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PAIR DISTRIBUTION FUNCTION AND SCATTERING PHENOMENA

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A B S T R A C T

Consequences of familiar relations between the compressibility of a many-body system, its density fluctuations, its pair distribution and its scattering properties are discussed and are applied to the case of a system in its ground state, with special reference to atomic nuclei.

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1. Introduction

It was first pointed out by Gibbs that the compressibility of a medium is simply related to its density fluctuations. Since the density fluctuations are in turn expressible in terms of a volume integral of the two-particle correlation function, a knowledge of the compressibility of a medium imposes an important and well-known restriction on the form of its pair correlation function. In addition, the usual sum rule for the potential scattering of projectiles by the medium depends on the Fourier transform (with respect to momentum transfer) of the same pair correlation function. Consequently, the compressibility determines the scattering sum rule for all momentum transfers q such that q^{-1} is large compared with the interparticle distance and small compared with the dimensions of the system.

These relations have proved of considerable use, e.g., in the investigation of the properties of solids and liquids by neutron scattering²⁾. Our purpose in discussing them here in revised and more detailed form is to show that they have interesting consequences for many-particle systems in their ground state, a case for which they have not been commonly used. In particular, for atomic nuclei, they impose a non-trivial condition on the pair correlation function in "nuclear matter", and so provide a consistency check on calculations of the nuclear matter wave function. They also provide information in the small momentum transfer region on the scattering of various projectiles by large nuclei. The condition for their applicability to a many-particle system is that the minimum dimension of the system be large compared to the interparticle distance. For scattering problems one must also assume that the scattering amplitude for the system is a sum of single particle, momentum and spin independent scattering amplitudes. This is often the case for small momentum transfers if the Born approximation or the impulse (or pseudo-potential) approximation is valid.

2.

2. Pair correlation function and density fluctuations

We recall briefly the main formulae concerning pair correlation functions and density fluctuations for a system at temperature T ; further details can be found in the review article of de Boer ¹⁾. We then discuss in more detail the case of a system in its ground state, obtained by letting T approach zero. We assume the system to contain only one kind of particles. The extension to several kinds is straightforward.

The 1-particle and 2-particle distribution functions are defined as expectation values at temperature T of the corresponding operators

$$n_1(\underline{r}) = \left\langle \sum_{j=1}^N \delta(\underline{r} - \underline{r}_j) \right\rangle \quad (2.1)$$

$$n_2(\underline{r}', \underline{r}'') = \left\langle \sum_{j \neq k=1}^N \delta(\underline{r}_j - \underline{r}') \delta(\underline{r}_k - \underline{r}'') \right\rangle \quad (2.2)$$

N is the number of particles. We are interested in a large system : $N \rightarrow \infty$ with fixed density $n = N/V$ (V is the volume of the system and tends to ∞ proportionally to N). In the homogeneous and isotropic case we can then assume

$$n_1(\underline{r}) = n + \omega_N(\underline{r}) \quad (2.3)$$

$$n_2(\underline{r}', \underline{r}'') = n_1(\underline{r}') n_1(\underline{r}'') + n^2 \left[g(r) + \varphi_N(\underline{r}', \underline{r}'') \right] \quad (2.4)$$

where for $N \rightarrow \infty$ we have $\omega_N(r) \rightarrow 0$, $\varphi_N(\underline{r}', \underline{r}'') \rightarrow 0$ except for $\underline{r}, \underline{r}'$ or $\underline{r}''$ near the boundary. $g(r)$ depends only on the distance $r = |\underline{r}' - \underline{r}''|$. It is the usual 2-particle correlation function which, if we exclude the case of long range order, goes to zero for r beyond a certain finite range r_0 .

