More about exactly massless quarks on the lattice.

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Abstract

In a previous publication [hep-lat/9707022] I showed that the fermion determinant for strictly massless quarks can be written on the lattice as \( \det D \), where \( D \) is a certain finite square matrix explicitly constructed from the lattice gauge fields. Here I show that \( D \) obeys the Ginsparg-Wilson relation \( D \gamma_5 D = D \gamma_5 + \gamma_5 D \).
In a recent publication [1] I showed that the overlap led to a simple definition of a lattice gauge theory with exactly massless quarks on the lattice. The vector-like character of the theory makes it possible to represent the lattice Dirac operator for strictly massless quarks by a matrix $D$ of finite size and fixed shape.

$D$ is defined as follows: Start from,

$$X(m) = \begin{pmatrix} B + m & C \\ -C^\dagger & B + m \end{pmatrix},$$

(1)

where,

$$(C)_{x\alpha i, y\beta j} = \frac{1}{2} \sum_{\mu=1}^{4} \sigma_\mu^{\alpha\beta} [\delta_{y, x+\hat{\mu}} (U_\mu(x))_{ij} - \delta_{x, y+\hat{\mu}} (U_\mu^\dagger(y))_{ij}],$$

(2)

$$(B)_{x\alpha i, y\beta j} = \frac{1}{2} \delta_{\alpha\beta} \sum_{\mu=1}^{4} [2\delta_{xy} \delta_{ij} - \delta_{y, x+\hat{\mu}} (U_\mu(x))_{ij} - \delta_{x, y+\hat{\mu}} (U_\mu^\dagger(y))_{ij}],$$

and

$$\begin{pmatrix} 0 \\ \sigma_\mu^\dagger \\ \sigma_\mu \end{pmatrix} = \gamma_\mu.$$ 

(3)

The $\gamma_\mu$ are Euclidean Dirac matrices, $x, y$ are sites on the lattice, $\alpha, \beta$ are Weyl spinor indices and $i, j$ are color indices. The $U_\mu(x)$ are lattice link matrices. Set the parameter $m$ to some number in the range $(-1, 0)$. Define the unitary matrix $V$ by

$$V = X \frac{1}{\sqrt{X^\dagger X}}.$$ 

(4)

This definition is valid except for exceptional configurations with $\det(X) = 0$. We shall assume that these gauge backgrounds can be ignored statistically. Now define $D = 1 + V$.

In ref. [1] I first argued that $\det D$ was a good lattice regularization of the fermion determinant for exactly massless quarks, and then showed that $D$ represented the effects of instantons correctly, by robust zeros. Other nice features of the $D$ were also pointed out. Obviously, the spectrum of $D$ is concentrated on the circle $1 + e^{i\theta}$, $\theta \in [0, 2\pi]$ in the complex plane. In odd Euclidean dimensions this property also holds and is instrumental in checking that the correct global anomalies are reproduced on the lattice [2].

Here (see also [3]) I wish to add the rather trivial observation that, in view of

$$\gamma_5 X \gamma_5 = X^\dagger,$$ 

(5)

we also have

$$\gamma_5 \frac{1}{1 + V} \gamma_5 = \frac{1}{1 + \gamma_5 V \gamma_5} = \frac{1}{1 + V^\dagger} = \frac{V}{1 + V} = 1 - \frac{1}{1 + V}.$$ 

(6)
This means
\[ \gamma_5 D^{-1} + D^{-1} \gamma_5 = \gamma_5, \]
(7)
an equation Ginsparg and Wilson first wrote down many years ago [4], as a way one
may represent exact masslessness on the lattice, while, at the same time, preserving the
continuum anomaly. In ref. [4] eq. (7) (actually a slight generalization of equation (7)
which is immaterial here) was derived as a “remnant of chiral symmetry on the lattice”
after blocking a chirally symmetric continuum theory with a necessarily chirality breaking
local renormalization group kernel, of the type studied thoroughly in ref. [5] for example.

As shown in ref. [1], D nicely reproduces instanton effects. Once it is understood
that D can be used to define topological charge, since the latter is an integer valued,
nonconstant function over the compact space of gauge configurations (we are assuming
a finite lattice size), we know that exceptional configurations invalidating some of the
definitions must exist. We identified them above as those configurations for which X
becomes non-invertible. Thus, what we had to designate as an “exceptional” configuration
turns out to be very close to the definition one adopts in lattice QCD with “ordinary”
Wilson fermions. The topological charge defined from D has been shown to produce
reasonable quantitative and qualitative results in [6].

A recent paper [7] presents a very implicit definition of another matrix D which also
obeys eq. (7), and has a similar spectrum in its simplest variant. Strictly speaking, this
matrix is infinite, but probably admits some truncations that would have no discernible
effect numerically. The matrix D in this paper and the related methods of extracting
a topological charge from lattice gauge field configurations mentioned above, have been
arrived at in the overlap framework, independently of ref. [7]. On the other hand, the au-
thors of ref. [7] appear oblivious of the overlap. It is therefore a quite amusing coincidence
that relatively similar solutions to the problem of putting strictly massless quarks on the
lattice have been arrived at from quite different starting points, independently. The result
of ref. [7] provides further support to the overlap, although it is unclear whether any such
support is still needed, given the impressive numerical results of [8]. (Actually, since full
implementation of the matrix D on the lattice would be expensive due to the square root
factor, one needs to truncate, essentially approximating the overlap. The truncation is
studied in refs. [9], but has been used in [8] before [9] appeared, since it was proposed
before, in [10], as an improvement over [11].)

The most obvious differences between the two ways of defining a D matrix is that the
one given here and in ref. [1] is explicit. The matrix elements of the D-matrix of ref. [7] are
defined implicitly as the solution of a nontrivial recursion relation, which, in turn, includes
internally a nontrivial minimization in the space of gauge fields. In practice it would seem easier to use the overlap (via the flow methods of [6]) for getting the topological charge. For many more uses of $D$ (more precisely, its truncation) I refer again to [8].

The claim of reference [7] about the absolute absence of exceptional configurations, in the sense that I use the term, cannot hold, since they are a logical necessity for any definition of topological charge on the lattice, as argued above. Let us consider the vicinity of an exceptional gauge configuration: Changing the background ever so slightly we can get topological charge zero or one and it is quite plausible that some of the zero modes we find are better viewed as lattice artifacts. The space of all continuum connections over a compact manifold is not connected while the replacement of this space in the lattice approximation to the manifold clearly is. There is no way around this and and the price to pay will always be in accepting the presence of some exceptional configurations.

I should add a word of caution here: Clearly, identity (7) would hold even had we picked the parameter $m$ in (1) positive. This would eliminate all exceptional configurations since one can easily prove (see first paper in [6]) that $\det X \neq 0$ for any gauge field. However, the new matrix $D$ does not describe massless quarks, and, if “asked” what the topology of the gauge background is, would always return zero for an answer.

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References: