The fractal structure of the universe: a new field theory approach, H.J. de Vega\textsuperscript{(a)}, N. Sánchez\textsuperscript{(b)} and F. Combes\textsuperscript{(b)}

(a) Laboratoire de Physique Théorique et Hautes Energies, Université Paris VI, UA 280, Tour 16, 1er étage, 4, Place Jussieu 75252 Paris, Cedex 05, France

(b) Observatoire de Paris, DEMIRM, 61, Avenue de l’Observatoire, 75014 Paris, France

Abstract

While the universe becomes more and more homogeneous at large scales, statistical analysis of galaxy catalogs have revealed a fractal structure at small-scales ($\lambda < 100h^{-1}$ Mpc), with a fractal dimension $D = 1.5 - 2$ (Sylos Labini et al 1996). We study the thermodynamics of a self-gravitating system with the theory of critical phenomena and finite-size scaling and show that gravity provides a dynamical mechanism to produce this fractal structure. We develop a field theoretical approach to compute the galaxy distribution, assuming them to be in quasi-isothermal equilibrium. Only a limited, (although large), range of scales is involved, between a short-distance cut-off below which other physics intervene, and a large-distance cut-off, where the thermodynamic equilibrium is not satisfied. The galaxy ensemble can be considered at critical conditions, with large density fluctuations developing at any scale. From the theory of critical phenomena, we derive the two independent critical exponents $\nu$ and $\eta$ and predict the fractal dimension $D = 1/\nu$ to be either 1.585 or 2, depending on whether the long-range behaviour is governed by the Ising or the mean field fixed points, respectively. Both set of values are compatible with present observations. In addition, we predict the scaling behaviour of the gravitational potential to be $r^{-\frac{1}{2}(1+\eta)}$. That is, $r^{-0.5}$ for mean field or $r^{-0.519}$ for the Ising fixed point. The theory allows to compute the three and higher density correlators without any assumption or Ansatz. We find that the $N$-points density scales as $r_1^{(N-1)(D-3)}$, when $r_1 >> r_i$, $2 \leq i \leq N$. There are no free parameters in this theory.
1 Introduction

One obvious feature of galaxy and cluster distributions in the sky is their hierarchical property: galaxies gather in groups, that are embedded in clusters, then in superclusters, and so on. (Shapley 1934, Abell 1958). Moreover, galaxies and clusters appear to obey scaling properties, such as the power-law of the two point-correlation function:

$$\xi(r) \propto r^{-\gamma}$$

with the slope $\gamma$, the same for galaxies and clusters, of $\approx 1.7$ (e.g. Peebles, 1980, 1993). This scale-invariance has suggested very early the idea of fractal models for the clustering hierarchy of galaxies (de Vaucouleurs 1960, 1970; Mandelbrot 1975). Since then, many authors have shown that a fractal distribution indeed reproduces quite well the aspect of galaxy catalogs, for example by simulating a fractal and observing it, as with a telescope (Scott, Shane & Swanson, 1954; Soneira & Peebles 1978). Sometimes the analysis has been done in terms of a multi-fractal medium (Balian & Schaeffer 1989, Castagnoli & Provenzale 1991, Martinez et al 1993, Dubrulle & Lachieze-Rey 1994).

There is some ambiguity in the definition of the two-point correlation function $\xi(r)$ above, since it depends on the assumed scale beyond which the universe is homogeneous; indeed it includes a normalisation by the average density of the universe, which, if the homogeneity scale is not reached, depends on the size of the galaxy sample. Once $\xi(r)$ is defined, one can always determine a length $r_0$ where $\xi(r_0) = 1$ (Davis & Peebles 1983, Hamilton 1993). For galaxies, the most frequently reported value is $r_0 \approx 5h^{-1}$ Mpc (where $h = H_0/100$ km s$^{-1}$Mpc$^{-1}$), but it has been shown to increase with the distance limits of galaxy catalogs (Davis et al 1988). $r_0$ is called ‘correlation length’ in the galaxy literature. [The notion of correlation length $\xi_0$ is usually different in physics, where $\xi_0$ characterizes the exponential decay of correlations ($\sim e^{-r/\xi_0}$). For power decaying correlations, it is said that the correlation length is infinite].

