CM-P00056809

ON POSSIBLE SIMPLE FORMS OF THE COLLISION MATRIX FOR STRONGLY
INTERACTING PARTICLES AT VERY HIGH ENERGIES

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A B S T R A C T

In multi-GeV collisions of strongly interacting particles the most readily measurable quantities like total and elastic cross-sections, multiplicity and momentum distribution of secondaries have been found to have a very simple energy dependence suggestive of an asymptotic region. It seems natural to expect that the whole S-matrix would become simple in this region. As a first exploration of what this might imply we discuss some possible simple forms of the S-matrix compatible with the existence of inelasticity, the purely imaginary character of the elastic scattering amplitude, and the requirements of unitarity and invariance. The first criterion of simplicity we choose to study is the condition that S have as few eigenvalues $\neq 1$ as possible, i.e., that the number of non-vanishing eigenphases be minimum. This minimum number turns out to be two for each angular momentum and, if it is realized, the operator S is completely determined by the knowledge of how it acts on the initial state. The second criterion considered is a reality condition more comprehensive than the observed imaginary character of the elastic scattering amplitude. Together with the first it means that the two non-vanishing eigenphases of S for given angular momentum become opposite of each other, a property which has some relation to recent speculations on what may be regarded as the number of "elementary" particles. The two criteria considered are presented solely as theoretical possibilities; we do not know whether they are fulfilled by the actual S-matrix of strongly interacting particles.

1. Introduction

The rapidly increasing experimental evidence indicates that all dominant features of collisions of strongly interacting particles in the multi-GeV region have a very simple and smooth dependence on incident energy, suggesting that this region is of an asymptotic nature, without marked structure effects. Thus total cross-sections become practically constant with values approximately satisfying the relations predicted by the Pomeranchuk theorem ¹⁾, simple transverse and longitudinal momentum distributions are found for the secondaries formed in inelastic collisions ²⁾, the elastic scattering is dominated by a strong forward peak with a momentum transfer distribution which is almost energy independent, and the forward scattering amplitude seems to be purely imaginary ³⁾.

Clearly, in very high energy collisions, the various characteristic parameters which have been investigated up to now reveal surprisingly simple properties. While theoretical understanding of these properties is still largely lacking, it seems to us natural to expect that in the multi-GeV region a very simple asymptotic behaviour would prevail for the complete S-matrix of strongly interacting particles. As a first exploration of what such a general behaviour might be we propose to discuss in the present paper what may be called, in a sense to be defined below, the simplest asymptotic forms which are possible for S, given a few empirically established features like the existence of inelastic, unitarity and the invariance properties of strongly interacting particles.

Our discussion, of course, will depend crucially on what we choose to consider as a simple asymptotic form for S. The criterion of simplicity we have adopted is in terms of the eigenvalues s_n of the S-matrix, and consists in requiring that the number of s_n different from 1 be as small as possible. We certainly lack well founded reasons to adopt this criterion. It seems to us rather natural for an exploratory investigation, however, because it is expressed

in terms of intrinsic properties of S independently of any special representation. In addition, it turns out a posteriori to have the virtue of leading to rather definite results : given the final state $S|i\rangle$ of a collision with initial two-particle state $|i\rangle$, the minimum number of eigenvalues $\neq 1$ is two for each angular momentum, and S is uniquely determined by the condition that this minimum number be realized, so that it can then be constructed explicitly from the knowledge of $|i\rangle$ and $S|i\rangle$. This result, established in section 2, has mathematically a non-trivial contents. Whether the S -matrix of strongly interacting particles approaches at very high energies the simple form verifying this minimum property is of course another question which we must leave completely unanswered. This possibility exists, however, and if realized in nature it would be quite significant for any dynamical theory of high energy collisions.

As is well known the available data are compatible with a purely imaginary character of the elastic scattering amplitude at very high energies. This means reality of $\langle i, \ell | S | i, \ell \rangle$, where $|i, \ell\rangle$ is the angular momentum ℓ component of the initial two-particle state $|i\rangle$. One can wonder whether this feature is a reflection of a more comprehensive reality property of the whole S -matrix. A reality property of this type is suggested in section 3. It states the reality of $\langle i, \ell | S^m | i, \ell \rangle$ for all m , or equivalently the fact that the eigenvalues of S for each ℓ are two by two complex conjugate of each other. If, as suggested above, for each ℓ all eigenvalues but two are equal to 1, this reality property becomes equivalent to the statement that the sum of the eigenphases of S for given ℓ approaches an integral multiple of π at infinite energy. Recent conjectures have related this sum to the number of "elementary" particles of given angular momentum ⁴⁾.

Throughout our discussion we completely neglect spin and isospin, under the assumption that spin - and isospin - dependent effects become small at very high energies. The angular momentum ℓ is meant to designate always the orbital angular momentum.

2. Minimum number of non-vanishing eigenphases

We study the following mathematical question. $|i\rangle$ being the initial two-particle state, consider all matrices S verifying unitarity and the usual invariance conditions of strong interactions and transforming $|i\rangle$ into a given final state $|f\rangle = S|i\rangle$. Among these, determine the matrices S with the minimum number of eigenvalues different from 1.

We work in the centre of mass system. Since spin is neglected we have from rotational invariance

$$|i\rangle = \sum_{\ell} c_{\ell} |i\ell\rangle \quad |f\rangle = \sum_{\ell} c_{\ell} |f\ell\rangle \quad (1)$$

$$|f\ell\rangle = S |i\ell\rangle \quad (2)$$

where $|i\ell\rangle, |f\ell\rangle$ are states of angular momentum $\ell = 0, 1, 2, \dots$. We select the norm and phase of $|i\ell\rangle$ such that

$$\langle i\ell | i\ell \rangle = 1, \quad \mathcal{K} |i\ell\rangle = |i\ell\rangle \quad (3)$$

where $\mathcal{K}$ is the time reversal operator. Let S_{ℓ} be the restriction of S to the subspace of states of same total parity and angular momentum as $|i\ell\rangle$.

We know that S_{ℓ} must satisfy the following conditions

$$S_{\ell}^* = S_{\ell}^{-1} \quad (4)$$

$$\mathcal{K} S_{\ell} \mathcal{K} = S_{\ell}^{-1} \quad (5)$$

$$S_{\ell} |i\ell\rangle = |f\ell\rangle \quad (6)$$

Since the collision is supposed to be inelastic we can suppose $|i\ell\rangle$ and $|f\ell\rangle$ to be linearly independent. We shall prove that S_ℓ then has at least two eigenvalues different from 1, and that it is entirely determined by the requirement that it has exactly two such eigenvalues.

The proof is quite simple when the time reversed state of $|f\ell\rangle$ is in the two-dimensional space $\mathcal{H}_2$ spanned by $|i\ell\rangle$ and $|f\ell\rangle$:

$$\mathcal{K}|f\ell\rangle = \mathcal{K} S_\ell |i\ell\rangle = S_\ell^{-1} \mathcal{K} |i\ell\rangle = S_\ell^{-1} |i\ell\rangle = \alpha |i\ell\rangle + \beta |f\ell\rangle.$$

We have used $\mathcal{K}^2 = 1$. Note that $\beta \neq 0$. Clearly S_ℓ then leaves $\mathcal{H}_2$ invariant and its restriction to $\mathcal{H}_2$ is completely determined by the knowledge of $|i\ell\rangle$, $|f\ell\rangle$ and the ensuing knowledge of $\mathcal{K}|f\ell\rangle$. It has two eigenvalues which will be calculated below. Since S_ℓ transforms $\mathcal{H}_2$ into itself, we are not particularly interested in the space $\mathcal{H}'$ of states orthogonal to $\mathcal{H}_2$. (We remark, nevertheless, that this space is also invariant for S_ℓ and that one minimizes the number of eigenvalues of S_ℓ different from 1 by defining the restriction of S_ℓ to $\mathcal{H}'$ to be the unit operator.) The eigenvalues of S_ℓ in $\mathcal{H}_2$ are easily determined as follows. One can write

