CM-P00056801

THEORY OF HIGH-ENERGY SCATTERING AND MULTIPLE PRODUCTION ⁺)D. Amati, S. Fubini ^{*)} and A. Stanghellini

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A B S T R A C T

This paper contains a general mathematical treatment of a model for high-energy interactions, in which the high-energy physical properties are determined as a function of only low energy quantities. It is shown that many general predictions can be derived which are independent of the detailed form of the low energy input. The results refer both to high-energy elastic scattering and multiple production. For the inelastic processes we obtain simple general predictions for measurable quantities such as multiplicities, inelasticity and spectra of secondaries. For the elastic scattering we find the characteristic Regge pole behaviour for all scattering amplitudes. The relation between the bound state problem and diffraction can therefore be understood in a relativistic model which contains in itself both phenomena. We finally discuss the possible corrections of the model by using approximatively the unitarity condition and we find indications of the possible existence of continuous power distribution, or equivalently, of cuts in the complex angular momentum variable.

⁺) To be published in Nuovo Cimento.

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PART A

THE PHYSICAL BASIS OF THE MULTIPERIPHERAL MODEL

I. INTRODUCTION

We shall discuss in this paper a theoretical model for the study of high-energy collisions (above the GeV region). The experimental information in this field - provided by big accelerators and, for even higher energies, by cosmic rays - is rich and many features are rather independent of the nature of the colliding particles (we restrict the discussion to strong interactions). Let us point out some characteristic behaviours :

- a) total cross-sections seem to be energy independent or, at least, to vary slowly with it;
- b) multiplicities of secondaries increase slowly with the energy;
- c) the major part of secondaries are pions (K mesons, hyperons and anti-baryons are in small number compared with pions);
- d) the mean transverse momenta of secondaries seem to be small (~ 300 MeV) and rather energy independent;
- e) the only important diffractive channel is the elastic one : the shape of the diffraction peak depends weakly on the energy. It shows perhaps a slight shrinking at increasing energies.

These features, as well as more detailed information on spectra, on the nature of secondaries, etc., form a bulk of information whose theoretical interpretation is necessary in order to understand the dynamics of high-energy processes.

Let us discuss briefly which theoretical models were proposed :

1) the statistical model

Different versions of this model were proposed : it consists essentially in saying that cross-sections for any particular process are mainly given by the corresponding phase space available. Experiments above the GeV region already show a backward-forward asymmetry of secondaries that is in disagreement with the statistical models. Besides, it would be difficult to conciliate the energy dependence of phase-space with properties a), b), c) and d) listed above.

2) the peripheral model

This model gives particularly good results in the 1-3 GeV region ¹⁾ and is very useful at higher energies in studying some particular kind of events. It is, however, not adequate in order to study the main average features of high-energy physics ²⁾ like total cross-sections or diffraction scattering.

For example, constant total cross-sections are inconsistent with the peripheral model. In a preceding paper ³⁾ we have shown that a dispersive approach (essentially a strip approximation to the Mandelstam representation) gives a theoretical justification to the peripheral model if the energy is not too high. At higher energies it suggested, however, a different model that we shall call, hereafter, multiperipheral ⁴⁾. Such a model has a very simple physical meaning and can easily be understood and used without needing any particular knowledge of dispersion relations. Thus, we think it is useful to restate the results obtained in "I", with a more simple language, as the consistent generalization to high energy of the peripheral ideas.

Before coming down to details of the proposed model, we want to stress that it gives definite predictions for both the fundamental high-energy phenomena, i.e., the multiple production and the diffractive scattering. In order to understand better this fact, let us recall that already in optics any absorption is

accompanied by a shadow scattering. This shadow scattering is a typical coherent effect, so that in order to calculate it one must know the absorptive amplitudes and not only their modulus squared (absorptive cross-sections). This is better illustrated by the following equations :

$$\sigma_{tot} = 2g \sqrt{s} A(s,0) \propto \sum_n |\langle n | T | i \rangle|^2 \propto \text{Im} \langle i | T | i \rangle \quad (I.1)$$

$$d\sigma_{el} = \frac{1}{\pi s} \left(\frac{A(s,t)}{8g} \right)^2 \propto \left| \sum_n \langle f | T^\dagger | n \rangle \langle n | T | i \rangle \right|^2 \propto \left| \text{Im} \langle f | T | i \rangle \right|^2 \quad (I.2)$$

where $\langle n | T | i \rangle$ is the amplitude for producing n particles. Our model gives a general expression for this amplitude and, therefore, allows to compute both σ_{tot} and $d\sigma_{el}$.

As already pointed out, high-energy physics shows very regular and simple features. In this paper we shall show that a smooth behaviour is indeed predicted as the result of an averaging effect over the sum of very complicated graphs. The main idea is to reduce the sum over all individual multiperipheral effects to the solution of linear integral equations. The equations become particularly simple in the high-energy limit, allowing therefore for simple solutions. This reflects the fact that the sum over many complicated processes can indeed become rather simple.

This paper is subdivided into four parts. We shall conclude Part A by restating and discussing critically our model. In Part B we shall deal with the problem of multiple production. First of all we shall analyze the structure of the single multiperipheral graphs, summing afterwards over them by means of an integral equation. The solution of this equation will give a simple form for the total cross-section showing a power dependence on the energy. The exponent will be related only to low-energy parameters. We shall then use that solution in order to evaluate all the main features of amplitude production, such as multiplicities and different spectra of secondaries.

In Part C we shall study the elastic scattering process. We shall reduce the problem to a linear integral equation for the absorptive amplitude. The asymptotic solution of such an equation shows again the characteristic energy power behaviour where, this time, the exponent depends explicitly on the transfer momentum. The calculation of the real part of the scattering amplitude through a fixed momentum transfer dispersion relation, gives for the whole elastic amplitude the characteristic Regge pole behaviour ⁵⁾. The relation between the asymptotic behaviour of elastic scattering and the bound state problem is then stated in an even clearer way by understanding the intimate relation between our integral equation and the Bethe-Salpeter equation in the ladder approximation. In Part D, we shall show how our model - as well as any reasonable relativistic theory where production processes can be present - predicts not only the Regge poles, but also other contributions equivalent to cuts in the angular momentum variable. The contribution of such cuts is then discussed and it is shown to be indeed important.

II. THE MULTIPERIPHERAL MODEL

The basic idea of the peripheral model is that the one-pion exchange amplitudes [see Fig. 1] dominate the high-energy processes. These amplitudes can be written formally

$$T_{xy}(p_x, p_y; q) = \frac{T_{x\pi}(p_x, q; k_{x_i}) T_{y\pi}(p_y, q; k_{y_i})}{q^2 - \mu^2} \quad (\text{II.1})$$

Fig. 1

where $T_{x\pi}(T_{y\pi})$ is the amplitude describing the reaction between the x (y) particle and the virtual pion, giving system X (Y) as final state (k_{xi} indicate the momenta of the i particle in the X system). Let us call $k_x = \sum_i k_{xi}$ and define the following variable

$$s = (p_x + p_y)^2 = (k_x + k_y)^2$$

$$s_x = (p_x + q)^2 = k_x^2$$

$$s_y = (p_y - q)^2 = k_y^2$$

$$\sqrt{s} \geq \sqrt{s_x} + \sqrt{s_y}$$

where s is the square energy in the general centre-of-mass system, s_x and s_y the square energy in the X and Y c.m.s. At intermediate incident energies $\sqrt{s}$, when either s_x and s_y are small due to the limited extension of the phase space, the peripheral model works rather well if one neglects the q^2 (mass) dependence of the low-energy scattering amplitudes. We wish to show that, by making use of the following two hypotheses which are the basic assumption of the peripheral model, i.e.,

- the dominance of the one-pion exchange at high energy;
- the negligible dependence of the low-energy amplitudes on the mass of the virtual pion;

one is led to a well-defined set of amplitudes for inelastic processes. Let us consider the graph of Fig. 1.

Several kinematical situations will appear at a sufficiently high energy $\sqrt{s}$; the two groups of particles X and Y may have low or high energy in their centre-of-mass system. If s_x and/or s_y are not small in their turn, we shall extract the one-pion exchange contribution from the corresponding amplitudes $T_{X\pi}$ and/or $T_{Y\pi}$. Energy conservation forces $\sqrt{s_1}$ and $\sqrt{s_2}$ (the energy in c.m.s. of the two sub-processes which appear at the end of the virtual pion) to be smaller than the initial energy $\sqrt{s_x}$ and/or $\sqrt{s_y}$.

$$\sqrt{s_x} \geq \sqrt{s_1} + \sqrt{s_2}$$

Therefore any subdivision of an amplitude, using the one-pion exchange for it, degrades the energy of each group of particles in the final state. Repeating the same procedure for all the high-energy amplitudes, we shall arrive at a situation in which we have a chain of low-energy amplitudes linked by virtual pions, as shown in Fig. 2.

The number of low-energy vertices is evidently limited by the incident energy and they increase for increasing energy.