The same problem occurs for the two-point correlation function of galaxy clusters; the corresponding $\xi(r)$ has the same power law as galaxies, their length $r_0$ has been reported to be about $r_0 \approx 25h^{-1}$ Mpc, and their correlation amplitude is therefore about 15 times higher than that of galaxies.
(Postman, Geller & Huchra 1986, Postman, Huchra & Geller 1992). The latter is difficult to understand, unless there is a considerable difference between galaxies belonging to clusters and field galaxies (or morphological segregation). The other obvious explanation is that the normalizing average density of the universe was then chosen lower.

This statistical analysis of the galaxy catalogs has been criticized by Pietronero (1987), Einasto (1989) and Coleman & Pietronero (1992), who stress the uncomfortable dependence of $\xi(r)$ and of the length $r_0$ upon the finite size of the catalogs, and on the a priori assumed value of the large-scale homogeneity cut-off. A way to circumvent these problems is to deal instead with the average density as a function of size (cf §2). It has been shown that the galaxy distribution behaves as a pure self-similar fractal over scales up to $\approx 100h^{-1}$ Mpc, the deepest scale to which the data are statistically robust (Sylos Labini et al 1996; Sylos Labini & Pietronero 1996). This is more consistent with the observation of contrasted large-scale structures, such as superclusters, large voids or great walls of galaxies of $\approx 200h^{-1}$ Mpc (de Lapparent et al 1986, Geller & Huchra 1989). After a proper statistical analysis of all available catalogs (CfA, SSRS, IRAS, APM, LEDA, etc.. for galaxies, and Abell and ACO for clusters) Pietronero et al (1997) state that the transition to homogeneity might not yet have been reached up to the deepest scales probed until now. At best, this point is quite controversial, and the large-scale homogeneity transition is not yet well known.

Isotropy and homogeneity are expected at very large scales from the Cosmological Principle (e.g. Peebles 1993). However, this does not imply local or mid-scale homogeneity (e.g. Mandelbrot 1982, Sylos Labini 1994): a fractal structure can be locally isotropic, but inhomogeneous. The main observational evidence in favor of the Cosmological Principle is the remarkable isotropy of the cosmic background radiation (e.g. Smoot et al 1992), that provides information about the Universe at the matter/radiation decoupling. There must therefore exist a transition between the small-scale fractality to large-scale homogeneity. This transition is certainly smooth, and might correspond to the transition from linear perturbations to the non-linear gravitational collapse of structures. The present catalogs do not yet see the transition since they do not look up sufficiently back in time. It can be noticed that some recent surveys begin to see a different power-law behavior at large scales ($\lambda \approx 200 - 400h^{-1}$ Mpc, e.g. Lin et al 1996).
There are several approaches to understand non-linear clustering, and therefore the distribution of galaxies, in an infinite gravitating system. Numerical simulations have been widely used, in the hope to trace back from the observations the initial mass spectrum of fluctuations, and to test postulated cosmologies such as CDM and related variants (cf Ostriker 1993). This approach has not yet yielded definite results, especially since the physics of the multiple-phase universe is not well known. Also numerical limitations (restricted dynamical range due to the softening and limited volume) have often masked the expected self-similar behavior (Colombi et al 1996). A second approach, which should work essentially in the linear (or weakly non-linear) regime, is to solve the BBGKY hierarchy through closure assumptions (Davis & Peebles 1977; Balian & Schaeffer 1989). The main assumption is that the \( N \)-points correlation functions are scale-invariant and behave as power-laws like is observed for the few-body correlation functions. Crucial to this approach is the determination of the void probability, which is a series expansion of the \( N \)-points correlation functions (White 1979). The hierarchical solutions found in this frame agree well with the simulations, and with the fractal structure of the universe at small-scales (Balian & Schaeffer 1988). A third approach is the thermodynamics of gravitating systems, developed by Saslaw & Hamilton (1984), which assumes quasi thermodynamic equilibrium. The latter is justified at the small-scales of non-linear clustering, since the expansion time-scale is slow with respect to local relaxation times. Indeed the main effect of expansion is to subtract the mean gravitational field, which is negligible for structures of mean densities several orders of magnitude above average. The predictions of the thermodynamical theory have been successfully compared with N-body simulations (Itoh et al 1993), but a special physical parameter (the ratio of gravitational correlation energy to thermal energy) had to be adjusted for a better fit (Bouchet et al 1991, Sheth & Saslaw 1996, Saslaw & Fang 1996).

We present in this article a new approach based on field theory and the renormalisation group to understand the clustering behaviour of a self-gravitating expanding universe. We also consider the thermodynamics properties of the system, assuming quasi-equilibrium for the range of scales concerned with the non-linear regime and virialisation. We find an exact mapping between the self-gravitating gas and a continuous field theory for a single scalar field with an exponential self-coupling. This allows us to use statistical field theory and the renormalisation group to determine the scal-
The small-scale fractal universe can be considered critical with large density fluctuations developing at any scale. We derive the corresponding critical exponents. They are very close to those measured on galaxy catalogs through statistical methods based on the average density as function of size; these methods reveal in particular a fractal dimension $D \approx 1.5 - 2$ (Di Nella et al 1996, Sylos Labini & Amendola 1996, Sylos Labini et al 1996). This fractal dimension is strikingly close to that observed for the interstellar medium or ISM (e.g. Larson 1981, Falgarone et al 1991). We show in the present paper that the theoretical framework based on self-gravity that we have developed for the ISM (de Vega, Sánchez & Combes 1996a,b, hereafter dVSC) is also the dynamical mechanism leading to the small scale fractal structure of the universe. This theory is powerfully predictive without any free parameter. It allows to compute the $N$-points density correlations without any extra assumption.

We first clarify the definition of the average density we use for a fractal medium in §2, present the dynamical equations of the comoving fractal in §3, and apply our field-theory approach in §4.

2 Correlation functions and mass density in a fractal

The use of the two point correlation function $\xi(r)$ widely spread in galaxy distributions studies, is based on the assumption that the Universe reaches homogeneity on a scale smaller than the sample size. It has been shown by Coleman, Pietronero & Sanders (1988) and Coleman & Pietronero (1992) that such an hypothesis could perturb significantly the results. The correlation function is defined as

$$\xi(r) = \frac{<n(r_i).n(r_i + r)>}{<n>^2} - 1$$

where $n(r)$ is the number density of galaxies, and $< ... >$ is the volume average (over $d^2r_i$). The length $r_0$ is defined by $\xi(r_0) = 1$. The function $\xi(r)$ has a power-law behaviour of slope $-\gamma$ for $r < r_0$, then it turns down to zero rather quickly at the statistical limit of the sample. This rapid fall leads to an over-estimate of the small-scale $\gamma$. Pietronero (1987) introduces the
conditional density
\[ \Gamma(r) = \frac{\langle n(r_i) n(r_i + r) \rangle}{\langle n \rangle} \]
which is the average density around an occupied point. For a fractal medium, where the mass depends on the size as
\[ M(r) \propto r^D \]
\( D \) being the fractal (Hausdorff) dimension, the conditional density behaves as
\[ \Gamma(r) \propto r^{D-3} \]
This is exactly the statistical analysis used for the interstellar clouds, since the ISM astronomers have not adopted from the start any large-scale homogeneity assumption (cf Pfenniger & Combes 1994).

The fact that for a fractal the correlation \( \xi(r) \) can be highly misleading is readily seen since
\[ \xi(r) = \frac{\Gamma(r)}{\langle n \rangle} - 1 \]
and for a fractal structure the average density of the sample \( \langle n \rangle \) is a decreasing function of the sample length scale. In the general use of \( \xi(r) \), \( \langle n \rangle \) is taken for a constant, and we can see that
\[ D = 3 - \gamma . \]
If for very small scales, both \( \xi(r) \) and \( \Gamma(r) \) have the same power-law behaviour, with the same slope \( -\gamma \), then the slope appears to steepen for \( \xi(r) \) when approaching the length \( r_0 \). This explains why with a correct statistical analysis (Di Nella et al 1996, Sylos Labini & Amendola 1996, Sylos Labini et al 1996), the actual \( \gamma \approx 1 - 1.5 \) is smaller than that obtained using \( \xi(r) \). This also explains why the amplitude of \( \xi(r) \) and \( r_0 \) increases with the sample size, and for clusters as well.