$$|f\ell\rangle = \gamma |i\ell\rangle + i e^{i\varphi} |\psi\rangle$$

with γ complex, φ real, and $|\psi\rangle$ satisfying

$$\langle\psi|i\ell\rangle = 0, \quad \mathcal{K}|\psi\rangle = |\psi\rangle, \quad \langle\psi|\psi\rangle = 1 - |\gamma|^2 > 0.$$

S_ℓ in $\mathcal{H}_2$ is then defined by the two equations

$$S_\ell |i\ell\rangle = \gamma |i\ell\rangle + i e^{i\varphi} |\psi\rangle$$

$$S_\ell^{-1} |i\ell\rangle = \gamma^* |i\ell\rangle - i e^{-i\varphi} |\psi\rangle$$

One easily checks the validity of (4) and (5). Calculation of the eigenvalues is now immediate and gives

$$s_{1,2} = e^{i\varphi} (x \pm i \sqrt{1-x^2}), \quad x = \operatorname{Re}(\gamma e^{-i\varphi}).$$

These eigenvalues are in general both $\neq 1$. In actual cases the elastic scattering amplitude is known to be purely imaginary; hence γ is real, $x = \gamma \cos \varphi$ with $|\gamma| < 1$ and both s_1 and s_2 are $\neq 1$ (and $\neq -1$).

We go over to the more general case in which the three vectors $|i\ell\rangle$, $|f\ell\rangle$ and $\mathcal{K}|f\ell\rangle$ are linearly independent. They span a three-dimensional space $\mathcal{H}_3$. From (3), (5) and (6) we have

$$S_\ell^{-1} |i\ell\rangle = \mathcal{K} |f\ell\rangle. \quad (7)$$

(6) and (7) are now of course insufficient to determine S_ℓ . Furthermore the smallest space containing $|i\ell\rangle$ and invariant for S_ℓ , while containing $\mathcal{H}_3$, will in general be much larger than $\mathcal{H}_3$. Let $|\psi_n\rangle$ be a complete set of eigenvectors of S_ℓ , and s_n the corresponding eigenvalues. Consider the expansion

$$|i\ell\rangle = \sum_n \lambda_n |\psi_n\rangle \quad (8)$$

From (6) and (7)

$$|f\ell\rangle = \sum_n \lambda_n s_n |\psi_n\rangle \quad (9)$$

$$\mathcal{K} |f\ell\rangle = \sum_n \lambda_n s_n^{-1} |\psi_n\rangle \quad (10)$$

We can always select the $|\psi_n\rangle$ so that for any two λ_n which are $\neq 0$ the corresponding eigenvalues s_n are different from each other. Let this be done. If the vectors (8), (9) and (10) are to be linearly independent, at least 3 of the λ_n must be $\neq 0$. Consequently at least two of the corresponding eigenvalues s_n must be $\neq 1$. Conversely there exists one and only one operator S_ℓ verifying (4), (5), (6) and (7) and having two eigenvalues $\neq 1$. This unique operator transforms $\mathcal{H}_3$ into itself and leaves invariant every state vector orthogonal to $\mathcal{H}_3$. The proof of these facts is somewhat too long to be given here. They follow from slightly more general results established in the appendix.

A remark is in order before closing this section. In the two cases discussed above, the condition of minimum number of eigenvalues $\neq 1$ has led to the following results :

- 1) there is a subspace $\mathcal{H}_N$ of low dimension N ($N=2$ or 3) containing the initial two-particle state $|i\ell\rangle$ and invariant for S_ℓ ;
- 2) knowledge of $|i\ell\rangle$ and $S_\ell|i\ell\rangle$ completely determines S_ℓ in this subspace ;
- 3) S_ℓ is unity in the subspace $\mathcal{H}'$ orthogonal to $\mathcal{H}_N$.