For our purpose we need somewhat more specific properties of $\varphi_N(\underline{r}', \underline{r}'')$. For a large system it seems natural to assume that the variation of this function with $\underline{r}'$ and $\underline{r}''$ will be very slow when $|\underline{r}' - \underline{r}''| \gg r_0$ and $\underline{r}', \underline{r}''$ are away from the boundary. Actually, as a discussion of the density fluctuations will show presently, the range of $\varphi_N(\underline{r}', \underline{r}'')$ in $|\underline{r}' - \underline{r}''|$ must be of the order of the over-all dimensions of the system, i.e. of order $V^{1/3}$. These properties imply that for $\underline{q}$ fixed $\neq 0$

$$\lim_{N \rightarrow \infty} V^{-1} \int_V d\underline{r}' \int_V d\underline{r}'' e^{i\underline{q} \cdot (\underline{r}' - \underline{r}'')} \varphi_N(\underline{r}', \underline{r}'') = 0 \quad (2.5)$$

because the presence of the exponential entails that the total contribution to the integral comes from finite values of $|\underline{r}' - \underline{r}''|$. The relation (2.5) is of great importance for scattering: it implies that for fixed non-vanishing momentum transfer $\underline{q}$ the scattering by a large system is entirely controlled by $g(r)$, the asymptotic value of the 2-particle correlation.

The density fluctuations in a subvolume Ω of V are obtained by considering the number of particles in Ω

$$N_\Omega = \int_\Omega d\underline{r} \sum_j \delta(\underline{r} - \underline{r}_j) \quad (2.6)$$

4.

Its average is

$$\overline{N}_{\Omega} = \int_{\Omega} d_{\underline{3}}\underline{r} n_1(\underline{r}) = n \Omega \quad (2.7)$$

(the latter value holding for a large system) and its mean square fluctuation

$$\overline{\Delta N_{\Omega}^2} = \overline{N_{\Omega}^2} - \overline{N}_{\Omega}^2 = \overline{N}_{\Omega} + \int_{\Omega} d_{\underline{3}}\underline{r}' \int_{\Omega} d_{\underline{3}}\underline{r}'' \left[n_2(\underline{r}', \underline{r}'') - n_1(\underline{r}') n_1(\underline{r}'') \right] \quad (2.8)$$

For $\Omega = V$, one has $\overline{N}_{\Omega} = N$, $\overline{\Delta N_{\Omega}^2} = 0$ and (2.8) becomes an identity between n_1 and n_2 . This identity is a special consequence of the more general relations

$$\int_V d_{\underline{3}}\underline{r} n_1(\underline{r}) = N \quad (2.9)$$

$$\int_V d_{\underline{3}}\underline{r}' n_2(\underline{r}', \underline{r}'') = (N-1) n_1(\underline{r}'') \quad (2.10)$$

which follow immediately from the definitions.

By substituting (2.3) and (2.4) and keeping only non-vanishing terms we rewrite (2.8) for a large system and a volume $\Omega \gg r_0^3$

$$\begin{aligned} \overline{\Delta N_{\Omega}^2} / \overline{N}_{\Omega} &= 1 + 4\pi n \int_0^{\infty} g(r) r^2 dr + \\ &+ n \Omega^{-1} \int_{\Omega} d_{\underline{3}}\underline{r}' \int_{\Omega} d_{\underline{3}}\underline{r}'' \psi_N(\underline{r}', \underline{r}'') \end{aligned} \quad (2.11)$$

When Ω remains finite as the system goes to infinity the last term approaches zero and we obtain the familiar relation between $g(r)$ and the density fluctuations

$$1 + 4\pi n \int_0^\infty g(r)r^2 dr = \overline{\Delta N_\Omega^2} / \overline{N}_\Omega \quad \text{for } \Omega \text{ finite } \gg r_0^3 \quad (2.12)$$

The right-hand side of (2.12), on the other hand, can be expressed in thermodynamic terms by applying the grand ensemble theory to the volume Ω ³⁾. One finds the well-known Gibbs relation

$$\overline{\Delta N_\Omega^2} / \overline{N}_\Omega = kT (\partial \overline{N}_\Omega / \partial \mu) / \overline{N}_\Omega = nkT [\Omega (\partial^2 F / \partial \Omega^2)]^{-1} \quad (2.13)$$

where μ is the chemical potential and F the free energy (for several kinds of particles one considers only the first relation and remembers that there are as many chemical potentials as kinds of particles).

The relations (2.12), (2.13) have been extensively applied to the scattering of X-rays and neutrons by macroscopic systems ^{1),2)}. Our purpose is to discuss their implications when the system is in its ground state. This is done very simply by letting $T \rightarrow 0$ in (2.13). For general systems the intensive quantity $(\partial \overline{N}_\Omega / \partial \mu) / \overline{N}_\Omega$ then approaches a finite non-vanishing limit. This fact, i.e. the non-vanishing of $\Omega (\partial^2 F / \partial \Omega^2)$ for $T \rightarrow 0$, is immediately verified for the perfect Fermi gas. By continuity it must then hold in general for a Fermi gas with forces, irrespective of whether its ground state is of the "normal" or the "superconducting" type. The

same property holds for the Bose gas with forces ^{*}) (but neither for the perfect Bose gas nor for the perfect classical gas). It seems natural to assume its validity also for nuclear matter, despite the fact that $\partial F/\partial \Omega$ vanishes in this case ^{**}). Combining (2.12) and (2.13) we then obtain the identity

$$1 + 4\pi n \int_0^{\infty} g(r) r^2 dr = 0 \quad (2.14)$$

This equation is an important restriction on the form which the 2-particle correlation function of a large system in its ground state can have. In nuclear matter for example, accepting the existence of a hard core of radius $r_c \approx 0.4 f$ in the nucleon-nucleon potential, we have $g(r) = -1$ for $r < r_c$. Hence

$$4\pi n \int_{r_c}^{\infty} g(r) r^2 dr = -1 + \frac{4\pi}{3} n r_c^3 \quad (2.15)$$

Noting that the last term in the right-hand side is not larger than about 0.04, we conclude that the rise of $g(r)$ from -1 to 0 for $r > r_c$ must be spread over a considerable distance, of the order of the nearest neighbour

^{*}) For the low temperature properties of a dilute Bose gas of hard spheres, see T.D. Lee and C.N. Yang ⁴⁾. The case of liquid helium is discussed on a more phenomenological basis by R.P. Feynman and M. Cohen ⁴⁾, who in appendix B of their paper predict equation (2.14) below and discuss its relation to the identity (2.10). We are grateful to Dr. E.P. Gross for bringing this point to our attention.

^{**}) We thank Dr. J.S. Bell for an interesting discussion on this point.

distance (this is not in contradiction with the fact that the healing distance ⁵⁾ is quite short ; the latter refers to the values of r for which $g(r)$ differs from its Fermi gas value ^{*}). Another restriction on the 2-particle correlation $g(r)$, derivable from scattering theory, will be mentioned at the end of the next section.

We now consider the fluctuation formula (2.11) in the case where, as the system increases, one lets the subvolume Ω also tend to infinity. As long as the ratio Ω/V tends to zero, we expect $\overline{\Delta N_\Omega^2} / \overline{N}_\Omega$ to retain its grand canonical value (2.13) because the subvolume Ω is then still very small compared to the particle reservoir formed by $V-\Omega$. Consequently the expression

$$n \Omega^{-1} \int_{\Omega} d\mathbf{r}' \int_{\Omega} d\mathbf{r}'' \varphi_N(\mathbf{r}', \mathbf{r}''), \quad (2.16)$$

which forms the only Ω dependent part of (2.11), must then remain zero. If on the contrary $\Omega \rightarrow \infty$ so fast that Ω/V remains above a fixed positive number, the ratio subvolume to reservoir is finite and we expect $\overline{\Delta N_\Omega^2} / \overline{N}_\Omega$ to have a value smaller than (2.13) (for a classical ideal gas one has $n_1 = n$, $n_2 = (1-N^{-1})n^2$ and (2.8) gives the simple value $1-(\Omega/V)$ for