In the following, we adopt the framework of the fractal medium that we used for the ISM (dVSC), and will not consider any longer \( \xi(r) \).

3 Equations in the comoving frame

Let us consider the universe in expansion with the characteristic scale factor \( a(t) \). For the sake of simplicity, we modelise the galaxies by points of equal
masses $m$, although they have a mass spectrum (it may be responsible for a multi-fractal structure, see Sylos Labini & Pietronero 1996).

The present analysis can be generalised to galaxies of different masses following the lines of sec. IV in de Vega, Sánchez & Combes (1996b). We expect to come to this point in future work.

If the physical coordinates of the particles are $\vec{r}$, we can introduce the comoving coordinates $\vec{x}$ such that

$$\vec{r} = a(t) \vec{x}$$

The Lagrangian for a system of $N$ particles interacting only by their self-gravity can be written as

$$L_N = \sum_{i=1}^{N} \left[ \frac{m}{2} a(t)^2 \dot{\vec{x}}_i^2 - \frac{m}{a(t)} \phi(\vec{x}_i(t)) \right],$$

(1)

where $\phi(\vec{x})$ is the gravitational potential in the comoving frame, determined by the Poisson equation

$$\nabla^2 \phi(\vec{x}) = 4\pi G \rho(\vec{x},t),$$

(2)

and $\rho(\vec{x},t)$ is the mass density. For our system of point particles,

$$\rho(\vec{x},t) = m \sum_{i=1}^{N} \delta(\vec{x} - \vec{x}_i(t))$$

(3)

and therefore the solution of the Poisson equation takes the form

$$\phi(\vec{x}) = -Gm \sum_{i=1}^{N} \frac{1}{|\vec{x} - \vec{x}_i(t)|}.$$  

(4)

The canonical momenta and Hamiltonian of the system are

$$\vec{p}_i = m a(t)^2 \dot{\vec{x}}_i$$

$$H_N = \sum_{i=1}^{N} \left[ \frac{1}{2ma(t)^2} \vec{p}_i^2 + \frac{m}{a(t)} \phi(\vec{x}_i(t)) \right]$$

$$= \frac{1}{2ma(t)^2} \sum_{i=1}^{N} \vec{p}_i^2 - \frac{Gm^2}{a(t)} \sum_{1 \leq i < j \leq N} \frac{1}{|\vec{x}_i - \vec{x}_j|}$$

(5)
We see that the \( N \)-particle Hamiltonian in cosmological spacetime eq.(5) can be obtained from the Minkowski Hamiltonian \([a(t) = 1]\) by making the replacements
\[ m \rightarrow m a(t)^2, \quad G \rightarrow G a(t)^{-5}. \] 

(6)

As a first approximation, we shall assume in the following that the characteristic time of the particle motions under the gravitational self-interaction are shorter than the time variation of \( a(t) \). We can then consider that this system of self-gravitating particles is at any time in approximate thermal equilibrium. This hypothesis is true of course for structures that have already decoupled from the expansion, and are truly self-gravitating and virialised. It could be also valid for the whole non-linear regime of the gravitational collapse. As for the linear regime, we know already that the primordial fluctuations are not forgotten in the large-scale structures, and therefore the resulting correlations will depend on initial conditions, and not be entirely determined by self-gravity.

The above assumption introduces a natural upper limit in the scales concerned by the theory developed below. Similarly to the case of the interstellar medium, the fractal structure considered is bounded by a short distance cut-off and by a large-scale limit as well (dVSC).

The short distance cut-off corresponds to the appearance of other physics at short scale, essentially dissipative, which we do not need to introduce. In addition, the short distance cut-off avoids the gravo-thermal catastrophe. For the ISM, the cut-off was naturally the size of the smaller fragments, of the order of the Jeans length. Here the cut-off corresponds also to the size of the ‘particles’ considered, i.e. the galaxy size, below which another physics steps in, related to stellar formation and radiation.