Instead of minimizing the number of eigenvalues $\neq 1$ for S_ℓ , we could have minimized, always for given $|i\ell\rangle$ and $S_\ell|i\ell\rangle$, the smallest subspace $\mathcal{H}_N$ invariant for S_ℓ and containing $|i\ell\rangle$. This would have led to the same $\mathcal{H}_N$ as before, with $N = 2$ or 3 . For $N = 2$, S_ℓ would have been fully known in $\mathcal{H}_2$. For $N = 3$ on the contrary the remaining arbitrariness of S_ℓ would have been lifted by the condition that one of the three eigenvalues of S_ℓ in $\mathcal{H}_3$ be $= 1$. No statement would have been obtained concerning the action of S_ℓ in the subspace $\mathcal{H}'$ orthogonal to $\mathcal{H}_N$.

This latter point is in fact irrelevant, because result 3) above cannot be physically true if one includes in $\mathcal{H}'$ the totality of states having the same quantum numbers as $|i\ell\rangle$ and orthogonal to $\mathcal{H}_N$, like e.g., arbitrary

3 - and more - particle states which of course lead to a non-trivial S-matrix. With regard to such collisions with more than 2 particles in the initial state $|i'\ell\rangle$ one would simply expect that at very high energies the smallest subspace $\mathcal{H}_N'$ invariant for S_ℓ and containing $|i'\ell\rangle$ would be some low-dimensional space entirely orthogonal to $\mathcal{H}_N$, i.e., contained in $\mathcal{H}'$.

3. Reality condition

There is evidence that the elastic scattering amplitude of strongly interacting particles becomes purely imaginary at very high energies. In our notation this means that the number $\langle i\ell | S_\ell | i\ell \rangle$ becomes real. We now discuss what seems to be a rather natural extension of this reality condition. Here again, in absence of further experimental evidence, our choice is based on grounds of simplicity. We assume that $\langle i\ell | S_\ell^m | i\ell \rangle$ is real for all integers m . Let us first discuss this condition in general. We shall afterwards consider it under the additional restriction that S_ℓ has the minimum number 2 of eigenvalues $\neq 1$.

As shown in the appendix, reality of $\langle i\ell | S_\ell^m | i\ell \rangle$ for all m has the following consequences. Let us restrict S_ℓ to the smallest Hilbert space possible containing $|i\ell\rangle$, i.e., to the space spanned by the eigenvectors $|\psi_n\rangle$ for which $\lambda_n \neq 0$ in (8) (we assume as before that $s_n \neq s_{n'}$ whenever both λ_n and $\lambda_{n'}$ are $\neq 0$). For each eigenvalue $s_{n_0} \neq 1$ there is then another eigenvalue s_{n_0}' equal to $s_{n_0}^* = s_{n_0}^{-1}$, and $|\lambda_{n_0}| = |\lambda_{n_0}'|$ if ψ_{n_0} and ψ_{n_0}' have the same norm. Conversely, the latter properties imply reality of $\langle i\ell | S_\ell^m | i\ell \rangle$. We discuss these properties somewhat further. From time-reversal invariance we can select the eigenvectors ψ_n so that $K\psi_n = \psi_n$ and the λ_n are then real. The λ_n can be made positive by changing the sign of ψ_n if necessary. We

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further norm all ψ_n to unity. Then we can assert $\lambda_{n_0} = \lambda_{n'_0}$.

Consider now the unitary operator U defined by

$$U \psi_{n_0} = \psi_{n'_0} \quad \text{if} \quad s_{n_0} \neq 1 \quad (11)$$

$$U \psi_n = \psi_n \quad \text{if} \quad s_n = 1. \quad (12)$$

It verifies

$$USU = S^{-1}, \quad U^2 = 1, \quad U|i\ell\rangle = \mathcal{K}|i\ell\rangle = |i\ell\rangle, \quad U|f\ell\rangle = \mathcal{K}|f\ell\rangle \quad (13)$$

and is therefore very similar to the time reversal operator $\mathcal{K}$, except that it is unitary rather than anti-unitary. It is interesting to speculate whether U could have any physical significance. The form of the Schrödinger equation suggests that a unitary operator simulating time reversal should be an operator changing the sign of all energy differences with respect to some energy value E . Such an operator U' would thus act as follows on the Hamiltonian H :

$$U'(H-E)U' = -(H-E), \quad U'^2 = 1. \quad (14)$$

The S -matrix on the energy shell E can be defined as the limit of $\exp[-i(H-E)t]$ for $t \rightarrow +\infty$. It will therefore verify

$$U'SU' = S^{-1} \quad (15)$$

in analogy with (13). We suggest that for very large E such an "energy inversion" symmetry U' may exist among those virtual states which contribute appreciably to collisions at energy E , and that its restriction to the energy

shell of energy E may be the operator U considered above. The reality condition we have proposed would thereby be a manifestation of the asymptotic validity of an energy inversion symmetry at very high energies.

We have still to discuss our reality condition under the assumption that S_ℓ has only two eigenvalues $\neq 1$. In this case the reality condition is fulfilled as soon as the two numbers $\langle i\ell | S_\ell | i\ell \rangle$ and

$$\langle i\ell | S_\ell^2 | i\ell \rangle = \langle f\ell | \mathcal{K} | f\ell \rangle \quad (16)$$

are real. Reality of the first is already indicated by the experimental facts ; the unverified contents of the condition is therefore reality of (16). If s_1 and s_2 are the two eigenvalues of S_ℓ which are different from 1, and if δ_1 and δ_2 are the corresponding eigenphases

$$s_n = \exp(2i \delta_n) \quad (17)$$

the reality condition is equivalent to the statement

$$\delta_1 + \delta_2 = m \pi, \quad m \text{ integer},$$

or, since all other eigenphases are themselves integral multiples of π , it is equivalent to the statement that the sum $\sum \delta_n$ of all eigenphases of S_ℓ approaches an integral multiple of π as the energy E tends to infinity. At low energy this sum reduces of course to the elastic phase-shift, which is known to tend to an integral multiple of π for $E \rightarrow 0$, so that our reality condition now amounts to

$$\left(\sum_{E=\infty} \delta_n \right) - \left(\sum_{E=0} \delta_n \right) = m \pi, \quad m \text{ integer} \quad (18)$$

This is an interesting relation which has also been encountered in a completely different context, namely in speculations concerning an observable characterization of the number of particles which should be regarded as "elementary". One proposal in this respect relates the number m_{el} of elementary particles of given quantum numbers to the integer m in Eq. (18) where $\sum \delta_n$ is the sum of eigenphases of the S-matrix for the same quantum numbers and is treated as a continuous function of the energy E . The relation is

$$m_{el} = m_{st} + m$$

where m_{st} is the number of discrete stationary states, i.e., of stable particles observed with the given set of quantum numbers. This relation, known as the Robinson theorem in Schrödinger theory ⁵⁾, has also been found to hold in models with multi-channel reactions ⁴⁾.

4. Concluding remarks

In concluding we should emphasize again the entirely speculative character of the relation between actual phenomena and the above considerations. Although we are quite convinced that some form of asymptotic simplicity must exist for the complete collision matrix of strongly interacting particles in the multi-GeV region, we have no reason to believe that the simple forms we have proposed should be the ones realized in nature. In view of the fact, however, that an experimental determination of the full S-matrix seems to be a highly impractical task while its knowledge is bound to be of interest for theoretical understanding of high-energy collisions, we feel that a theoretical exploration of possible simple asymptotic forms should be undertaken, and present our considerations as an attempt in this direction.