^{*}) In nuclear physics it has been rather common to consider as 2-particle distribution the function $P_2 = n_2/N(N-1)$ which is normalized to one, and to use the Ansatz $P_2(\mathbf{r}', \mathbf{r}'') = V^{-2} \left[1 + G(|\mathbf{r}' - \mathbf{r}''|) \right]$ with $G(r) \rightarrow 0$ for $r \rightarrow \infty$ (see e.g. Fowler and Watson ⁶⁾ and Kerman, McManus and Thaler ⁷⁾). In this case $\int_V G(r) d\mathbf{r} = 0$ is required for normalization but, since for a large system $G(r)$ is essentially our function $g(r)$, the equations (2.14) or (2.15) are violated. A slightly modified Ansatz which satisfies (2.14) and therefore avoids this difficulty is $P_2 = N(N-1)^{-1} V^{-2} [1 + G(r)]$. For large but finite systems it has, however, no more than formal value.

$\overline{\Delta N_\Omega^2} / \overline{N_\Omega}$). Correspondingly (2.16) has then a non-vanishing negative value. Call r_N the range of $\varphi_N(\underline{r}', \underline{r}'')$ in the relative distance $|\underline{r}' - \underline{r}''|$. Clearly the variation of (2.16) with Ω occurs for values $\Omega \sim r_N^3$. From our discussion we therefore conclude that r_N is of order $V^{1/3}$. We still note that in the extreme case $\Omega = V$ the density fluctuation vanishes so that (2.16) becomes equal and opposite to (2.13); this is also a consequence of the identity

$$n \int_V d\underline{r}' \varphi_N(\underline{r}', \underline{r}'') = -1 - 4\pi n \int_0^\infty g(r) r^2 dr$$

which is easily derived from (2.10).

Thus, in the case of general temperature, we see that when Ω increases from a finite volume to V the quantity (2.16) decreases from zero to its minimum value, equal and opposite to (2.13). Specializing to the ground state, $T \rightarrow 0$, and assuming again that $(\partial \overline{N_\Omega} / \partial \mu) / N$ approaches a finite non-vanishing limit, we obtain the identity

$$\Omega^{-1} \int_\Omega d\underline{r}' \int_\Omega d\underline{r}'' \varphi_N(\underline{r}', \underline{r}'') = 0 \quad (2.17)$$

valid in the limit of a large system, irrespective of the behaviour of Ω in the limiting process. A slight extension of (2.17) is obtained by considering a volume $\Omega = \Omega' + \Omega''$ formed of two non-overlapping parts Ω' , Ω'' . Remembering that n_2 and therefore φ_N are symmetric in $\underline{r}'$ and $\underline{r}''$ we obtain from (2.17)

$$\begin{aligned} \Omega^{-1} \left[\int_{\Omega'} d\underline{r}' \int_{\Omega'} d\underline{r}'' \varphi_N(\underline{r}', \underline{r}'') + \int_{\Omega''} d\underline{r}' \int_{\Omega''} d\underline{r}'' \varphi_N(\underline{r}', \underline{r}'') + \right. \\ \left. + 2 \int_{\Omega'} d\underline{r}' \int_{\Omega''} d\underline{r}'' \varphi_N(\underline{r}', \underline{r}'') \right] = 0 \end{aligned}$$

From (2.17) the two first terms are zero. We obtain

$$\Omega^{-1} \int_{\Omega'} d_{\underline{z}} r' \int_{\Omega''} d_{\underline{z}} r'' \varphi_N(r', r'') = 0 \quad (2.18)$$

Combining this result with (2.17) one immediately concludes that (2.18) is valid for any subvolumes Ω', Ω'' of V , overlapping or not; Ω is then the volume of the union of Ω' and Ω'' . Here again the extension of our identities to several kinds of particles is straightforward.