The present treatment can be generalized when thermal equilibrium only holds region by region (de Vega, Sánchez & Combes, in preparation). In such case we are lead to a quenched average over the temperature and we argue that the scaling properties are the same as in exact thermal equilibrium provided the temperature variations are smooth over the structure at the considered scale.

The fact that in the catalogs, we are observing in projection large-scale structures at different epochs, with different values of the scale factor \( a(t) \), could slightly modify the fractal dimension. Even though fractal structures are self-similar, and scale-independent, the largest scales are systematically
observed at a younger epoch where the contrast has not grown up as high as today. This evolution effect however should be significant only at high redshift (> 1), and the present catalogs are not yet statistically robust so far back in time (the average redshift of optical catalogs is about 0.1).

4 Application of renormalization group theory

As in all scale-independent problems, where the fluctuations cannot be represented by analytical functions, the renormalization group theory developed in the 1970’s for the study of critical phenomena, appears here perfectly adapted (e.g. Wilson & Kogut 1974). We can consider the fractal structure of the Universe as the critical state of a system, where fluctuations develop at any scale, with a very large correlation length (asymptotically infinite). The fluctuations that are distributed as a fractal of dimension $D$ are the large-scale structures of the universe (cf Totsuji & Kihara 1969).

We have recently begun to tackle, with the tools of statistical field theory, the study of an $N$-body system only interacting through their own self-gravity (dVSC). We have found an exact mathematical connection between this statistical system and a scalar field with exponential self-coupling. We then used for it the powerful methods of field theory (e.g. Itzykson & Drouffe 1989, Parisi 1988, Zinn-Justin 1989). Using the renormalisation group, the critical behaviour of this gravitational system has been described and its critical exponents identified. This theory explains both the origin of the fractal structure, and predicts its fractal dimension $D$. This has been successfully applied to the interstellar medium (dVSC). Another approach has been proposed for galaxy correlations (not for ISM) (Hochberg & Pérez Mercader 1996), but it yields different critical exponents.

Let us apply the theory to the system of galaxy points, already defined in the previous section. Since they are considered in approximate thermal equilibrium, we will use the grand canonical ensemble, that also allows a variable number of particles. The grand partition function of the system can be written as
\[ Z = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int \cdots \int \prod_{l=1}^{N} \frac{d^3p_l d^3x_l}{(2\pi)^3} e^{-\beta H_N} \] (7)

where \( z \) is the fugacity, which for an ideal gas of number density \( \rho_0 \) is:
\[ z = \rho_0 \left( \frac{\hbar^2}{2\pi \hbar m kT} \right)^{3/2} \] (h is the Planck constant).

In dVSC we found a functional integral representation for the grand partition function
\[ Z = \int \int D\phi e^{-S[\phi(.)]} \] (8)
i.e., \( Z \) can be written as the partition function for a single scalar field \( \phi(\vec{x}) \) with local action
\[ S[\phi(.)] \equiv \frac{1}{T_{eff}} \int d^3x \left[ \frac{1}{2} (\nabla \phi)^2 - \mu^2 e^{\phi(\vec{r})} \right] . \] (9)

where
\[ \mu^2 = \frac{\pi^{5/2}}{\hbar^3} z G (2m)^{7/2} \sqrt{kT}, \quad T_{eff} = 4\pi \frac{G m^2}{kT} \] (10)

In the \( \phi \)-field representation, the mass density eq.(3) is expressed as
\[ \rho(\vec{x}) = -\frac{m}{T_{eff}} \nabla^2 \phi(\vec{r}) = \frac{m \mu^2}{T_{eff}} e^{\phi(\vec{r})} . \] (11)

and the mass contained in a region of size \( R \) is
\[ M(R) = \frac{m \mu^2}{T_{eff}} \int_{R} e^{\phi(\vec{x})} d^3x . \] (12)