We still wish to make two points regarding the possible relation of our attempt with experimental facts. Firstly, as mentioned already at the end of section 2, it is not truly the full S -matrix which we have discussed, but its restriction to the smallest subspace of states which it leaves invariant and which contains the initial two-particle state. A rather similar behaviour may hold for 3 and more particle collisions. Another point which one may have to keep in mind is that a very simple asymptotic form of S , as proposed in our considerations, might well hold only in terms of final "particles" which are not the usual particles stable for strong interactions but might for example be some of the unstable baryons and mesons recently discovered as "resonances".

A P P E N D I X

For brevity we write $|0\rangle$ instead of $\mathcal{K}|fl\rangle = S_\ell^{-1}|il\rangle$ and S instead of S_ℓ . $|0\rangle$ is a vector of norm one, S a unitary operator. We are interested in the smallest subspace $\mathcal{H}_N$ invariant for S and containing $|0\rangle$. Let $|\psi_n\rangle$, s_n be the eigenvectors and eigenvalues of S . Consider the expansion

$$|0\rangle = \sum \lambda_n |\psi_n\rangle. \quad (A1)$$

As stated before the eigenvectors $|\psi_n\rangle$ can be so selected that for any two non-vanishing coefficients $\lambda_{n'}$, $\lambda_{n''}$ in (A1) one has $s_{n'} \neq s_{n''}$. The space $\mathcal{H}_N$ is then spanned by the set of $|\psi_n\rangle$ which have non-vanishing coefficient in (A1). The dimension of $\mathcal{H}_N$ is denoted by N ; we are especially interested in cases where N is finite.

$\mathcal{H}_N$ is also the space spanned by all vectors

$$|m\rangle = S^m |0\rangle, \quad m=0, \pm 1, \pm 2, \dots \quad (A2)$$

The vectors $|0\rangle, |1\rangle, \dots, |N-1\rangle$ are linearly independent and any $|m\rangle$ is a linear combination of them. From the unitarity of S we have

$$\langle m'|m\rangle = \langle 0|S^{m-m'}|0\rangle = a_{m-m'}, \quad (A3)$$

$$a_{-m} = a_m^*, \quad a_0 = 1 \quad (A4)$$

Consider the determinant

$$D_{m,n} \equiv \begin{vmatrix} \langle m'' | m' \rangle & \langle m'' | m'+1 \rangle & \dots & \langle m'' | m'+n \rangle \\ \langle m''+1 | m' \rangle & \langle m''+1 | m'+1 \rangle & \dots & \langle m''+1 | m'+n \rangle \\ \vdots & \vdots & & \vdots \\ \langle m''+n | m' \rangle & \langle m''+n | m'+1 \rangle & \dots & \langle m''+n | m'+n \rangle \end{vmatrix} = \begin{vmatrix} a_m & a_{m+1} & \dots & a_{m+n} \\ a_{m-1} & a_m & \dots & a_{m+n-1} \\ \vdots & \vdots & & \vdots \\ a_{m-n} & a_{m-n+1} & \dots & a_m \end{vmatrix} \quad (A5)$$

where $m = m' - m''$. From the above properties

$$\left. \begin{aligned} D_{m,n} &\neq 0 \quad \text{for } n < N, m \text{ arbitrary} \\ D_{m,n} &= 0 \quad \text{for } n \geq N, m \text{ arbitrary} \end{aligned} \right\} \quad (A6)$$

We discuss now the eigenvalues s_n of S which belong to the eigenvectors $|\psi_n\rangle$ with non-vanishing coefficient λ_n in (A1). By relabelling indices we denote then by $s_1, \dots, s_N$, the corresponding eigenvectors being $\psi_1, \dots, \psi_N$. The linear relationship between $|0\rangle, |1\rangle, \dots, |N\rangle$ is clearly given by

$$\prod_{i=1}^N (S - s_i) |0\rangle = |N\rangle - (s_1 + s_2 \dots) |N-1\rangle + \dots + (-1)^N s_1 \dots s_N |0\rangle = 0 \quad (A7)$$