At this stage we can use the second hypothesis, i.e., neglect the pion mass dependence of the low-energy amplitudes. The neglect of the mass dependence of the low-energy absorptive amplitude is essentially due to the small range of the low-energy interaction. It is, however, clear that the high-energy amplitude will still have an important dependence on the mass of the incoming particle, due simply to the dependence of the phase space on such a mass. This dependence, which will be shown to be physically meaningful, will be one of the results of our paper. This is one of the reasons why at high energy the simple peripheral formula is not adequate. The matrix element corresponding to the n^{th} order peripheral diagram becomes

$$T(p_x p_y; k_i) = \frac{T^R(p_x q_1; k_{0i}) T^R(q_1 q_2; k_{1i}) \cdots T^R(q_n p_y; k_{ni})}{(q_1^2 - \mu^2) (q_2^2 - \mu^2) \cdots (q_n^2 - \mu^2)} \quad (\text{II.2})$$

T^R are low-energy amplitudes ^{*)} and are taken for $q_i^2 = \mu^2$, i.e., are the true low-energy scattering amplitudes. In this manner we have a definite and calculable contribution for any high-energy inelastic process. In particular, we can calculate the total cross-section

$$\sigma_{\text{tot}}(s) = \sum_n \sigma_n(s)$$

where σ_n is the contribution of the n^{th} peripheral process that is given by

^{*)} The index R stands for resonance in the sense that the low-energy region is defined (as in paper "I") to be the one dominated by the few-channels physics, in contraposition to the many-channels physics (multiple production and diffraction).

$$\sigma_n(s) = \frac{1}{29\sqrt{s}(8\pi^4)^n} \iint \dots \int \frac{d^4 q_1 d^4 q_2 \dots d^4 q_n ds_0 ds_1 \dots ds_n}{(q_1^2 - \mu^2)^2 (q_2^2 - \mu^2)^2 \dots (q_n^2 - \mu^2)} \times$$

$$\times \delta(k_0^2 - s_0) 2|\vec{k}_0| \sqrt{s_0} \sigma(s_0) \dots \delta(k_n^2 - s_n) 2|\vec{k}_n| \sqrt{s_n} \sigma(s_n) \quad (\text{II.3})$$

The n^{th} contribution to $A(s,t) = \sum_n A_n(s,t)$, from which the elastic diffractive cross-section can be calculated using (I.1), is given by

$$A_n(s,t) = \frac{1}{(8\pi^4)^n} \iint \dots \int \frac{d^4 q_1 d^4 q_2 \dots d^4 q_n ds_0 ds_1 \dots ds_n}{(q_1^2 - \mu^2) [(q_1 - \Delta)^2 - \mu^2] (q_2^2 - \mu^2) [(q_2 - \Delta)^2 - \mu^2] \dots (q_n^2 - \mu^2) [(q_n - \Delta)^2 - \mu^2]} \times$$

$$\times \delta(k_0^2 - s_0) A(s_0, t) \delta(k_1^2 - s_1) A(s_1, t) \dots \delta(k_n^2 - s_n) A(s_n, t) \quad (\text{II.4})$$

where

$$\Delta = p'_x - p_x = p'_y - p_y$$

$$\Delta^2 = t$$

p'_x and p'_y are the momenta of the scattered particles.

$A_n(s,t)$ is the absorptive part of the graph in Fig. 3

In particular, in the GeV region where s_x and s_y in (II.1) are already small, we obtain the following expression for the inelastic cross-sections and the differential scattering cross-section :

$$\frac{d\sigma(s)}{ds_1 ds_2 dq^2} = \frac{1}{16\pi^2} \frac{k_1 \sqrt{s_1} T^R(s_1) k_2 \sqrt{s_2} T^R(s_2)}{p^2 s (q^2 - \mu^2)^2} \quad (\text{II.5a})$$

$$\frac{d\sigma_{el}}{dt} = \frac{1}{\pi s} \left| \frac{1}{32\pi^2 p} \int_{4\mu^2}^{\infty} dt_0 g(t_0, t) F\left(\frac{\sqrt{t}}{t_0}\right) \right|^2 \quad (\text{II.5b})$$

where

$$g(t_0, t) = \frac{1}{t_0} \iint ds_1 ds_2 A^R(s_1, t) A^R(s_2, t) T_H(s, s_1, s_2) \times \quad (\text{II.6})$$

$$\times \mathcal{J}(t_0 - t_0(s, s_1, s_2))$$

$$F(x) = \frac{1}{x \sqrt{1+x^2}} \log(x + \sqrt{1+x^2}) \quad (\text{II.7})$$

$t_0(s, s_1, s_2)$ is a function that is given in Eq. (8') of "I" for different reactions, and $T_H(s, s_1, s_2)$ is a hyperbolic triangle function that shall appear frequently in this paper and is defined by

$$T_H(abc) = \frac{\Theta(a^2 + b^2 + c^2 - 2ab - 2bc - 2ac)}{\sqrt{a^2 + b^2 + c^2 - 2ab - 2bc - 2ac}} \quad (\text{II.8})$$

Equations (II.5a) and (II.5b) are those of the simple peripheral model : i.e., the one proposed by Drell and Salzman⁶⁾, where just physical amplitudes are introduced in the two bubbles of Fig. 1. Eq. (II.5a) was previously given by Salzman⁶⁾; Eq. (II.5b) gives the shadow scattering corresponding to the absorption whose cross-section is given by (II.5a). We have derived our multiperipheral model as a generalization of the simple peripheral one whose soundness at not too high energies has been undoubtedly established. It is easy, however, to see that the results for the total cross-sections, as well as for diffractive scattering $\overline{\text{Eq. (II.3)}}$ and $\overline{\text{(II.4)}}$, are closely analogous to the corresponding equations of "I", $\overline{\text{(24)}}$ and $\overline{\text{(14)}}$ which were obtained by iterative solution of the strip approach to the Mandelstam representation. In fact, it is easy to see that the singularities of Eq. (II.4) in the strip are just the same as those predicted by the Mandelstam representation, and obtained in "I". The reason is clear : the multiperipheral model is based on the idea of considering only - but consistently - the more peripheral processes and this means, mathematically, to consider the most important singularities of the scattering amplitude that are nearest to the physical region (i.e., strip approximation).

Let us try to discuss critically our approach and to single out the major limitations thereof. These limitations are mainly due to the following reasons :

- a) the possible exchange of a heavier particle or system of particles at some stage in the chain of Fig. 2 or 3 (out of the strip effect in the language of "I");
- b) to having neglected the q^2 -dependence of low energy amplitudes $\overline{\text{Eq. (II.2)}}$. This is equivalent to the approximation done in Eq. (10) of "I", in which the t' -dependence of $A^R(s', t')$ was neglected at low energies. This approximation can be particularly questioned for low energy interactions in partial waves higher than the s-wave : the corresponding Legendre polynomial gives a dependence in q^2 that enhances the big values of q^2 . But this would be in contradiction with the spirit of our model, as well as, however, with the experimental information that is provided to us - both by inelastic scattering⁷⁾ and by the shape of the diffraction peak.

This means that it is an important point (appearing in many other fields of research) to understand how to handle vertices of graphs, as the one of Fig. 3, when a state with a higher angular momentum is emitted. In conclusion, we can say that our model - at least in the present form - is only really completely consistent and adequate when the low-energy interaction is in an s-wave state.

- c) an important limitation of the model previously discussed is due to the fact that we have neglected the elastic effects as compared to the inelastic ones. In fact, in order to construct our model, we have iterated the peripheral formula (II.1) that we know to be good at intermediate energies. This is true if we neglect diffractive scattering, which - being at least a 2π -exchange effect - is not automatically contained in such a formula. We know, however, that at 2 GeV, diffraction is indeed important : it is nearly a half of the inelastic (peripheral) contribution. At higher energies the more important corrections to our model will probably come not from pure diffraction, but from mixed diffractive and peripheral effects. One of these mixed effects is, for instance, the diffractive scattering from a pion of the nucleon cloud, considered by Drell and Hiida⁸⁾ whose graph can be represented as in Fig. 4 :

Fig. 4

This effect, even if less peripheral than the one considered before [Fig. 2], can nevertheless be important. And this continues to be true at even higher energies because a diffractive "unity" (or several ones) can be inserted at any stage in the chain of the multiperipheral graph of Fig. 2. We want to point out, however, that the study of such effects is not extraneous to our model, because the diffraction is itself given by the model, through unitarity [Eq. (1.2)]. This will be discussed further in Part D, where it will be shown how the model can be extended in order to include the effects discussed above. It is, however, clear that to be able to carry on this more ambitious approach, we must know the solutions of the simple multiperipheral model. In any case, many of the predictions of it, especially the trends and energy dependences, will probably not vary too much, due to the inclusion of diffractive effects in the multiperipheral chain.

The principal aim of the present investigation will be to understand which are the predictions of the multiperipheral model at rather high energies; this means at energies where many links of the multiperipheral chain can be present. We note now that all the chain, except - at the maximum - two links (i.e., the first and the last ones), is constituted by pions. Therefore, the fundamental study is the pion chain (i.e., multiperipheral for $\pi - \pi$ scattering); when this is understood, all other phenomena (πN , NN , KN , etc.) can be simply obtained by adding the corresponding extreme links. This operation consists of elementary calculations involving only low-energy quantities. Therefore, in the following, we shall study systematically $\pi - \pi$ scattering, and in Section IX we shall apply these results to other problems. We shall also forget, for the time being, the isotopic indices due to the pion charge. Inclusion of it is a simple matter, and we shall do it in Section IX.

PART B

ASYMPTOTIC PROPERTIES OF TOTAL CROSS-SECTIONS AND INELASTIC PROCESSES

III. BEHAVIOUR OF A MULTIPERIPHERAL CONTRIBUTION

We shall investigate in this Section the properties of each particular multiperipheral contribution. We shall find a recurrence relation that will allow us to calculate the $n+1$ contribution once the n^{th} is known. For the simplicity of future notation, let us consider the forward absorptive amplitudes which are simply related to total cross-sections through the optical theorem.

