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Dispersion Relations and Stability Conditions

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A B S T R A C T

In the axiomatic formulation it is shown that, in the non-exothermic case, the elastic scattering dispersion relations up to some positive momentum transfer follow only from the condition of stability of the external particles.

1. Introduction

The proof of the dispersion relations in the axiomatic formulation with the Bogoliubov ¹⁾ method by the introduction of the Jost-Lehmann-Dyson representation ²⁾ in the unitarity condition is guaranteed if the conditions (3) are satisfied.

In this paper we have investigated the mathematical problem of finding under what conditions Eq. (3) is valid in the elastic scattering process :

$$A + B \rightarrow A + B$$

A and B being two particles with arbitrary masses m_A and m_B which are restricted only by the stability conditions.

In what follows we need no hypothesis about the internal quantum numbers of these particles. One finds that the conditions (3) entail the condition (A) and (A') to be fulfilled in order to prove dispersion relations. A closer examination of (A) and (A') shows that the validity of (A) entails the validity of (A'). We stress that only the stability of the external particles is used in order to be left with the condition (A) and (A'), and to show further that (A') is fulfilled if (A) is fulfilled.

It follows the special result that, if all the reactions starting from the initial states $A+B$ and $\bar{A}+B$ are both non-exothermic (which means that the threshold of the absorptive parts of the scattering processes $A+B \rightarrow A+B$ and $\bar{A}+B \rightarrow \bar{A}+B$ ^{*}) occurs at the baricentric total energy m_A+m_B), then the stability of the external particles is necessary and sufficient to guarantee the validity of dispersion relations up to a certain positive momentum transfer. For exothermic processes a sufficient condition is given in order that (A) is fulfilled.

2. Discussion of the condition (3)

The proof of the dispersion relations is reduced to the discussion of the minimum of the function ¹⁾ :

$$X = \frac{\bar{u}^2 + k_c^2 - (u_0 + \frac{m_B - m_A}{2W})^2}{2|\bar{u}|} \quad (1)$$

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^{*}) $\bar{A}$ = antiparticle of A.

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where :

$$k_c^2 = \frac{(W^2 + m_B^2 - m_A^2)^2}{4W^2} - m_B^2$$

W is the centre-of-mass energy, m_A and m_B are the masses of the two particles, and $|\bar{u}|$, u_0 , x must move inside the domain S :

S :

$$\begin{cases} 0 \leq |\bar{u}| \leq \frac{W}{2} \\ |u_0| \leq \frac{W}{2} - |\bar{u}| \\ \mathcal{E} = \text{Max} \left\{ 0; M_A - \sqrt{\left(\frac{W}{2} + u_0\right)^2 - \bar{u}^2}; M_B - \sqrt{\left(\frac{W}{2} - u_0\right)^2 - \bar{u}^2} \right\} \end{cases}$$

M_A and M_B being the lowest masses of the states having the same quantum number (and containing at least two particles) of the A and B particles respectively.

The stability conditions of the A and B particles impose that :

$$\begin{aligned} M_A^2 &\geq m_A^2 \\ M_B^2 &\geq m_B^2 \end{aligned} \quad (2)$$

According to the Bogoliubov ¹⁾ method, we must keep in (1) one of the particles at its physical mass value and the other squared mass is allowed to vary from some negative value up to its physical value.

Because (2) holds for the physical mass values, it will be true a fortiori when m_A^2 or m_B^2 are smaller than their physical values. We choose to keep on the mass shell the particle which is congruent to the smallest of the two masses M_A and M_B . We suppose

$$M_A \geq M_B$$

and we let $m_A^2 \equiv \xi$ run from its negative value up to its physical value.

The energy W in (1) ranges from the minimum mass of the states congruent to the $A+B$ and $\bar{A}+B$, up to infinity.

The dispersion relations hold if :

$$\begin{aligned} X &> 0 \\ X^2 &> k_c^2 \end{aligned} \quad (3)$$

for all W running from zero to infinity, $|\vec{u}|$, u_0 and αe varying in S and ξ going from some negative values to its physical value.

If (3) holds for the minimum value of X $X_{\min}$, it holds always. It has been shown that $X_{\min}^2$ is also the upper limit of the momentum transfer for which dispersion relations can be proved. The research of $X_{\min}$ gives the following results :

a) For : $W > M_A + M_B$

$$X_{\min} = \left\{ k_c^2 + \frac{(M_B^2 - m_B^2)(M_A^2 - \xi)}{W^2 - (M_A - M_B)^2} \right\}^{1/2} \quad (4)$$

b) For : $\bar{W} < W < M_A + M_B$

where $\bar{W} = \left[\frac{(M_A - M_B)(M_A M_B - m_B^2)}{M_B} \right]^{1/2}$

$$X_{\min} = \left\{ k_c^2 + \frac{(M_B^2 - m_B^2)(M_A^2 - \xi)}{W^2 - (M_A - M_B)^2} \right\}^{1/2} \quad (5)$$

provided that the additional condition

$$W^3 - W^2(M_A + M_B) - W(m_B^2 + \xi - 2M_A M_B) + (\xi - m_B^2)(M_A - M_B) > 0 \quad (6)$$

