum}} = \sqrt{1 + \frac{\mu^2}{k'^2}} \sqrt{1 + \frac{(n+1)^2 \mu^2}{k^2}} + \frac{(n+1) \mu^2}{k k'} \quad (73)$$

The region of regularity of R_n is an ellipse with foci $-1 + 1$ and semi major axis c increasing with n . One can make this ellipse arbitrarily large. Inside this ellipse the only singularities are those given by the Born expansion, i.e. a series of cuts starting at

$$\cos \theta = \frac{k^2 + k'^2 + \mu^2}{2kk'} \quad \dots \quad \cos \theta = \frac{k^2 + k'^2 + n^2 \mu^2}{2kk'} \quad \dots$$

(or $t = \mu^2 \quad \dots \quad t = n^2 \mu^2 \quad \dots$)

We have now proven that outside the cuts there is no singularity. The only question we cannot answer is the problem of the behaviour at infinity. As we increase the dimensions of the ellipse we must take into account more and more Born terms and it looks impossible to obtain in this way an idea on the asymptotic behaviour.

At a given point of the cut the discontinuity can be in principle calculated since it consists of a finite number of contributions. As was done in the case of the analytic properties of the partial wave, one can here also try to construct $\Gamma(\alpha)$, the weight function appearing in the potential from the discontinuity in the t plane²⁰⁾. The reasoning is very similar to the one we have already done: $\text{int} = \mu^2 t = 4\mu^2$ interval, the only contribution to the discontinuity comes from the first Born term

$$T_1 = \text{const.} \int \frac{\Gamma(\alpha) d\alpha}{t - \alpha^2}$$

The discontinuity is easily seen to be proportional to $\frac{\Gamma(\sqrt{t})}{\sqrt{t}}$. Therefore $\Gamma(\alpha)$ can be constructed in the range $\alpha = \mu \quad \alpha = 2\mu$ because T_2 and the remaining terms have singularities starting at least at $t = 4\mu^2$. Let us now generalise this.

Let us write the potential as

$$V = V_{1n} + V_{nn+1} + V_{n+1\infty}$$

where

$$V_{1n} = \int_{\mu}^{n\mu} \Gamma(\alpha) \frac{e^{-\alpha r}}{r} d\alpha \quad V_{nn+1} = \int_{n\mu}^{(n+1)\mu} \Gamma(\alpha) \frac{e^{-\alpha r}}{r} d\alpha \quad V_{n+1\infty} = \int_{(n+1)\mu}^{\infty} \Gamma(\alpha) \frac{e^{-\alpha r}}{r} d\alpha$$

Assume now that we have been able to construct V_{1n} . We want to get V_{nn+1} from the discontinuity of T in the range $t = n^2 \mu^2$ to $t = (n+1)^2 \mu^2$. In this range the discontinuity is given by

$$\Delta T_1 + \Delta T_2 + \dots + \Delta T_n$$

Now the terms in which $V_{n+1\infty}$ appear do not contribute to the discontinuity because they give rise to a cut starting at least at $t = (n+1)^2 \mu^2$. The contributions $T_2 + \dots + T_n$ in which V_{nn+1} appears not to contribute to the discontinuity either, because they are combined at least with one potential of range $\frac{1}{\mu}$ and the singularities start at least at $t = (\mu + n\mu)^2 = (n+1)^2 \mu^2$. Therefore the discontinuity ΔT is given by

$$\Delta T = \Delta T_1 + \Delta (T_2 + \dots + T_n) (V_{1n})$$

where $T_2 \dots T_n$ are calculated from V_{1n} alone which is supposed to be known. This equation determines ΔT_1 in the range $t = n^2 \mu^2$ to $t = (n+1)^2 \mu^2$ and, because of the special form of T_1 it determines $\Gamma(\alpha)$ in the range $\alpha = n\mu$ to $\alpha = (n+1)\mu$. One can then iterate. Eventually one can construct $\Gamma(\alpha)$ from $\alpha = \mu$ to $\alpha = N\mu$ if ΔT is known in the range $t = \mu^2 \rightarrow t = N^2 \mu^2$. Here again the farther we know the discontinuity the better we know the potential.

This procedure furnishes an unambiguous definition of an energy dependent potential fitting at a given energy a scattering amplitude having the properties implied by Mandelstam representation. In the low energy region, for the nucleon-nucleon case, Fubini, Charap and Tausner²¹⁾ have shown that at least the external part of the potential (one and two pion exchange) is energy independent.