The n^{th} peripheral contribution to the absorptive forward amplitude for the scattering of two pions of four-momenta p and p' , respectively, is

$$A_n(pp') = \iint \cdots \int d^4 q_1 d^4 q_2 \cdots d^4 q_n G_n(q_1, q_2, \dots, q_n; p, p') \quad (\text{III.1})$$

where

$$G_n(q_1, q_2, \dots, q_n; p, p') = \frac{1}{(8\pi^4)^n} \iint \cdots \int ds_0 ds_1 \cdots ds_n \quad (\text{III.2})$$

$$\delta[(p+q_1)^2 - s_0] \delta[(q_2 - q_1)^2 - s_1] \cdots \delta[(q_n - p')^2 - s_n] A^R(s_0) A^R(s_1) \cdots A^R(s_n)$$

$A^R(s_0)$ being the resonant forward amplitude, related to the resonant cross-section of Section II by

$$A^R(s_i) = 2 |\vec{k}_i| \sqrt{s_i} \frac{\sigma_{\pi\pi}^R(s_i)}{8\pi^2}$$

(III.3)

From Eqs. (III.1) and (III.2), it is clear that $A_n(p, p')$ satisfies the recurrence relation

$$A_n(p, p') = \int ds_0 \frac{A^R(s_0)}{8\pi^4} \int d^4 p'' \frac{A_{n-1}(p, p'') \delta[(p' - p'')^2 - s_0]}{(p''^2 - \mu^2)^2} \quad (\text{III.4})$$

So if $A_{n-1}(p, p'')$ would be known, then A_n could be calculated from (III.4) using only low-energy physics data [concentrated in $A^R(s_0)$]. But we see that we would need to know $A_{n-1}(p, p'')$ for all values of p'' , this means also values that are not on the mass shell $p''^2 = \mu^2$. Therefore (III.4) gives us a recurrence relation for amplitudes outside the mass shell. Let us define as $p'^2 = -u$ the formally arbitrary mass of the incident pion (the target pion does not need to be taken out from the mass shell). $A(p, p')$ shall then depend on s and u , through the definition

$$A(p, p') = \iint A(s, u) \delta[s - (p + p')^2] \delta(u + p'^2) ds du \quad (\text{III.5})$$

The recurrence relation can then be written

$$A_n(s, u) = \int ds_0 \frac{A^R(s_0)}{8\pi^4} \iint ds' du' \frac{Q(s, u; s', u'; s_0) A_{n-1}(s', u')}{(u' + \mu^2)^2} \quad (\text{III.6})$$

where

$$Q(s,u;s',u';s_0) = \int d^4 p'' \delta[(p'-p)^2 - s_0] \delta[(p''+p)^2 - s'] \delta(p''^2 - u) \quad (\text{III.7})$$

We note that the procedure needed to obtain $A_n(s,u)$ from $A_{n-1}(s',u')$ is independent of n . The coupling between s,u and s',u' is only due to the kinematical limits of integration over phase space, as shown by Eq. (III.7). So it is clear that, starting from the knowledge of the zeroth iteration, i.e.,

$$A_0(s,u) \equiv A^R(s) \quad (\text{III.8})$$

it is possible, in principle, to obtain the properties of $A_n(s,u)$ by iterating n times the same operation, and therefore the cross-section due to the n^{th} peripherism, i.e.,

$$\sigma_n(s) = 2 |\vec{k}| \sqrt{s} A_n(s, -\mu^2) \quad (\text{III.9})$$

This is what we shall investigate here, leaving for the next Section the problem of summing over all multiperipherisms.

Performing the integration of (III.7), one finds the explicit expression for Q . In order to simplify the expression, it is better to introduce the variables

$$\begin{aligned}
 S &= s - u - \mu^2 \\
 S' &= s' + u' - \mu^2 \\
 S_0 &= s_0 + u + u'
 \end{aligned}
 \tag{III.10}$$

then

$$Q(s, u, s', u'; s_0) = \frac{\pi}{2\sqrt{S^2 + 4\mu^2 u}} \Theta(u S'^2 + u' S^2 - \mu^2 S_0^2 - S S' S_0 - 4\mu^2 u u') \times \Theta(\sqrt{S} - \sqrt{S'} - \sqrt{S_0})
 \tag{III.11}$$

where Θ is the usual step function whose role is to point out which is the phase space allowable. This means that for every value of s_0 and u , there is a region in the s', u' plane over which the integration in (III.6) takes place. The second Θ in (III.11) guarantees that of the two branches of the curve $s' = s'(u')$ coming from the vanishing of the argument of the first function, the lowest branch - which is actually plotted in Fig. 5 - is the actual limit of integration.

Fig. 5

For high values of the energy s (high with respect to μ^2 and u), we can write (III.6) as ^{*})

$$A_n(s, u) = \frac{1}{s} \int ds_0 \frac{A_n^R(s_0)}{16\pi^3} \int_{s'_{\min}}^{s'_{\max}} ds' \int_{u'_{\min}}^{u'_{\max}} du' \frac{A_{n-1}(s', u')}{(u' + \mu^2)^2} \quad (\text{III.12})$$

where

$$u'_{\max, \min} = \frac{s-s'}{2} \pm \sqrt{\left(\frac{s-s'}{2}\right)^2 - s's_0 - \frac{s'}{s} u(s-s')} \quad (\text{III.13})$$

and $s'_{\max}$ is given by the vanishing of the radical of Eq. (III.13), i.e., roughly

$$s'_{\max} = s - 2\sqrt{ss_0} \quad (\text{III.14})$$

In order to have a clearer insight into the situation we note that, except for the highest values of s' , the upper and lower boundaries given by (III.12) are well expressed by

$$\begin{aligned} u'_{\max} &\sim s - s' \\ u'_{\min} &\sim x \left(u + \frac{s_0}{1-x} \right) \end{aligned} \quad (\text{III.15})$$

where $x = \frac{s'}{s}$.

^{*}) In this formula, and in the following, when we study asymptotic behaviour, we shall neglect terms of the order $\frac{u}{s}$ or $\frac{\mu^2}{s}$ compared to 1, being careful, however, that s' can be nearly as big as s .

The first property to note from (III.12) and (III.13) is that $A_n(s, u)$ is a decreasing function of u . This property used for $A_{n-1}(s', u')$, together with the denominator $(u' + \mu^2)^2$, makes that the higher values of the integration in (III.12) count only very little.

By using A_0 , given by (III.8), the asymptotic behaviour of A_1 , obtained from (III.12), will be given by ^{*)}

$$A_1 \sim \frac{1}{s} \quad (\text{III.16})$$

We now discuss the high-energy limit of the n th order term. Let us introduce a value $\bar{s}$ such that for $s > \bar{s}$ the asymptotic expression of the $n-1$ term can be used. If one now splits in (III.12) the integral over s' into two parts separated by $\bar{s}$, we see that for very large s the lower integration contributes just a constant, whereas the second part can be evaluated using the asymptotic formula. This part of the integral behaves as $\log s$ and therefore determines the asymptotic behaviour, i.e.,

$$A_2 \sim \frac{\log s}{s}$$

Generalizing the preceding discussion, it turns out that for the purpose of obtaining the asymptotic behaviour of A_n we are allowed to use the asymptotic form for A_{n-1} in the recurrence equation (III.12). Iterating therefore n times this procedure, we get

$$A_n \sim \frac{(\log s)^{n-1}}{s} \quad (\text{III.17})$$

*) The arguments we shall develop apply equally well to a finite superposition of s_0 [i.e., $A^R(s_0) = 0$ for, $s_0 > s_{0\max}$] or to an infinite superposition of s_0 , if $A^R(s_0)$ goes to 0 faster than $\frac{1}{s_0}$.

On the basis of (III.17) we can understand, in this model, the basis of high-energy scattering. At low energy, we shall just have $A_0 = A^R$; by increasing the energy, the simple peripherism A_1 becomes important, but, after having reached a maximum, it will die out as $1/s$. At this stage already the double peripherism A_2 will be important; it will die out - further up - as $\frac{\log s}{s}$ [see Fig. 6] :

Fig. 6

At any energy, a particular peripherism will be predominant over the others, but nevertheless all the "tails" of the lower contributions will compete. The behaviour of $A(s) = \sum_n A_n(s)$ and therefore of the total cross-section

$$\sigma_{tot} \sim \frac{\sum A_n(s)}{s} \quad (\text{III.18})$$

will just depend on the coefficient with which every peripherism shall die out. This is the problem we shall treat in the next Section.

IV. THE ASYMPTOTIC BEHAVIOUR OF THE TOTAL CROSS-SECTION

In the last Section we have discussed the high-energy behaviour of each multiperipheral graph. In order to investigate the behaviour of the total cross-sections, we will study in this Section the sum of all multiperipheral contributions:

$$A(s, u) = \sum_n A_n(s, u) \quad (\text{IV.1})$$

where $A_n(s, u)$ are given by Eqs. (III.1), (III.2) and (III.5). It is clear that $A(s, u)$ satisfies the integral equation

$$A(s, u) = A^R(s) + \int ds_0 \frac{A^R(s_0)}{8\pi^4} \left(\int \frac{ds' du'}{(u' u')^2} Q(s, u; s', u'; s_0) A(s', u') \right) \quad (\text{IV.2})$$

where Q is given by Eqs. (III.7) or (III.11). In fact, solving (IV.2) by iteration, we reconstitute the series of Eq. (IV.1) in which A_{n-1} and A_n satisfy the recurrence relation (III.6). All the discussions of the preceding Section on the limits of integration due to the function Q apply clearly to Eq. (IV.2). Eq. (IV.2) can be interpreted as the calculation of the graphs of Fig. 7 :

Fig. 7

We now wish to derive the high-energy limit of A . We shall simplify the kernel in Eq. (IV.2), on the same lines as in the preceding Section, considering all masses (including the variables u, u') to be small as compared to s and s' . Therefore the region of integration is given by Eq. (III.15). Remembering, besides, that $A^R(s)$ is vanishingly small at high energies, the reduced asymptotic equation for $A(s, u)$ is

$$A(s, u) = \frac{1}{16\pi^3 s} \int_{s_0}^{s-2\sqrt{s}s_0} ds_0 A^R(s_0) \int_{s_{\min}}^{s'} ds' \int_{\frac{s'}{s}(u + \frac{s_0}{1-s/s})}^{s-s'} du' \frac{A(s', u')}{(u' + \mu')^2} \quad (\text{IV.3})$$

However, we must be careful in using this procedure since we now have to deal with an integral equation. We have thus to show that the solution of the reduced equation does indeed represent the high-energy behaviour of the true solution. This fact can be checked self-consistently once the solution of the reduced equation is obtained. Introducing it in the original integral equation (IV.2), it must happen that the variable region in which this equation was mistreated, in order to obtain the asymptotic equation (s' small and u' large), does not contribute to the asymptotic behaviour. We shall show later that this condition will be satisfied by our solution. This result is due to the fact that the energy range in s' , where the true integral equation is mistreated, remains constant for increasing s , whereas the whole integration range increases with it. In other words, one neglects surface effects - as compared to volume effects - when the volume increases. Moreover, it is to the presence of the denominators $(u' + \mu')^2$ that we owe the fact that the integrations in u' will be dominated by the small values of this variable. By virtue of this fact, the u' integration shall not depend on the upper limit. Therefore Eq. (IV.3) can be rewritten as

$$A(s, u) = \frac{1}{16\pi^3} \int ds_0 A^R(s_0) \int_0^\infty \frac{ds'}{s} \int_{\frac{s'}{s}(u + \frac{s_0}{1-s/s})}^\infty du' \frac{A(s', u')}{(u' + \mu')^2} \quad (\text{IV.3}')$$

The kernel of Eq. (IV.3') has the important property of being invariant under the group of multiplicative transformation

$$s \rightarrow c s \quad s' \rightarrow c s'.$$

This suggests that the solution of this equation can be expressed by means of the irreducible representations of the group given by s^α .

So we write

$$A(s, u) = s^\alpha \varphi_\alpha(u) \quad (\text{IV.4})$$

where $\varphi_\alpha(u)$ must satisfy the integral equation

$$\varphi_\alpha(u) = \frac{1}{16\pi^3} \int_0^1 ds_0 A(s_0) \int_0^1 x^\alpha dx \int_0^\infty du' \frac{\varphi_\alpha(u')}{(u' + \mu^2)^2} \frac{x(u + \frac{s_0}{1-x})}{(u' + \mu^2)^2} \quad (\text{IV.5})$$

or,

$$\varphi_\alpha(u) = \frac{1}{16\pi^3(1+\alpha)(2u)^{\alpha+1}} \int_0^1 ds_0 A(s_0) \int_0^\infty \frac{(s_0 + u + u' - \sqrt{(s_0 + u + u')^2 - 4uu'})^{\alpha+1}}{(u' + \mu^2)^2} \varphi_\alpha(u') du' \quad (\text{IV.6})$$

This equation is meaningful only if $\alpha > -1$. This is a Fredholm homogeneous integral equation, and is therefore a typical eigenvalue problem. This means that Eq. (IV.6) will have a solution only if $A^R(s_0)$ and α satisfy a definite relation. In other words, in order to have a solution, α will be determined as a functional of $A^R(s_0)$, i.e.,

$$\alpha = \alpha(A^R(s_0)) \quad (\text{IV.7})$$

As $A^R(s_0)$ is a low energy physics quantity, we have that the high-energy behaviour of total cross-sections will be controlled - in the multiperipheral model - by low energy physics.

Eqs. (IV.4), (IV.5) and (IV.7) are the fundamental points in our investigation. They are :

- 1) the sum of all multiperipheral graphs averages at high energy to a simple form , i.e., energy power behaviour.
- 2) the exponent of this power law is determined by the solution of a homogeneous linear integral equation of the Fredholm type, which gives rise to an eigenvalue problem.
- 3) the parameters appearing in such equations are only low energy physical quantities.
- 4) the dependence on the unphysical mass of the particle can be factorized and is given by the eigenfunctions of the integral equation. It will be shown that such a "wave function" has a physical meaning, since it will allow to compute different average properties of multiple production. The reduction of the asymptotic limit of amplitudes to the solution of an eigenvalue problem, is of the utmost importance, and will be generalized to non-forward scattering, allowing to understand and justify certain guesses on high-energy diffraction ⁹⁾.

The explicit solution of Eq. (IV.6) has been studied by Ceolin, Duimio, Stroffolini and one of us ¹⁰⁾ : we refer to their work for the details, summarizing here the properties of the solution and the method used to obtain it.

Using for ψ_α the spectral representation

$$\frac{\psi(u)}{(u+\mu^2)^2} \equiv \psi(u) = \int \frac{f(v)}{v+u} dv \quad (\text{IV.8})$$

from Eq. (IV.6) an integral equation for $g(v)$ can be obtained :

$$g(v) = N_1 \delta(v-\mu') - N_2 \delta(v-\mu') + \frac{1}{(v-\mu')^2} \int K(vv') g(v') dv'$$

where

$$K(vv') = \int \frac{A(s_0)}{16\pi^3} ds_0 \int_0^1 x^x dx \Theta\left(v - \frac{v'}{x} - \frac{s_0}{1-x}\right) \quad (\text{IV.9})$$

and

$$N_1 = \iint \frac{K(vv')}{v-\mu'} g(v') dv dv'$$

$$N_2 = \iint \frac{K(vv')}{(v-\mu')^2} g(v') dv dv'$$

plus the following conditions

$$\begin{aligned} \int g(v') dv' &= 0 \\ \int (v'-\mu') g(v') dv' &= 0 \end{aligned} \quad (\text{IV.10})$$

The interesting property of Eq. (IV.9) is that after every iteration the kernel is zero for a bigger region in the v, v' plane. This property implies that for every value of v only a finite number of iterations (smaller for smaller v) must be performed in order to obtain the exact solutions and the corresponding eigenvalue condition. The first iteration gives the eigenvalue equation

$$\int \frac{A(s_0)}{16\pi^3} ds_0 \int_0^1 \frac{x^x (1-x) dx}{s_0 x + \mu^2 (1-x)^2} = 1 \quad (\text{IV.11})$$

We omit here the expression for the corresponding eigenfunction, due to the fact that it is rather complicated. Notwithstanding, one sees from it that $\varphi_\alpha(u)$ is a decreasing function of u behaving for big u as

$$\varphi(u) \propto \frac{1}{(u+N)^{1+\alpha}} \quad (\text{IV.12})$$

where N is a functional of $A^R(s_0)$.

In the light of Eq. (IV.12), it is possible to see that our solution (IV.4) reproduces itself, for big s and small u , independently on the s' lower integration limit, and on the u' upper one. This confirms, as was discussed before, the asymptotic limit in the integral equation. Let us make a very rough evaluation of the order of magnitude of α from (IV.11). Unfortunately, we have not a very good knowledge of low energy $\pi\pi$ scattering, except for the rather well known $T=J=1$ resonance. For s_0 quite bigger than μ^2 , Eq. (IV.11) reduces approximately to

$$\alpha(\alpha+1) \approx \frac{1}{16\pi^3} \int \frac{A^R(s_0)}{s_0} ds_0 \approx \frac{1}{16\pi^3} \int \sigma_{\pi\pi}^R(s_0) ds_0 \quad (\text{IV.13})$$

The $T=J=1$ resonance contribution to the integral in (IV.13) would be ^{*)} 0.3. It is clear, from the fact that all other states must contribute to (IV.13) (in particular low-energy $\pi\pi$ s-wave), and from the fact that (IV.13) is itself a roughly approximated expression of the eigenvalue problem, that the only thing we can say from the previous evaluation is that our eigenvalue condition seems to be consistent with the experimental value $\alpha \sim 1$.

V. MULTIPLICITY OF SECONDARIES AND INELASTICITY

Let us find out now some immediate consequences of our solution (IV.5) for $A(s,u)$ from which we can understand some physical meaning of the functions $\varphi(u)$ and $\alpha(A^R(s_0))$. We shall first investigate a very interesting physical problem, i.e., the multiplicity of secondaries in high-energy collisions. Our multiplicity will really be concerned with the multiplicity of final (resonant) states, from which the multiplicity of pions can be obtained, by multiplying by the mean number of pions contained in the final "resonant" state.

*) In the actual evaluation, we have taken into account the pion charges, whose introduction is treated generally in Section IX.

Any n^{th} multiperipherism contributes with $n+1$ final states, this means that the average number of final states at an energy s shall be given by

$$\langle N \rangle = \frac{\sum (n+1) A_n(s, -\mu')}{\sum A_n(s, -\mu')} \quad (\text{V.1})$$

From Eqs. (III.1) and (III.2), we see that the explicit expression for A contains $n+1$ times the function A^R ; therefore it is clear that

$$(n+1) A_n(s, -\mu') = \int \frac{\int A_n(s, -\mu')}{\int A^R(s_0)} A^R(s_0) ds_0 \quad (\text{V.2})$$

and, therefore

$$\langle N \rangle = \frac{1}{A(s, -\mu')} \int \frac{\int A(s, -\mu')}{\int A^R(s_0)} A^R(s_0) ds_0 \quad (\text{V.3})$$

In order to find the asymptotic behaviour of $\langle N \rangle$, we use Eq. (IV.4) for A - remembering that both α and $\varphi_\alpha(u)$ depend on $A^R(s_0)$ - and we have

$$\langle N \rangle \sim \int A^R(s_0) \left(\frac{\int \alpha}{\int A^R(s_0)} \log s + \frac{\int \log \varphi_\alpha(-\mu')}{\int A^R(s_0)} \right) ds_0 \quad (\text{V.4})$$

This means that the multiplicity of secondaries in the multiperipheral model grows logarithmically with the energy, at high energy. Such a behaviour is quite compatible with experimental results. Another interesting property of Eq. (V.4) is that, in order to calculate the coefficient of $\log s$, we need only to calculate $d\alpha/dA^R$ and we do not need to use the explicit solution of the integral equation for $\varphi_\alpha(u)$.

This gives a physical meaning to the dependence on A^R of the eigenvalue α . In order to study a first example in which the eigenfunction $\varphi_\alpha(u)$ plays a role, let us consider the spectra of that final system which is directly connected with the incident particle (first system of the multiperipheral chain).

The spectra of the first system can be easily obtained by introducing a $\delta(p+q-k)$ in the expression (III.1) for each multiperipheral contribution and then summing over them, k being the four momenta of the system in question. It is then easy to obtain the following spectra :

$$dn(k) = \frac{A^R(k')}{8\pi^4} \frac{A(s', u')}{(u' + \mu^2)^2} d^4k \quad (V.5)$$

where

$$\begin{aligned} s' &= (p + p' - k)^2 \\ u' &= -(p' - k)^2 \end{aligned} \quad (V.6)$$

[see Fig. 8] :

Fig. 8

We shall now use the explicit expression for the spectra as given in (V.5) in order to calculate the average energy carried away by the primary, which is given by

$$\bar{E} = \frac{\int E_k d u(k)}{\int d u(k)} \quad (V.7)$$

where $E_k = \frac{p \cdot k}{\mu}$. Using Eq. (V.6)

$$E_k = \frac{s - s' - u' - \mu^2}{2\mu} \quad (V.8)$$

Introducing (V.5) and (V.8) into (V.7), and transforming, as done before, to s' , u' and k^2 integration variables, we obtain

$$\bar{E} = \frac{1}{2\mu A(s, -\mu^2) 8\pi^4} \int d k^2 A^R(k^2) \iint \frac{d s' d u'}{(u' + \mu^2)^2} A(s' u') \times (s - s' - u' - \mu^2) Q(s, -\mu^2; s', u'; k^2) \quad (V.9)$$

Q being given in Section III. Asymptotically, using $A(s', u') = s'^x \varphi_x(u')$, we obtain

$$\begin{aligned} \bar{E} &= \frac{1}{2\mu s^x 16\pi^3 \varphi_x(-\mu^2)} \int d k^2 A^R(k^2) \int_0^1 \frac{d s'}{s} (s - s') s'^x \int_{\frac{s'(-\mu^2 + \frac{k^2}{1-s'})}{s}}^{\infty} d u' \frac{\varphi_x(u')}{(u' + \mu^2)^2} = \\ &= \frac{s}{32\pi^3 \mu \varphi_x(-\mu^2)} \int d k^2 A^R(k^2) \int_0^1 d x (1-x) x^x \int_{x(-\mu^2 + \frac{k^2}{1-x})}^{\infty} \frac{\varphi_x(u) d u}{(u + \mu^2)^2} \end{aligned} \quad (V.10)$$

The integrations in (V.10) do not depend on s , therefore

$$\bar{E} = \alpha s \quad (V.11)$$

where α is a constant whose evaluation would need the explicit knowledge of the eigenfunction $\varphi_\alpha(u)$. Eq. (V.11) means that in our model the average energy carried away by the first system is a fixed proportion of the incident energy.

As we shall discuss in Section IX, the nature of the incident and/or target particle does not change the trend of our results and, therefore, the form of Eq. (V.11). The spectra we are studying are particularly easy to measure when the incident particle is distinguishable from the secondary pions. This is the case for nucleon collisions, which have been extensively studied either with accelerators or cosmic rays. The analysis of high-energy jets gives indeed as a result that the average energy that the nucleon carries away, in the lab. system, is nearly a constant fraction of the incident energy ¹¹⁾ (constancy of the inelasticity), which is just the result predicted by our model.

The three results obtained in this chapter, i.e.,

- cross-sections behave as a power of the energy;
- multiplicities grow logarithmically with the energy;
- constant inelasticity;

are general properties of the multiperipheral model in the asymptotic limit.

They do not depend on the actual solution of our integral equation and, therefore, on the details of low-energy physics (A^R). This situation will be found in all phenomena (for spectra and diffraction, this will be shown in the next sections) and, in our opinion, reflects the general invariance property we discussed before.

VI. SPECTRA OF SECONDARIES

We shall discuss now the spectra of secondaries; actually, we shall be interested in the spectra of final states from which the spectra of final pions can be obtained. We shall be interested in finding which is the number of final states at energy s

$$N_s(k) d^4 k$$

whose four momenta shall be comprised between k and $k+dk$.

Clearly, using the notation of Eqs. (III.1) and (III.2) :

$$N_s(k) = \sum_{n=2}^{\infty} \sum_{i=2}^n \int \cdots \int d^4 q_1 \cdots d^4 q_n G_n(q_1 \cdots q_n; p, p') \delta^4(q_i - q_{i-1} - k) \quad (\text{VI.1})$$

In (VI.1) we are leaving out the first and the final link whose spectra were already treated in Section V. This means that

$$\int N_s(k) d^4 k = \langle N \rangle - 2 \quad (\text{VI.2})$$

From Eq. (III.2) we see that

$$G_n(q_1 \cdots q_n; p, p') = \frac{1}{8\pi^4} \int A(s_0) ds_0 \times \quad (\text{VI.3})$$

$$\times \frac{G_{i-2}(q_1 \cdots q_{i-2}; q_{i-1}, p) G_{n-i}(q_{i+1} \cdots q_n; q_i, p') \delta[(q_1 \cdots q_i)^2 - s_0]}{(q_1^2 - \mu^2)^2 (q_{i-1}^2 - \mu^2)^2}$$

The double sum in (VI.1) can be transformed as

$$\sum_{n=2}^{\infty} \sum_{i=2}^n = \sum_{n-i=0}^{\infty} \sum_{i=2=0}^{\infty}$$

We obtain

$$\begin{aligned} N_s(k) &= \frac{A^R(k^2)}{8\pi^4} \iint \frac{d^4 q' d^4 q'' \delta^4(q' - q'' - k)}{(q'^2 - \mu^2)^2 (q''^2 - \mu^2)^2} \sum_{n-i=0}^{\infty} \sum_{i=2=0}^{\infty} \left\{ \int d^4 q_1 \dots d^4 q_{i-2} \right. \\ &\quad \times G_{i-2}(q_1 \dots q_{i-2}; q' p) \left. \int d^4 q_{i+1} \dots d^4 q_n G_{n-i}(q_{i+1} \dots q_n; q'' p') \right\} = \\ &= \frac{A^R(k^2)}{8\pi^4} \iint \frac{d^4 q' d^4 q'' \delta^4(q' - q'' - k)}{(q'^2 - \mu^2)^2 (q''^2 - \mu^2)^2} A(q', p) A(q'', p') \end{aligned} \quad (\text{VI.4})$$

and using the definition, Eq. (III.5), we find :

$$N_s(k^2) = \frac{A^R(k^2)}{8\pi^4} \iint \frac{du du''}{(u^2 - \mu^2)^2 (u''^2 - \mu^2)^2} \iint ds' ds'' A(s', u') A(s'', u'') I \quad (\text{VI.5})$$

where

$$I = \iint d^4 q' d^4 q'' \delta^4[s' - (q' + q'')] \delta^4[s'' - (q' - q'')] \delta^4(u + q') \delta^4(u'' + q'') \delta^4(q' - q'' - k) \quad (\text{VI.6})$$

The separation (VI.3), giving rise to (VI.5), corresponds to the graph of Fig. 9, in which any intermediate link of the chain is isolated.

Fig. 9

In the following, we shall evaluate $N_s(k)$ for very high energies, fixing our attention on the whole spectrum, except for the secondaries that have maximum longitudinal momentum in both senses in the c.m.s. This is because these limits of the spectrum shall just be given by the initial or final links of the chain, i.e., situations in which s' or s'' are small. This shall allow us to use for $A(s', u')$ and $A(s'', u'')$ the asymptotic expressions, and consistently disregard all the masses when compared to s, s' and s'' .

I is an invariant function of the vectors p_1, p_2 and k and, therefore, of the invariants $s', s'', u', u'', s = (p+p')^2, k^2, s_1 = (p+k)^2$ and $s_2 = -(p'+k)^2$. In order to compute it, let us work in the c.m.s. and define

$$\left\{ \begin{array}{l} k_+ = k_0 + k_L \\ k_- = k_0 - k_L \\ \vec{k}_T \end{array} \right. \quad \begin{array}{l} q_+ = q_0 + q_L \\ q_- = q_0 - q_L \\ \vec{q}_T \end{array} \quad (\text{VI.7})$$

where k_L is the longitudinal component of k with respect to the incident direction, and k_T is the two-dimensional vector, orthogonal to such a direction. In this system

$$\begin{aligned} s_1 &= k_+ \sqrt{s} & s_2 &= k_- \sqrt{s} \\ s_1 s_2 &= s k_+ k_- = s (k^2 + k_T^2) \end{aligned} \quad (\text{VI.8})$$

As

$$d\vec{q}' d\vec{q}'' = \frac{1}{4} dq'_+ dq''_+ dq'_- dq''_- d\vec{q}'_T d\vec{q}''_T \quad (\text{VI.9})$$

the integrations over q'_+ , q''_+ , q'_- , q''_- in Eq. (VI.6) give just $1/s$ and fix such variables to be given by

$$\begin{aligned} q'_+ &= k_+ x & q'_- &= -k_- (1+y) \\ q''_+ &= k_+ (1+x) & q''_- &= -k_- y \end{aligned} \quad (\text{VI.10})$$

where

$$x = \frac{s'_1}{s_1} \quad y = \frac{s''_2}{s_2} \quad (\text{VI.11})$$

We are therefore left with

$$\begin{aligned} I &= \frac{1}{4s} \iint d\vec{q}'_T d\vec{q}''_T \int_2 (\vec{q}'_T - \vec{q}''_T - \vec{k}_T) \delta(u' - (k^2 + k_T^2)x(1+y) - \vec{q}'_T^2) \times \\ &\quad \times \delta(u'' - (k^2 + k_T^2)(1+x)y - \vec{q}''_T^2) \end{aligned} \quad (\text{VI.12})$$

and performing the remaining integrations

$$I = \frac{1}{8s} T \left[u' - (k^2 + \vec{k}_T^2) \chi (1+\chi) ; u'' - (k^2 + \vec{k}_T^2) \chi (1+\chi) ; \vec{k}_T^2 \right] \quad (\text{VI.13})$$

where the T is the triangle function given by

$$T(a,b,c) = \frac{\Theta(-a^2 - b^2 - c^2 + 2ab + 2bc + 2ac)}{\sqrt{-a^2 - b^2 - c^2 + 2ab + 2bc + 2ac}} \quad (\text{VI.14})$$

Substituting (VI.14) into (VI.5), and using - as discussed above - the asymptotic form (IV.4) for $A(s',u')$ and $A(s'',u'')$, we find

$$N_s(k) = \frac{A^R(k^2)}{64\pi^4 s} \iint \frac{du' du''}{(u'+\mu^2)^2 (u''+\mu^2)^2} \varphi_x(u') \varphi_x(u'') \iint ds' ds'' (s' s'')^\alpha \times \\ \times T \left[u'' - (k^2 + \vec{k}_T^2) \chi (1+\chi) ; u' - (k^2 + \vec{k}_T^2) \chi (1+\chi) ; \vec{k}_T^2 \right] \quad (\text{VI.15})$$