The consequences of the results just obtained for scattering are discussed in the next section.

3. Application to scattering sum rule

Consider the scattering of projectiles by the system discussed in the previous section and assume that the interaction between projectile and system can be written as a sum

$$\sum_{j=1}^N V(\underline{r}_{\underline{p}} - \underline{r}_{\underline{j}})$$

($\underline{r}_{\underline{p}}$ position of the projectile) where $V(\underline{r})$ is a spin-independent potential which can be treated to 1st Born approximation, or a pseudo-potential which taken to 1st order represents the interaction under the impulse approximation ^{*)}. To fix the ideas we assume the projectile to be non-relativistic ^{**)}. The differential cross-section (elastic or inelastic), taken per particle in the system, is then given by

$$\frac{d^2\sigma}{d\Omega d\varepsilon_2} = \left| \frac{m}{2\pi} \tilde{V}(\underline{q}) \right|^2 \left(\frac{\varepsilon_2}{\varepsilon_1} \right)^{\frac{1}{2}} S(\underline{q}, \varepsilon_1 - \varepsilon_2) \quad (3.1)$$

where m is the mass of the projectile, ε_1 and ε_2 its initial and final energies, $\underline{q}$ the momentum transferred by it to the system, $\tilde{V}(\underline{q})$ the Fourier transform of $V(\underline{r})$ and

^{*)} The pseudo-potential case is familiar from the theory of slow neutron scattering by atomic nuclei.

^{**)} Our considerations also apply to scalar interactions of relativistic particles. An analogous discussion of the Coulomb part in the scattering of relativistic electrons by a nucleus is given elsewhere ⁸⁾.

$$S(\underline{q}, \varepsilon_1 - \varepsilon_2) = N^{-1} \sum_{n_2, n_1} \left| \langle n_2 | \sum_{j=1}^N e^{i\mathbf{q} \cdot \mathbf{r}_j} | n_1 \rangle \right|^2 \delta(E_{n_2} - E_{n_1} + \varepsilon_2 - \varepsilon_1) p_{n_1} \quad (3.2)$$

Here $|n\rangle$ is the n -th eigenstate of the system, E_n its energy and p_n its statistical weight at temperature T . If the system is in its ground state p_n is the Kronecker symbol δ_{n0} .

It is often convenient to eliminate the final states by integrating the experimental values of (3.2) over ε_2 at fixed $\underline{q}$. The function $S(\underline{q})$ so obtained can be expressed by means of a familiar sum rule in terms of the equilibrium properties of the system at temperature T (its ground state properties if $T = 0$):

$$S(\underline{q}) = \int_{-\infty}^{\infty} S(\underline{q}, \varepsilon_1 - \varepsilon_2) d\varepsilon_2 \quad (3.3)$$

$$= N^{-1} \sum_{n_2, n_1} \left| \langle n_2 | \sum_{j=1}^N e^{i\mathbf{q} \cdot \mathbf{r}_j} | n_1 \rangle \right|^2 p_{n_1} \quad (3.4)$$

$$= 1 + N^{-1} \sum_{n_1} p_{n_1} \langle n_1 | \sum_{j \neq k=1}^N e^{i\mathbf{q} \cdot (\mathbf{r}_j - \mathbf{r}_k)} | n_1 \rangle$$

$$= 1 + N^{-1} \int_V d\mathbf{r}' \int_V d\mathbf{r}'' e^{i\mathbf{q} \cdot (\mathbf{r}' - \mathbf{r}'')} n_2(\mathbf{r}', \mathbf{r}'') \quad (3.5)$$

where the definition (2.2) has been used. In the case of a large system we substitute for n_2 the expression (2.4) and obtain a result which can be written ^{*)}

$$S(q) = N \left| F(q) \right|^2 + S_{in}(q) \quad (3.6)$$

$$F(q) = N^{-1} \int_V e^{iq \cdot \underline{r}} n_1(\underline{r}) d\underline{r} \quad (3.7)$$

$$S_{in}(q) = s_{in}(q) + S_{in}^N(q) \quad (3.8)$$

$$s_{in}(q) = 1 + 4\pi n q^{-1} \int_0^\infty g(r) r \sin qr dr \quad (3.9)$$

$$S_{in}^N(q) = nV^{-1} \int_V d\underline{r}' \int_V d\underline{r}'' e^{iq \cdot (\underline{r}' - \underline{r}'')} \varphi_N(\underline{r}', \underline{r}'') \quad (3.10)$$