4.

is satisfied.

c) For : $M_A - M_B < W < \bar{W}$

$$X_{\min} = \left\{ k_c^2 + \frac{(M_A^2 - \xi)(M_B^2 - m_B^2)}{W^2 - (M_A - M_B)^2} + m_A^2 \left[\frac{W^2 - \bar{W}^2}{W^2 - (M_A - M_B)^2} \right]^2 \right\}^{1/2} \quad (7)$$

provided that condition (6) is satisfied.

d) For : $W < M_A - M_B$

$$X_{\min} = \left\{ k_c^2 + M_A^2 + \frac{(M_A^2 - \xi)(M_B^2 - m_B^2)}{W^2} \right\}^{1/2} \quad *) \quad (8)$$

If (6) is satisfied one sees that the condition $X^2 > k_c^2$ is verified for all W ; for what concerns the condition $X > 0$ we distinguish two cases :

if $m_A + m_B < \bar{W}$

X is positive for all W -values.

if $m_A + m_B > \bar{W}$

the stability conditions guarantee that $X > 0$ everywhere except for the ranges :

$$\bar{W} < W < m_A + m_B$$

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*) The $X_{\min}$ has been previously discussed in the Appendix, Chapter IV of A. Salam's lectures on the "Analytic Properties of Expectation Values of Products of Field Operators" - University of Rochester - NYO-8796 (1958). The results of the quoted paper are correct apart from the $X_{\min}$ in case c).

So we are left with the two conditions :

$$(A) \quad W^3 - W^2(M_A + M_B) - W(m_B^2 + \xi - 2M_A M_B) + (\xi - m_B^2)(M_A - M_B) > 0$$

for $M_A - M_B < W < M_A + M_B$

$$(A') \quad k_c^2 + \frac{(M_B^2 - m_B^2)(M_A^2 - \xi)}{W^2 - (M_A - M_B)^2} > 0$$

for $\bar{W} < W < m_A + m_B$

We investigate now what relation exists between (A) and (A').

One can show that :

$$\begin{aligned} & \left[W^2 - (M_A - M_B)^2 \right] \left\{ k_c^2 + \frac{(M_B^2 - m_B^2)(M_A^2 - \xi)}{W^2 - (M_A - M_B)^2} \right\} = \\ & = W^3 - W^2(M_A + M_B) - W(m_B^2 + \xi - 2M_A M_B) - (\xi - m_B^2)(M_A - M_B) \quad (9) \end{aligned}$$

Therefore because from stability conditions the right-hand side of (9) is positive for all W between $M_A - M_B$ and $M_A + M_B$, then the fulfilment of condition (A) entails the fulfilment of (A').

Finally because $M_A - M_B < m_A + m_B$ *) we arrive at the conclusion that the

*) It holds :

$$M_A \leq m_A + 2m_B$$

because a state containing the A particle plus a pair $(B, \bar{B})$ is surely congruent to A whatever are the charges and the parity involved and M_A is the lightest of such states. $M_B > m_B$ then gives

$$M_A - M_B < m_A + m_B$$

6.

dispersion relations in the axiomatic formulation hold provided that the condition (A) is fulfilled in the range :

$$M_A - M_B < W < m_A + m_B$$

when ξ attains its physical value. The maximum momentum transfer is the over-all minimum in W among the four $X_{\min}^2$ (4), (5), (6) and (7).

3. Non-exothermic case

In this case the condition (A) is always verified because $W \geq m_A + m_B$. The maximum momentum transfer allowed is the minimum in W of :

$$k_c^2 + \frac{(M_B^2 - m_B^2)(M_A^2 - m_A^2)}{W^2 - (M_A - M_B)^2}$$

if

$$\bar{W} < m_A + m_B$$

when

$$\bar{W} > m_A + m_B$$

the maximum momentum transfer is the smallest of the two minima :

$$k_c^2 + \frac{(M_B^2 - m_B^2)(M_A^2 - m_A^2)}{W^2 - (M_A - M_B)^2} \quad \text{for} \quad W > \bar{W}$$

and

$$k_c^2 + \frac{(M_B^2 - m_B^2)(M_A^2 - m_A^2)}{W^2 - (M_A - m_B)^2} + m_B^2 \left[\frac{W^2 - \bar{W}^2}{W^2 - (M_A - M_B)^2} \right]^2 \quad \text{for} \quad m_A + m_B < W < \bar{W}$$

4. Exothermic case

In the general case a sufficient condition for the validity of (A) is given by :

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$$W_m^3 - W_m^2 (M_A + M_B) - W_m^2 (m_B^2 + m_A^2 - 2M_A M_B) + (m_A^2 - m_B^2)(M_A - M_B) > 0 \quad (10)$$

where :

$$W_m = \frac{1}{3} \left[M_A + M_B + \sqrt{(M_A + M_B)^2 + 3(m_A^2 + m_B^2 - 2M_A M_B)} \right]$$

In the particular case of equal particle scattering

$$m_A = m_B = m$$

$$M_A = M_B = M$$

(10) becomes :

$$M^2 > 2m^2$$

The well-known condition which forbids nucleon-nucleon dispersion relations to be proved in the axiomatic formulation.

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