The mass parameter \( \mu \) coincides at the tree level with the inverse of the Jeans length \( d_J \) (dVSC)
\[ \mu = \sqrt{\frac{12}{\pi}} \frac{1}{d_J} . \] (13)

The functional representation for the grand partition function can be easily generalized for an arbitrary scale factor \( a(t) \). After the changes specified above in eq.(6), the local action becomes
\[ S[\phi(.)] \equiv \frac{a(t)}{T_{eff}} \int d^3x \left[ \frac{1}{2} (\nabla \phi)^2 - \mu^2 a(t)^2 e^{\phi(\vec{r})} \right] \] (14)
Notice that all quantities depend on time through the scale factor $a(t)$ only. There is no integration over $t$.

The mass parameter $\mu$ in the $\phi$-theory gets effectively multiplied by the scale factor $a(t)$. Since the Jeans length $d_J \simeq \mu^{-1}$ according to eq.(13), in comoving coordinates $d_J$ effectively becomes

$$d_J = \sqrt{\frac{12}{\pi}} \frac{1}{\mu a(t)} ,$$

as one could have expected.

On the other hand, the dimensionless coupling constant

$$g^2 = \mu T_{\text{eff}}$$

is unchanged by the replacements of eq.(6).

Therefore, for any fixed time $t$ we find the same scaling behaviour, after making the replacement

$$\mu \rightarrow \mu a(t)$$

and keeping the coupling $g$ unchanged.

### 4.1 Scaling behaviour

As is well known in the theory of critical phenomena (e.g. Wilson 1975, 1983; Domb & Green 1976), physical quantities for infinite volume systems diverge at the critical point as $\Lambda$ to a negative power, where $\Lambda$ measures the distance to the critical point. The correlation length $\xi_0$ diverges as

$$\xi_0(\Lambda) \sim \Lambda^{-\nu} ,$$

and the specific heat $C$ behaves as

$$C \sim \Lambda^{-\alpha} .$$

(15)

The critical exponents $\nu$ and $\alpha$ are pure numbers that depend only on the universality class of the problem considered (e.g. Binney et al 1992).

For a finite volume system, all physical quantities are finite at the critical point. And for a system whose size $R$ is large, the physical quantities take large, but finite, values at the critical point. Thus, for large critical systems,
one can use asymptotically the infinite volume theory. In particular, the

correlation length $\xi_0$ can be identified with the relevant physical scale $R$:

$$\xi_0 \sim R.$$ 

This implies that

$$\Lambda \sim R^{-1/\nu}.$$ 

(16)

These concepts apply to the gravitational $\phi$-theory since it exhibits scaling in

a finite volume $\sim R^3$ (dVSC). Scaling behaviour was found for a continuoum

set of values of $\mu^2$ and $T_{\text{eff}}$.

We have previously shown (dVSC) that it is possible to identify

$$\Lambda \equiv \frac{\mu^2}{T_{\text{eff}}} = \frac{z}{\hbar^3} (2\pi m k T)^{3/2}.$$ 

(17)

Notice that the critical point $\Lambda = 0$, corresponds to zero fugacity. Then, the

partition function in the scaling regime can be written as

$$Z(\Lambda) = \int \int D\phi e^{-S^* + \Lambda \int d^3 x e^{\phi(x)}},$$ 

(18)

where $S^*$ stands for the action at the critical point.

We define the renormalized mass density as

$$m \rho(\vec{x})_{\text{ren}} \equiv m e^{\phi(\vec{x})}$$ 

(19)

and we identify it with the energy density in the renormalization group (also

called the ‘thermal perturbation operator’).

Since the $\phi$-theory exhibits scaling (dVSC), the non-analytical part of the

free energy is

$$\log Z(\Lambda) \propto \Lambda^{2-\alpha},$$

so that its second derivative is $C \sim \Lambda^{-\alpha}$. Calculating the logarithmic derivative of $Z(\Lambda)$ with respect to $\Lambda$ from eqs.(18), using the standard relation between critical exponents in a three dimensional space $\alpha = 2 - 3 \nu$, and eqs.

(19) and (12), we find that the mass fluctuations inside a volume of radius $R$

$$(\Delta M(R))^2 \equiv <M^2> - <M>^2,$$

will scale as

$$\Delta M(R) \sim R^{1/\nu}.$$ 

(20)

The scaling exponent $\nu$ can be identified with the inverse Haussdorf (frac-
tal) dimension $D$ of the system

$$D = \frac{1}{\nu}.$$
4.2 Critical exponents

As usual in the theory of critical phenomena, there are only two independent critical exponents. All exponents can be expressed in terms of two of them: for instance the fractal dimension $D = 1/\nu$, and the independent exponent $\eta$, which usually governs the spin-spin correlation functions. The exponent $\eta$ appears here in the $\phi$-field correlator (dVSC), describing the gravitational potential, that scales as

$$<\phi(\vec{r})> \sim r^{-\frac{1}{2}(1+\eta)}$$

The values of the critical exponents depend on the fixed point that governs the long range behaviour of the system.

The renormalization group approach applied to a single component scalar field in three space dimensions shows the presence of only two fixed points: the mean field point and the Ising fixed point. The scaling exponents associated to the Ising fixed point are $\nu_{Ising} = 0.631...$, $D_{Ising} = 1.585...$, and $\eta_{Ising} = 0.037...$. The mean field value for the critical exponents are $\nu_{meanf} = \frac{1}{2}$, $D_{meanf} = 2$ and $\eta_{meanf} = 0$.

The value of the dimensionless coupling constant $g^2 = \mu T_{eff}$ should decide whether the fixed point chosen by the system is the mean field (weak coupling) or the Ising one (strong coupling). At the tree level, we estimate $g \approx \frac{5}{\sqrt{N}}$, where $N$ is the number of points in a Jeans volume $d_3^2$. The coupling constant appears then of the order of 1, and we cannot settle this question without effective computations of the renormalisation group equations. At this point, the predicted fractal dimension $D$ should be between 1.585 and 2.

4.3 Three point and higher correlations

Our approach allows to compute higher order correlators without any extra assumption (de Vega, Sánchez & Combes, in preparation).

The two and three point densities,

$$D(\vec{r}_1, \vec{r}_2) \equiv <n(\vec{r}_1)n(\vec{r}_2)>$$

$$D(\vec{r}_1, \vec{r}_2, \vec{r}_3) \equiv <n(\vec{r}_1)n(\vec{r}_2)n(\vec{r}_3)>,$$  \hspace{1cm} (21)
can be expressed as follows in terms of the correlation functions:

\[ D(\vec{r}_1, \vec{r}_2) = n_1 n_2 + C_{12} \tag{22} \]

\[ D(\vec{r}_1, \vec{r}_2, \vec{r}_3) = n_1 n_2 n_3 + n_1 C_{23} + n_3 C_{12} + n_2 C_{13} + C_{123} . \]

Here,

\[ n_i \equiv < n(\vec{r}_i) > , \quad i = 1, 2, 3 , \]

and \( C_{ij} \) and \( C_{ijk} \) are the two and three point correlation functions, respectively,

\[ C_{ij} \equiv C(\vec{r}_i, \vec{r}_j) \]

\[ C_{ijk} \equiv C(\vec{r}_i, \vec{r}_j, \vec{r}_k) \]

The behaviour of \( n_i \), \( C_{ij} \) and \( C_{ijk} \) in the scaling regime follow from the renormalisation group equations at criticality (de Vega, Sánchez & Combes, in preparation). If we do not impose homogeneity at all scales, we find,

\[ < n(\vec{r}) > \simeq A r^{D-3} , \]

\[ C(\vec{r}_1, \vec{r}_2) \overset{r_1 \gg r_2}{\simeq} B r_1^{2(D-3)} , \]

\[ C(\vec{r}_1, \vec{r}_2, \vec{r}_3) \overset{r_1 \gg r_2, r_3}{\simeq} C r_1^{3(D-3)} \]

where \( A, B \) and \( C \) are constants and \( D = 1/\nu \).