The first term in (3.6) contains the form factor $F(q)$ of the system. If the system is large, this term represents the elastic scattering corresponding to the terms $n_1 = n_2$ of (3.4) ^{**)} . $S_{in}(q)$, which for a large system represents the inelastic scattering, contains a part $s_{in}(q)$ independent of the size V of the system and a size-dependent term $S_{in}^N(q)$.

^{*)} We assume V to be of spherical shape.

^{**)} The reason is that for a large system the average of the product $\langle n_1 | N^{-1} \sum \exp(iq \cdot \underline{r}_j) | n_1 \rangle \langle n_1 | N^{-1} \sum \exp(-iq \cdot \underline{r}_j) | n_1 \rangle$ can be replaced by the product of the averages. This replacement and the identification of $S_{in}(q)$ with the inelastic scattering are exact for all sizes of the system if it is in the ground state ($T = 0$).

The quantity $S_{in}(q)$ tends to zero for $q \rightarrow 0$. This fact follows immediately from the vanishing of the density fluctuation (2.8) for $\Omega = V$. It is also implied by the identification of $S_{in}(q)$ with the inelastic part of the scattering: the terms $n_2 \neq n_1$ of (3.4) clearly vanish for $q \rightarrow 0$. The results of the foregoing section allow us to discuss in detail how this limit is approached. As mentioned in (2.5), $S_{in}^N(q) \rightarrow 0$ for $N \rightarrow \infty$ at q fixed $\neq 0$. The system being large, for these values of q the whole inelastic scattering is given by the size-independent term $s_{in}(q)$. It tends to 1 (quasi-elastic scattering) for $q \gg n^{1/3}$ and to

$$s_{in}(0) = 1 + 4\pi n \int_0^{\infty} g(r)r^2 dr$$

for $q \ll r_0^{-1}$ (in practice the range r_0 of $g(r)$ is usually of order $n^{-1/3}$). The latter limit is given by the thermodynamic density fluctuation, see (2.12), (2.13). If, on the contrary, we let q tend to zero at the same time as we let the system become very large, the size-dependent part $S_{in}^N(q)$ of the inelastic scattering must be taken into account. It is obvious from (3.10) that it contributes when q^{-1} is of the same order $V^{1/3}$ as the range of $\varphi_N(r', r'')$. When $q=0$ its contribution is $-s_{in}(0)$ and exactly compensates the size-independent term. The behaviour of the inelastic scattering function $S_{in}(q)$ for a large system at general temperature is illustrated in Fig. 1a.

It is now a simple matter to see how these results specialize when the system is in its ground state. The variation for q fixed $\neq 0$ remains qualitatively the same, with the exception that (for a general system) the limit $s_{in}(0)$ reached for $q \ll r_0^{-1}$ vanishes (identity (2.14) of the previous section). As to the size-dependent part $s_{in}^N(q)$ of the inelastic scattering, it must vanish for all q , even when q^{-1} is of order $V^{1/3}$. In this case, in fact, we can decompose the volume V in a finite number of subvolumes

$\Omega', \Omega'', \dots$, each one of dimensions $\ll q^{-1/3}$. Subdividing V in this fashion in the integrals of (3.10) we obtain a sum

$$\sum_{\Omega' \Omega''} nV^{-1} \int_{\Omega'} d\mathbf{r}' \int_{\Omega''} d\mathbf{r}'' e^{iq \cdot (\mathbf{r}' - \mathbf{r}'')} \varphi_N(\mathbf{r}', \mathbf{r}'')$$

in each term of which the exponential can be treated as constant. Application of the identity (2.18) established above then shows that the whole expression vanishes. The remarkably simple behaviour of the inelastic scattering at zero temperature is shown in Fig. 1b. It is entirely size-independent. For comparison we note that in all cases the elastic scattering is concentrated in the region $q \sim V^{-1/3}$, see Fig. 1c.

The above results, based on the separation (2.4) of $n_2(\mathbf{r}', \mathbf{r}'')$ in $n_1(\mathbf{r}')n_1(\mathbf{r}'')$ and a correlation term, were derived on the assumption that the scattering system is very large. It may be worth noting that an analogous separation of n_2 also occurs naturally for systems of non-interacting fermions in a definite quantum state, irrespective of their size. In this case, an example of which is given by the shell model of atomic nuclei, the wave function is a Slater determinant and the 2-particle distribution function is

$$\begin{aligned} n_2(\mathbf{r}', \mathbf{r}'') &= \frac{1}{2} \sum_{\alpha, \beta=1}^N \left| \psi_{\alpha}(\mathbf{r}') \psi_{\beta}(\mathbf{r}'') - \psi_{\beta}(\mathbf{r}') \psi_{\alpha}(\mathbf{r}'') \right|^2 \\ &= n_1(\mathbf{r}') n_2(\mathbf{r}'') - n_{\text{ex}}(\mathbf{r}', \mathbf{r}'') \end{aligned}$$

where n_{ex} is the correlation due to exchange

$$n_{\text{ex}}(\mathbf{r}', \mathbf{r}'') = \left| \sum_{\alpha=1}^N \psi_{\alpha}^*(\mathbf{r}') \psi_{\alpha}(\mathbf{r}'') \right|^2$$

The ψ_α are the single-particle wave functions. One now finds for the inelastic part of the scattering function the expression

$$S_{\text{in}}(q) = 1 - N^{-1} \int_V d\mathbf{r}' \int_V d\mathbf{r}'' e^{i\mathbf{q} \cdot (\mathbf{r}' - \mathbf{r}'')} n_{\text{ex}}(\mathbf{r}', \mathbf{r}'') \quad (3.11)$$

from which the elastic form factor has disappeared. If, in addition, the system is large and in its ground state, the discussion given before applies to the q -dependence of (3.11). Here again the presence of several kinds of particles, like protons and neutrons in nuclei, gives rise only to straightforward modifications.

Before closing we come back for one moment to the conditions which can be derived for the 2-particle correlation function $g(r)$ of an infinite system. As shown in section 2, consideration of the density fluctuations leads to an equality involving the integral

$$\int_0^\infty g(r) r^2 dr,$$

namely equations (2.12) and (2.13) for general temperature and (2.14) for the ground state. It is worth noting that by using the property of the scattering function $S(q)$ to be positive one can derive the inequality

$$I = \int_0^\infty g(r) dr > -\frac{3}{4} n^{-1/3}$$

The proof has been given elsewhere ²⁾. For a system with hard core radius r_c this inequality becomes

$$- \int_{r_c}^{\infty} g(r) dr < \frac{3}{4} n^{-1/3} - r_c$$

and can be an additional, useful source of information for approximate determinations of the pair correlation in nuclear matter for example. Explicit calculation in special cases shows that the integral I can have either sign : it is positive for a perfect Bose gas and negative for a perfect Fermi gas.

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FIGURE CAPTION

Fig. 1 The inelastic part of the scattering function, $S_{in}(q)$, as function of q for positive temperature (a) and ground state (b). The quantity $s_{in}(0)$ is equal to the density fluctuation $\Delta N_{\Omega}^2 / N_{\Omega}$ as calculated from grand ensemble theory; it vanishes for $T = 0$. Fig. 1(c) represents the elastic part of the scattering function.