Changing variables from s', s'' to x, y and using (VI.8) :

$$N_s(k) = s^\alpha \frac{A^R(k^2)}{64\pi^4} F \quad (\text{VI.16})$$

where

$$F = (k^2 + \vec{k}_T^2)^{1+\alpha} \iint \frac{du' du'' \varphi_x(u') \varphi_x(u'')}{(u'+\mu^2)^2 (u''+\mu^2)^2} \times \\ \times \iint dx dy T \left[u' - (k^2 + \vec{k}_T^2) \chi (1+\chi) ; u'' - (k^2 + \vec{k}_T^2) \chi (1+\chi) ; \vec{k}_T^2 \right] \quad (\text{VI.17})$$

We shall analyse the limits of integration in (VI.17) and show the important fact that, when the energy of the secondary (k_0) is small compared with the total energy $\sqrt{s}$ (i.e., a small fraction of it), the function F is independent both on the incident and the secondary energy, i.e.,

$$F = F(k, \vec{k}_T) \quad (\text{VI.18})$$

The function T in (VI.17) is different from 0 only in a region of the xy plane that is independent of s and k_0 . This region is in fact determined by u' , u'' , k_T^2 and $k_0^2 + k_T^2$ and is included in the rectangle

$$0 < x < \frac{u'}{k^2 + k_T^2}, \quad 0 < y < \frac{u''}{k^2 + k_T^2}$$

which is limited due to the fact that big u' and u'' are cut in the integration in (VI.17). It is possible to see that if k_0 is smaller than a certain fraction of the total energy ($k_0 < \frac{\sqrt{s}}{n}$) then the upper limit of the x, y integrations in (VI.17) (bigger for bigger u) overcomes the support of the T function discussed before. This implies that the integral in (VI.17) depends only on the variables contained in T , from which we obtain our result (VI.18). The limit imposed on k_0 is physically clear: it means that the spectra we obtained do not hold for secondaries that are very near to the ends of their energy spectrum. On the other hand, the spectra of secondaries that are energetic in the c.m.s. were already analysed in the preceding Section.

In order to understand the meaning of (VI.18), we note that, as

$$d^4k = \frac{1}{2} dk^2 d\vec{k}_T^2 \frac{dE_{\text{lab}}}{E_{\text{lab}}} \quad (\text{VI.19})$$

where E_{lab} is the energy of the secondary state in the lab. system, then

$$dN_s(k) = N_s(k) d^4k \propto s^\alpha \frac{A^R(k^2)}{64\pi^4} A^2 \frac{dE_{lab}}{E_{lab}} F(k, k_T) d\vec{k}_T^2 \quad (\text{VI.20})$$

This means that fixing the mass k^2 of the secondary state [it is restricted to be rather small, due to $A^R(k^2)$], the spectra of transverse momenta is independent of the initial energy and of the energy of the secondary we are looking at ¹²⁾, and is given by the function F of (VI.17). The mass spectra (k^2) of secondaries are given by $A^R(k^2)F(k^2, k_T^2)dk^2$. The high values of k^2 are depressed by two different reasons: one is due to the fact that $A^R(k^2)$ contain mainly low values of k^2 ; the other one is the fact that - as can be easily understood from the afore-mentioned discussion - $F(k^2, k_T^2)$ is a decreasing function of k^2 as well as k_T^2 . In any case, the mass spectrum is also independent of the incident energy and the secondary energy. This would imply, physically, that the branching ratios of different final states (low energy $\pi\pi$, ρ , $K\bar{K}$, etc.), are expected to be also energy independent. The secondary energy spectrum is simply given by dE_{lab}/E_{lab} independently of the primary energy ^{*)}. This extremely simple result is, from our point of view, very interesting. We note, in fact, that the factorization shown in (VI.20) depends only on the structure of the multiperipheral model. It does not depend indeed on the form of A^R and, therefore, not even on the solution of our integral equation. The eigenfunction $\gamma_\alpha(u)$ enters only in the explicit form of F as given by (VI.17).

*) We must remember here that this formula will not be suited for both ends of the spectra in E_{lab} , i.e., for those secondaries which have very low or very high energies in the lab. system.

PART C

ASYMPTOTIC PROPERTIES OF THE ELASTIC SCATTERING

VII. THE DIFFRACTION ELASTIC AMPLITUDE

We now turn to the application of the multiperipheral model to the computation of the absorptive part of the elastic scattering amplitude given by the ladder graphs of Fig. 10 :

Fig. 10

We denote by (p_1, p'_1) , (p_2, p'_2) the initial and final momenta, and define

$$P = \frac{p_1 + p_2}{2}, \quad Q = \frac{p'_1 + p'_2}{2}, \quad \Delta = \frac{p'_1 - p'_2}{2} = \frac{p_2 - p_1}{2}$$

$$(e.g.) \quad p_1'^2 = -u_1, \quad p_2'^2 = -u_2, \quad \Delta^2 = t \quad (\text{VII.1})$$

$$(Q + P)^2 = (p_1 + p'_1)^2 = (p_2 + p'_2)^2 = S$$

In order to compute the absorptive part $A(Q, P, \Delta)$ of the scattering amplitude, we shall follow the same line of reasoning as in Sections III and IV, in which the integral equation for the total cross-section was derived. First of all, we obtain the recurrence relation

$$A_n(Q, \Delta, P) = \frac{1}{8\pi^4} \int \frac{d^4 Q' A^R(s_0) \delta[(Q-Q')^2 - s_0] A_{n-1}(Q', \Delta, P)}{((Q'+\Delta)^2 - \mu^2)((Q'-\Delta)^2 - \mu^2)} \quad (\text{VII.2})$$

and from this, the integral equation

$$A(Q, \Delta, P) = A^R(Q, \Delta, P) + \frac{1}{8\pi^4} \iint \frac{d^4 Q' A^R(s_0) \delta[(Q-Q')^2 - s_0] A(Q', \Delta, P)}{[(Q'+\Delta)^2 - \mu^2][(Q'-\Delta)^2 - \mu^2]} \quad (\text{VII.3})$$

Expressed in terms of the invariants, Eq. (VII.3) reads

$$A(s, u, u_2, t) = A^R(s, t) + \frac{1}{8\pi^4} \iiint ds' du'_1 du'_2 ds_0 \frac{K_t(s, u, u_2; s', u'_1, u'_2; s_0) A(s', u'_1, u'_2, t)}{(u'_1 + \mu^2)(u'_2 + \mu^2)} \quad (\text{VII.4})$$

where

$$K_t(s, u, u_2; s', u'_1, u'_2; s_0) = \int d^4 Q' \delta[(Q-Q')^2 - s_0] \delta[(Q'+\Delta)^2 + u'_1] \times \\ \times \delta[(Q'-\Delta)^2 + u'_2] \delta[(Q'+P)^2 - s'] \quad (\text{VII.5})$$

We note that in Eq. (VII.4) s', u'_1 and u'_2 are integration variables, whereas t is a fixed parameter. The main difference between Eq. (VII.4) and the corresponding Eq. (IV.2) for the forward amplitude, is that now we are forced to extract two particles from the mass shell instead of one. Eq. (VII.4) has been studied by Bertocchi, Tonin, and one of us¹³⁾; the treatment is analogous (although more cumbersome !) to that of the equation for the total cross-section. For sake of completeness, we shall report here the most important points. Applying the high-energy limit to Eq. (VII.4), one obtains the homogeneous equation

$$A(s, u_1, u_2, t) = \int \frac{A(s_0 t)}{8\pi^4} ds_0 \iiint \frac{ds' du'_1 du'_2}{s (u'_1 + \mu^2) (u'_2 + \mu^2)} Q_t(u, u_2, u'_1, u'_2; x) \times A(s' u'_1 u'_2, t) \quad (\text{VII.6})$$

where the kernel Q is given by

$$Q_t(u, u_2, u'_1, u'_2) = \frac{1}{8} T(z_1, z_2, z_3) \quad (\text{VII.7})$$

$$z = -t/(1-x), \quad z_1 = u'_1 - u_1 x - \frac{s_0 x}{1-x}, \quad z_2 = u'_2 - u_2 x - \frac{s_0 x}{1-x}$$

T is the triangle function defined in Eq. (VI.14) and $x = \frac{s'}{s}$. Also here we find that our equation is invariant under the dilatation transformation

$$s \rightarrow c s \quad s' \rightarrow c s'$$

so that we can factorize the s dependence of the amplitude in the following way :

$$A(s, u_1, u_2, t) = C(t) s^{\alpha(t)} \varphi(u, u_2, t) \quad (\text{VII.8})$$

Substituting (VII.8) into (VII.6) we obtain

$$\varphi(u, u_2, t) = \int \frac{A(s_0)}{8\pi^3} ds_0 \int_0^1 dx x^{\alpha} \iint \frac{du'_1 du'_2 Q_t(u, u_2, u'_1, u'_2, x) \varphi(u'_1, u'_2, t)}{(u'_1 + \mu^2)(u'_2 + \mu^2)} \quad (\text{VII.9})$$

Eq. (VII.9) is a two-dimensional homogeneous integral equation of the Fredholm kind, giving rise to an eigenvalue problem: at a fixed value of the parameter t , the equation is satisfied for discrete real values of $\alpha(t)$: the largest of these values gives the dominating term at high energy.

In conclusion, Eq. (VII.8) shows that the energy dependence of the absorptive amplitude is of a power law; Eq. (VII.9) determines the exponent α as a function of the momentum transfer t . In BFT, Eq. (VII.9) is treated by means of an iteration method. In the lowest order approximation, the relation between α and t is given by

$$\int \frac{A(s_0)}{16\pi^3} ds_0 \int_0^1 dx x^{\alpha(t)} \int_0^1 \frac{dy}{1-x + \frac{s_0 x}{1-x} - t(1-x)y(1-y)} = 1 \quad (\text{VII.9}')$$

Eq. (VII.9') shows that the exponent $\alpha(t)$ decreases as t increases (remember that t is negative in the diffraction region!). This means that the width of the diffraction peak decreases as the energy increases. This decreasing behaviour of

$\alpha(t)$ is more general than the lowest perturbation order; in App. 1 it is proved that $\lim_{t \rightarrow 0} \left(\frac{d\alpha}{dt} \right)$ is in general positive. $\varphi(u_1, u_2, t)$ can be normalized so that

$$\varphi(-\mu^2, -\mu^2, t) = 1 \quad (\text{VII.10})$$

so from (VII.8) the absorptive scattering amplitude at high energy shall be given by

$$A(s, t) = C(t) s^{\alpha(t)} \quad (\text{VII.11})$$

The function $C(t)$, which gives the energy independent part of the diffraction peak, is of course not determined by the homogeneous Eq. (VII.6); its derivation can be made using a non-linear equation whose general lines will be illustrated in Appendix 1. We wish to note here that this derivation shows that $C(t)$ is a positive definite non-vanishing function.

Let us now consider the real part $D(s,t)$ of the scattering amplitude related to A by a fixed t dispersion relation

$$D(s,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} A(s',t) \left(\frac{1}{s'-s} \pm \frac{1}{s'-\bar{s}} \right) ds' \quad (\text{VII.12})$$

the upper (lower) sign corresponding to amplitudes even (odd) under crossing. In order to obtain convergent results, if we want to introduce Eq. (VII.11) into Eq. (VII.12), one has to perform m subtractions, where m is the minimum integer greater than $\alpha(t)$.

In the high-energy limit, Eqs. (VII.11) and (VII.12) give for $D(s,t)$

$$D(s,t) = -C(t) s^{\alpha(t)} \frac{\cos \pi \alpha(t) \pm 1}{\sin \pi \alpha(t)} \quad (\text{VII.13})$$

so that the scattering amplitude $T = D + iA$ will be given by

$$T(s,t) = -C(t) s^{\alpha(t)} \frac{e^{-i\pi\alpha(t)} \pm i}{\sin \pi \alpha(t)} = C(t) s^{\alpha(t)} \left[i + \left(\frac{-\cotg \frac{\pi \alpha(t)}{2}}{\tan \frac{\pi \alpha(t)}{2}} \right) \right] \quad (\text{VII.14})$$

Eq. (VII.14) has first been proposed by Gell-Mann, Frautschi and Zachariasen⁹⁾, by generalizing to relativistic theory the results of Regge on complex angular momentum. Next Section will contain a general discussion of the physical meaning of Eq. (VII.14), in particular its relation with the problem of the pion-pion bound states and resonances. Eq. (VII.14) leads to the following expression for the differential elastic cross-section :

$$d\sigma_{el}(s,t) \propto \frac{|T(s,t)|^2}{s} d\cos\theta \propto (C(t))^2 s^{2(\alpha(t)-1)} \begin{pmatrix} \sin^2 \frac{\pi\alpha(t)}{2} \\ \cos^2 \frac{\pi\alpha(t)}{2} \end{pmatrix} \quad (\text{VII.15})$$

showing the characteristic shrinking of the diffraction pattern.

The asymptotic behaviour of the total elastic cross-section is therefore

$$\sigma_{el}(s) \propto \frac{s^{2(\alpha(0)-1)}}{\log s} \quad (\text{VII.16})$$

so that the inelasticity ratio is given by

$$\rho = \frac{\sigma_{el}}{\sigma_{tot}} \propto \frac{s^{\alpha(0)-1}}{\log s} \quad (\text{VII.17})$$

If we take (as suggested by experiment) $\alpha(0) = 1$ we obtain that ρ is inversely proportional to the logarithm of the energy.

VIII. RELATION BETWEEN THE ASYMPTOTIC BEHAVIOUR OF THE SCATTERING AMPLITUDES AND BOUND STATES : REGGE POLES

In this Section we want to discuss in detail the physical meaning of the asymptotic expression for the scattering amplitude obtained in the preceding Section, and in particular its dependence on the momentum transfer t . Let us consider Eq. (VII.13) for $D(s,t)$ and assume that this formula is valid not only for space-like values of t (for which it has been originally derived), but also for time-like ones. In the first iterated solution of our integral equation,

Eq. (VII.9') will give $\alpha(t)$ also for positive values of t . $\alpha(t)$ starts at $t = -\infty$ with a value $\alpha_{\min}$; such a value depends on the form of $A^R(s)$ for large s : in particular, for $A^R(s) \lesssim \frac{1}{s}$, $\alpha_{\min} = -1$.

Starting from $\alpha_{\min}$, $\alpha(t)$ is an increasing function of t for all negative values of t . Eq. (VII.9') can easily be continued to positive values of t until $t = 4\mu^2$, where the denominator starts vanishing; for $t \geq 4\mu^2$ the equation can still have a meaning, if we add to the denominator a small imaginary part $i\varepsilon$. This means that for $t > 4\mu^2$, $\alpha(t)$ will become complex. Eq. (VII.13) shows that $D(s,t)$ has poles in the variable t for $\alpha(t) = 0, 2, 4$ or $1, 3, 5$ depending on whether the sign $+$ or $-$ holds. The pole term has a $s^{\alpha(t)}$ dependence, where $\alpha(t)$ is entire at the pole position. This behaviour of the real part $D(s,t)$ can be interpreted by saying that these poles correspond to the existence of $\pi-\pi$ bound states, the entire value of $\alpha(t)$ being the angular momentum of the state. This interpretation, which is closely analogous to the Regge result in potential scattering, can be easily understood if one considers the graph in Fig. 10 in the crossed channel where t represents the square of the total c.m. $\pi-\pi$ energy and

$$S = -\frac{t-4\mu^2}{2}(1-\cos\varphi)$$

is the momentum transfer.

Then the contribution of a bound state in an angular momentum α is multiplied by the coefficient

$$P_{\alpha}(\cos\varphi) = P_{\alpha}\left(1 + \frac{2S}{t-4\mu^2}\right)$$

which, in the $s \rightarrow \infty$ limit, gives rise to the factor s^{α} . For $t \geq 4\mu^2$, the values of t corresponding to $\text{Re } \alpha(t) = \text{entire}$, will correspond to the possible resonances in the $\pi-\pi$ system. The fact that only even (odd) values of α do actually give rise to poles, depends on the crossing properties of the amplitudes $D(s,t)$, which are an expression of the symmetry of the pion-pion wave function [i.e., due to the fact that we have only even ℓ for $T = 0, 2$ and odd ℓ for $T = 1$].

The validity of the continuation of Eqs. (VII.9') and (VII.13) to time-like values of t , together with the interpretation of the poles of Eq. (VII.13) in terms of bound states or resonances of the $\pi - \pi$ system, can be directly verified by considering the two-body Bethe-Salpeter equation, giving the pion-pion bound states.

$$\left[\left(\frac{\Delta}{2} - q \right)^2 - \mu^2 \right] \left[\left(\frac{\Delta}{2} + q \right)^2 - \mu^2 \right] \psi(q) = \iint \frac{A^R(s_0) ds_0}{(q-q')^2 - s_0} \psi(q') d^4 q' \quad (\text{VIII.1})$$

where Δ is the total momentum ($\Delta^2 = t$) and

$$\int \frac{A^R(s_0) ds_0}{(q-q')^2 - s_0}$$

represents the relativistic "potential" acting between the two pions.

It is possible to prove that the eigenvalue condition for the mass of a bound state having an angular momentum ℓ coincides with the condition $\alpha(t) = \ell$ given by Eq. (VII.9). The equivalence between Eq. (VIII.1) and Eq. (VII.9) is proved in Ref. ¹⁰ for the case $\Delta = 0$; and the general proof is sketched in BFT. We wish to stress that the equivalence between Eqs. (VII.9) and (VIII.1) is by no means trivial. Both equations do, of course, perform the sum on an infinite series of "ladder" graphs; however, Eq. (VII.9) is the asymptotic limit of an inhomogeneous equation for scattering, whereas Eq. (VIII.1) is the homogeneous equation for a bound state problem. The equivalence principle comes from the fact that $A(s, t)$ has the same poles in t (corresponding to bound states or resonances) for any value of s and, of course, also in the limit $s \rightarrow \infty$. So the equivalence between Eqs. (VII.9) and (VIII.1) is a check that the approximations used to pass from Eq. (VII.4) to (VII.9) and (VII.14) do indeed lead to the correct high-energy limit.

The preceding discussion shows that the analogy with the Regge results in potential scattering is very deep, and $\alpha(t)$ has indeed the meaning of the continuation of the angular momentum in the complex plane. Eqs. (VII.13) and (VII.14) might therefore be viewed as the generalization of the Regge formula to

the relativistic potential scattering. The method developed in this paper has the advantage of not needing the use of any technique of analytic continuation and of allowing a dynamical computation of $\alpha(t)$ and $\varphi(t)$, i.e., of the masses and of the coupling constants of the different $\pi\text{-}\pi$ bound states or resonances.

If we wish, however, to consider our initial programme more ambitiously, in order to compute masses and coupling constants of the $\pi\text{-}\pi$ resonances, we are immediately met with a difficulty already pointed out by Gell-Mann et al. In fact, considering the true elastic scattering trajectory which starts from $\alpha(-\infty) = -1$, and knowing that $\alpha(0) = 1$, from the experimental constancy of the high-energy cross-sections, α will be forced to be zero in connection with a negative value of t , giving rise to an unphysical particle of imaginary mass which could travel with a velocity larger than the velocity of light. It is impossible to get rid of this unwanted "ghost" by saying that its coupling constant to existing particles is zero, since we know that $C(t)$ is positive, at least in our model, for all negative values of t . The appearance of this "ghost" is a warning that the generalization of our theory outside its original range of validity can be very dangerous.

A first possibility to deal with such a difficulty is that our initial model, although well adapted for understanding diffraction, is completely inadequate as a starting point for a dynamical theory of resonances, in the sense that the pion itself cannot be treated as elementary, but behaves like a composite particle --- (Regge pole).

A second possibility is to try models for which $\alpha(-\infty)$ is larger than zero, in such a manner that the trajectory can never cross $\alpha = 0$. This second possibility, however, cannot be achieved by means of regular "potentials", i.e., potentials whose behaviour, for large values of the momenta, does not allow the application of Fredholm methods to Eq. (VIII.1). This would be analogous to potentials behaving like $1/r^2$ (or more) for small r in the non-relativistic case. They were studied recently by Predazzi and Regge¹⁴⁾. The study of such a singular relativistic potential is in progress; it is amusing to note that the

BS equation for two bosons or fermions, connected by a vector photon, is just of this singular kind. This singularity has forbidden, until now, all attempts of a satisfactory solution of these equations ^{*)}.