We can now derive the three point density behaviour when one point, say \( \vec{r}_1 \), is far away from the other two. We find from eqs.(22) and (23),

\[ D(\vec{r}_1, \vec{r}_2) \overset{r_1 \gg r_2}{\simeq} A r_1^{D-3} n_2 + B r_1^{2(D-3)} , \]

\[ D(\vec{r}_1, \vec{r}_2, \vec{r}_3) \overset{r_1 \gg r_2, r_3}{\simeq} A r_1^{D-3} (n_2 n_3 + C_{23}) + B r_1^{2(D-3)} (n_2 + n_3) + C r_1^{3(D-3)} \]

\[ + \]

Notice that this expression is dominated by the first term since \( D - 3 < 0 \).

Higher point distributions can be treated analogously in our approach. We find that the dominant behaviour in the \( N \)-points density is

\[ C(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) \overset{r_1 \gg r_i}{\simeq} \sum_{2 \leq i \leq N} r_i^{N(D-3)} \]

\[ \overset{r_1}{\simeq} \]

\[ 14 \]
Notice that when homogeneity is assumed to hold over all scales, the critical behaviour of the \( N \)-point correlation function involves a factor \( r_1^{(N-1)(D-3)} \), (Itzykson & Drouffe, 1989).

Eqs.\((24-25)\) are qualitatively similar, although not identical, to the behaviour inferred assuming the factorized hierarchical Ansatz \((fhA)\), (Balian & Schaeffer 1989). That is,

\[
D(\vec{r}_1, \vec{r}_2)_{fhA} = \bar{n}^2 \left( 1 + b r_{12}^{D-3} \right)^{r_1 \gg r_2} \bar{n}^2 \left( 1 + r_1^{D-3} \right)
\]

\[\Rightarrow r_{12} \equiv |\vec{r}_1 - \vec{r}_2|\) and so on. \( b \) and \( Q_3 \) are constants. Notice that in the factorized hierarchical Ansatz, the fractal dimension \( D \) is not predicted but it is a free parameter.

We see that the dominant behaviours in eqs.\((24)\) and \((26)\) are similar in case the scaling exponents \( D - 3 \) are the same.

5 Conclusions

The statistical analysis of the most recent galaxy catalogs, without the assumption of homogeneity at a scale smaller than the catalog depth, has determined that the universe has a fractal structure at least up to \( \approx 100h^{-1} \) Mpc (Sylos Labini et al 1996). The analysis in terms of conditional density has revealed that the fractal dimension is between \( D = 1.5 \) and \( 2 \) (Di Nella et al 1996, Sylos Labini & Amendola 1996). We apply a theory that we have developed to explain the fractal structure of the interstellar medium (dVSC), which has the same dimension \( D \). The physics is based on the self-gravitating interaction of an ensemble of particles, over scales limited both at short and large distances. The short-distance cut-off is brought by other physical processes including dissipation. The long-range limit is fixed by the expansion time-scale. In-between, the system is assumed in approximate
thermal equilibrium. The dynamical range of scales involved in this thermodynamic quasi-equilibrium is at present limited to 3-4 orders of magnitude, but will increase with time.

The critical exponents found in the theory do not depend on the conditions at the cut-off, which determine only the amplitudes. The theory is based on the statistical study of the gravitational field: it is shown that the partition function of the N-body ensemble is equivalent to the partition function of a single scalar field, with a local action. This allows to use field theory methods and the renormalisation group to find the scaling behaviour. We find scaling behaviour for a full range of temperatures and couplings. The theory then predicts for the system a fractal dimension $D = 1.585$ for the Ising fixed point, or $D = 2$ in the case of the mean-field fixed point. Both are compatible with the available observations. The $N$-points density correlators are predicted to scale with exponent $(N - 1)(D - 3)$ when $r_1 \gg r_i$, $2 \leq i \leq N$. That is, $-(N - 1)$ for the mean field, or $-1.415 (N - 1)$ for the Ising point.

We predict in addition a critical exponent $-\frac{1}{2}(1 + \eta)$ for the gravitational potential: that is, $-0.500$ for mean field or $-0.519$ for the Ising fixed point.

References


[34] Ostriker J.P.: 1993, ARAA 31, 689