A possible application of this is the study of the range of the imaginary potential in nucleon-antinucleon scattering²²⁾. For this one makes the preliminary remark that a purely elastic scattering amplitude (no other channel open) will be automatically associated with a purely real potential. In this case the unitarity condition reads

$$\int \psi_{\vec{k}'}^*(x) \operatorname{Im} [V(x)] \psi_{\vec{k}}(x) d^3x \equiv \phi(t) = 0$$

for $|\vec{k}| = |\vec{k}'|$. $\phi(t)$ can be easily seen to have the same analytic properties as the scattering amplitude in t ; since it has to vanish in the physical range it vanishes everywhere and one concludes from this that $\operatorname{Im} V(X) = 0$ (assuming for $\operatorname{Im} V(X)$ the same representation as for $V(X)$).

We can obviously apply our formalism to $N\bar{N} \rightarrow N\bar{N}$, but then the potential we get is clearly complex. Two kinds of graphs contribute to $N\bar{N} \rightarrow N\bar{N}$

T_a is purely elastic below the meson production threshold, i.e. if we use an energy such that the reaction $N\bar{N} \rightarrow N\bar{N} + \pi$ is impossible. T_a has the following analytic structure in the $\cos\theta$ plane (see Fig. 7).

Fig. 7

the singularities of T_b are symmetrical.

With such analytic properties we have to use a potential

$$\int \Gamma_D(\alpha) \frac{e^{-\alpha r}}{r} d\alpha + P \int \Gamma_E(\alpha) \frac{e^{-\alpha r}}{r} d\alpha$$

where P is a Majorana exchange operator. There is no difficulty to extend our treatment to this case : to construct Γ_D and Γ_E one has to progress symmetrically with respect to $\cos \theta$. The important point is that obviously $\Gamma_E(\alpha) = 0$ for $\alpha < 2M$, and $\Gamma_D(\alpha)$ is given by the discontinuity of T_a alone for $\alpha < 2M$. But since T_a is a purely elastic amplitude, from the above argument $\Gamma_D(\alpha)$ has to be purely real in the range $\alpha = \mu \quad \alpha = 2M$. Hence the range (defined by the behaviour $\exp -2Mr$) of the imaginary potential cannot exceed $\frac{1}{2M}$.

One could provide many other applications of these fixed energy properties, in particular the study of the deuteron photodisintegration matrix element which has already been mentioned. Some people are trying to make an explicit calculation of the discontinuity for a Yukawa potential to see from what point the oscillations predicted by Regge start.

R E F E R E N C E S

- 1) E.P. Wigner, Phys. Rev. 98, 145 (1955).
- 2) N.N. Khuri, Phys. Rev. 107, 1448 (1957);
See also : A. Klein and C. Zemach, Annals of Physics 7, 440 (1959).
- 3) S. Mandelstam, Phys. Rev. 112, 1344 (1958).
- 4) A. Klein, J. Math. Phys. 1, 41 (1960);
R. Blanckenbeckler, M. Goldberger, N. Khuri and S. Treiman, Annals of Physics 10, 62 (1960);
T. Regge, Nuovo Cimento 14, 951 (1959);
J. Bowcock and A. Martin, Nuovo Cimento 14, 516 (1959).
- 5) A. Martin, Nuovo Cimento 14, 403 (1959).
- 6) R. Jost, Helvetica Physica Acta 19, 256 (1946).
- 7) A. Martin, Nuovo Cimento 15, 99 (1959).
- 8) V. De Alfaro and M. Rossetti, Nuovo Cimento 18, 783 (1960).
- 9) A. Martin and R. Vinh Mau, Nuovo Cimento 20, 390 (1961).
- 10) H. Cornille, private communication. To be published.
- 11) L. Bertocchi, C. Ceolin and M. Tnin, Nuovo Cimento 18, 770 (1960).
- 12) R. Blanckenbeckler and Y. Nambu, Nuovo Cimento 18, 525 (1960).
- 13) A. Martin, Nuovo Cimento 19, 344 (1961).
- 14) A. Martin and R. Vinh Mau, Nuovo Cimento 20, 246 (1961).
- 15) R. Karplus, G. Sommerfield and M. Wichmann, Phys. Rev. 111, 1187 (1958);
Phys. Rev. 114, 376 (1959).

- 16) L. Bertocchi, S. Fubini and M. Tonin, private communication.
- 17) A. Martin, Nuovo Cimento 19, 1257 (1961).
- 18) H.P. Noyes and D.Y. Wong, Phys. Rev. Letters 3, 191 (1959).
- 19) R. Omnès, preprint. To be published in the proceedings of the Aix-en-Provence conference (1961).
- 20) A. Martin and Gy. Targonski, to be published in Nuovo Cimento.
- 21) J. Charap and S. Fubini, Nuovo Cimento 14, 540 (1959); Nuovo Cimento 15, 73 (1960).
J. Charap and J. Tausner, Nuovo Cimento 18, 316 (1960).
- 22) A. Martin, CERN preprint. To be published in Physical Review.