Multiplying on the left by $|m\rangle$ one obtains

$$a_{m-N} - (s_1 + s_2 \dots) a_{m-N+1} + \dots + (-1)^N s_1 \dots s_N a_m = 0 \quad (A8)$$

This provides us with the following N -th order equation for the eigenvalues $s_1, \dots, s_N$ (which, we remember, are all different from each other) in terms of $a_1, \dots, a_N$.

$$d_N(s) \equiv \begin{vmatrix} s^N & s^{N-1} & \dots & 1 \\ 1 & a_1 & \dots & a_N \\ a_{-1} & 1 & \dots & a_{N-1} \\ \vdots & \vdots & & \vdots \\ a_{-N+1} & a_{-N+2} & \dots & a_1 \end{vmatrix} = 0 \quad (\text{A9})$$

Once the eigenvalues are known the eigenvectors follow from the formula

$$\lambda_n |\psi_n\rangle = \prod_{i \neq n} \frac{s - s_i}{s_n - s_i} |0\rangle \quad (\text{A10})$$

which is an immediate consequence of (A1).

If one of the eigenvalues $s_1, \dots, s_N$ is $= 1$, Eq. (A9) with $s=1$ can be regarded as an equation determining a_N in terms of $a_1, \dots, a_{N-1}$. Indeed the coefficient of a_N in the relation $d_N(1) = 0$ is

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_{-1} & 1 & \dots & a_{N-2} \\ \vdots & \vdots & & \vdots \\ a_{-N+1} & a_{-N+2} & \dots & 1 \end{vmatrix} \quad (\text{A11})$$

Its complex conjugate can be written with (A4) as

$$\begin{vmatrix} 1 & a_1 & \dots & a_{N-1} \\ a_{-1} & 1 & \dots & a_{N-2} \\ \vdots & \vdots & & \vdots \\ a_{-N+2} & a_{-N+3} & & a_1 \\ 1 & 1 & \dots & 1 \end{vmatrix} \quad (\text{A12})$$

Vanishing of (A11) and (A12) is easily seen to be incompatible with (A6). The same reasoning, replacing $N-1$ by N in (A11) and (A12), shows that if a_N is so selected that $d_N(1) = 0$ the determinant $D_{0,N}$ is automatically $= 0$.

The above results can be summarized as follows. With the knowledge of the vectors

$$|0\rangle, S|0\rangle \dots S^{N-1}|0\rangle \quad (A13)$$

(which of course determines $a_1, \dots, a_{N-1}$) and the additional knowledge of $a_N = \langle 0 | S^N | 0 \rangle$, the operator S is completely determined in the space $\mathcal{H}_N$ spanned by (A13). Knowledge of a_N is unnecessary if S has in $\mathcal{H}_N$ an eigenvalue $= 1$. The two different cases considered in section 2 are covered by these results. In the first, $N = 2$ and a_2 is given by

$$a_2 = \langle f | \mathcal{K} | f \rangle^* ;$$

in the second, $N = 3$ and a_3 is determined by the equation $d_3(1) = 0$.

The reality condition of section 3 states simply, in the present notation, that all a_m are real. This implies reality of the coefficients of the eigenvalue equation (A9). The eigenvalues are consequently two by two complex conjugate of each other. If N is odd, one of the eigenvalues is $= 1$, so that a_N then automatically verifies $d_N(1) = 0$. If we return to the remarks at the end of section 2, this last result implies that, when $|i\ell\rangle$ and $S|i\ell\rangle$ are known, the reality condition and the condition of minimum dimension of $\mathcal{H}_N$ fully determine S in $\mathcal{H}_N$.

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