In Section D we shall discuss an alternative possibility for the elimination of the ghost difficulty which, in our point of view, looks very promising. In trying to introduce the unitarity in the s channel in our model, we shall realize that other singularities (cuts) can appear in the angular momentum variable. We shall discuss that one of the possible effects of the presence of these cuts will be to renormalize the pole trajectories. Roughly, we can argue that this effect might eliminate the ghost difficulty.

The preceding discussion has clearly shown that we still have to understand many important physical points before hoping to build any dynamical theory of elementary particles.

IX. EXTENSION TO PHYSICAL PROCESSES

As we said in Section II, we have up to now concentrated our attention on $\pi-\pi$ scattering, without considering their charge. In this Section we shall at first extend our previous results both for total cross-section and diffraction, to the scattering of other particles besides pions, and then discuss the introduction of charges and spin.

1) Extension to other particles

The model we deal with was constituted as a generalization of the peripheral one, i.e., uses the idea that the high-energy processes are dominated by the one-pion exchange. In the dispersive language this means to consider as fundamental these contributions that have low singularities in the transfer

^{*)} The equation solved by G.C. Wick, Phys.Rev. 96, 1124 (1954); and R.E. Cutkowski, Phys.Rev. 96, 1135 (1954), refers to two bosons connected by a spin zero photon; this equation is not of a singular type.

momentum (strip approximation). Therefore, the dynamic of the elastic process of any pair of particles is dominated by the exchange of the same chain, schematized in Fig. 10, that is responsible for $\pi-\pi$ scattering. In other words, the difference between the diagrams that give rise to pion-pion scattering, and scattering of any other two particles, stays only in the first and last links of the chain. Therefore, by having solved for $\pi-\pi$ scattering the problem of summing up all the series of multi-peripheral graphs, the same problem is solved once and for all for any incident particles. What remains is only to add one or two low-energy links to the high-energy pion-pion amplitudes. We shall now show how this is done : it is clear, however, from the beginning, that the asymptotic properties will be the same for all processes, i.e., the exponent of the energy (Regge pole) is independent of the colliding particles.

Let us consider the scattering of a pion against any spinless particle x . (We defer any considerations on spin to the next section.) Then the absorptive amplitude for this process in our model is given by all multi-peripheral graphs, like the one of Fig. 11 :

Fig. 11

which can be summarized as in Fig. 12.

Fig. 12

where the bubble represents the sum over all multiperipheral $\pi-\pi$ graphs, being therefore the absorptive amplitude out of mass shell for $\pi-\pi$ scattering. Therefore, similarly to Eq. (VII.4) :

$$A_{X\pi}(s,t) = A_{X\pi}^R(s,t) + \int ds_0 \frac{A_{X\pi}^R(s_0)}{8\pi^4} \iiint \frac{ds' du_1 du_2 K_+(s, \mu_{X_1}^2 - \mu_{X_2}^2; s' u_1 u_2)}{(u_1 + \mu^2)(u_2 + \mu^2)} \times A_{\pi\pi}(s' u_1 u_2, t) \quad (\text{IX.1})$$

where K is given by Eq. (VII.5). Having already solved the $\pi-\pi$ problem, the calculation of $A_{X\pi}(s,t)$ is reduced to integrations over known functions. Let us study the asymptotic behaviour of $A_{X\pi}(s,t)$ from (IX.1), using the same method as developed in the preceding sections : taking therefore for $A_{\pi\pi}$ the expression of Eq. (VII.8), i.e. :

$$A_{\pi\pi}(s u_1 u_2, t) = C(t) s^{\alpha(t)} \varphi(u_1 u_2, t) \quad (\text{IX.2})$$

where φ is normalized by (VII.10), we obtain :

$$A_{\pi\pi}(s, t) = C(t) s^{\alpha(t)} \varphi(t) \quad (\text{IX.3})$$

where

$$\varphi(t) = \int ds_0 \frac{A_{\pi\pi}^R(s_0)}{8\pi^4} \int_0^1 dx x^{\alpha(t)} \int du, dv \frac{Q_t(-\mu_x^2, -\mu_x^2; u, v, x) \varphi(u, v, t)}{(u+\mu^2)(v+\mu^2)} \quad (\text{IX.4})$$

Q being given by Eq. (VII.7).

Eqs. (IX.2) and (IX.3) show the same asymptotic behavior as was already guessable from intuitive arguments. This, of course, is true not only for the elastic scattering, but also for total cross-sections, as well as for all other properties of multiple production investigated in Sections V and VI. The total $\pi\pi$ amplitude in the asymptotic limit shall be given by

$$T_{\pi\pi}(s, t) = C(t) \varphi(t) s^{\alpha(t)} \begin{cases} -\cot \frac{\pi \alpha(t)}{2} + i & \text{for even amplitudes} \\ \tan \frac{\pi \alpha(t)}{2} + i & \text{for odd amplitudes} \end{cases} \quad (\text{IX.5})$$

For $t=0$, using the optical theorem, we obtain

$$\sigma_{x\pi}(s) = \sigma_{\pi\pi}(s) \varphi_{x\pi}(0) \quad (\text{IX.6})$$

while for the differential elastic cross-sections :

$$\sigma_{x\pi}(st) = \sigma_{\pi\pi}(st) \varphi_{x\pi}^2(t) \quad (\text{IX.7})$$

The calculation of $\varphi_{\pi x}(t)$ from (IX.4), once the solution of the integral equation (VII.6) is obtained, needs only the knowledge of low-energy πx amplitude. This means that the Regge pole behaviour is independent of the particles that exchange it. Really, the preceding argument was shown to be valid for πx scattering. We now show how it can be generalized for any x - y scattering. An important by-product of these arguments will be the possibility of obtaining a relation among the asymptotic cross-sections of different processes.

The absorptive amplitude for x - y scattering will be given in our model by the sum over all multiperipheral diagrams of Fig. 13, which can be summarized as in Fig. 14, where the bubble represents the $\pi - \pi$ absorptive amplitude with all 4 particles out of the mass shell.

Fig. 13

Fig. 14

Up to now we have investigated only the extension of two particles out of the mass shell. In order to extend, out of the mass shell, the other two particles, we note that in the asymptotic limit, the kernel Q of the integral equation for $\varphi(u_1, u_2, t)$ [Eqs. (VII.7) to (VII.9)] is indeed independent of the masses of the two particles that were kept on the mass shell. This would imply an asymptotic separability in the dependence on u_1, u_2 and v_1, v_2 of the scattering amplitude, i.e.,

$$A_{\pi\pi}(s, u_1, u_2, v_1, v_2, t) = S^{\alpha(t)}(t) \varphi(u_1, u_2, t) \varphi(v_1, v_2, t) \quad (\text{IX.8})$$

Due to the symmetry of the problem between the two incident (or final) pions, the functions φ and ψ are equal and given by (VII.9). This means

$$A_{\pi\pi}(s, u_1, u_2, v_1, v_2, t) = S^{\alpha(t)}(t) \varphi(u_1, u_2, t) \varphi(v_1, v_2, t) \quad (\text{IX.9})$$

Proceeding now in the way followed to calculate $A_{xy}(s, t)$, and using (IX.9), we obtain :

$$A_{xy}(s, t) = C(t) S^{\alpha(t)} \varphi_{x\pi}(t) \varphi_{y\pi}(t) \quad (\text{IX.10})$$

where $\varphi_{x\pi}(t)$ and $\varphi_{y\pi}(t)$ are given by (IX.4). The total amplitude will therefore be

$$T_{\pi\gamma}(s, t) = C(t) s^{\chi(t)} \begin{cases} \frac{\gamma_{\pi\pi}(t) \gamma_{\gamma\pi}(t)}{2} + i & \text{for even amplitudes} \\ \frac{\tan \frac{\pi \chi(t)}{2} + i}{2} & \text{for odd amplitudes} \end{cases} \quad (\text{IX.11})$$

supporting therefore the conclusions stated before on the independence of the asymptotic Regge pole behaviour on both incident particles.

For the particular case $t=0$, from Eqs. (IX.9) and (IX.10) and the optical theorem, we obtain

$$\sigma_{\pi\gamma}(s) = \sigma_{\pi\pi}(s) \frac{\gamma_{\pi\pi}(0) \gamma_{\gamma\pi}(0)}{2} \quad (\text{IX.12})$$

while for the differential cross-sections

$$\sigma_{\pi\gamma}(s, t) = \sigma_{\pi\pi}(s, t) \frac{\gamma_{\pi\pi}(t) \gamma_{\gamma\pi}(t)}{2} \quad (\text{IX.13})$$

Let us collect the results obtained up to now in this Section. First of all, we understood how in our model, with knowledge of low-energy physics, we can calculate any high-energy process. In fact, once the integral Eq. (VII.9) for γ is solved as in Section VII, and the normalisation function $C(t)$ is obtained - as discussed in Appendix 1 - (this implies only low-energy $\pi-\pi$ interaction), then any $\gamma_{\pi\pi}(t)$ can be calculated from (IX.4) in terms of low-energy $\pi\pi$ scattering. With it we can calculate, from (IX.11), any high-energy process. Besides, from Eqs. (IX.4) and (IX.13), we can obtain some relations among high-energy cross-sections, as :

$$\frac{\sigma_{\pi\gamma}}{\sigma_{\gamma\gamma}} = \frac{\sigma_{\pi\pi}}{\sigma_{\gamma\pi}} \quad (\text{IX.14})$$

where x, y, z, w stand for any kind of particles and σ represents either total or differential elastic cross-section.

Some examples of (IX.14) are ^{*)} :

$$\begin{aligned}\sigma_{\pi\pi} &= \frac{\sigma_{\pi N}^2}{\sigma_{NN}} \sim 15 \text{ mb} \\ \sigma_{\pi K} &= \frac{\sigma_{\pi N}}{\sigma_{NN}} \sigma_{KN} \sim 10 \text{ mb} \\ \sigma_{KK} &= \frac{\sigma_{KN}^2}{\sigma_{NN}} \sim 4 \text{ mb}\end{aligned}\tag{IX.15}$$

We shall show later that the introduction of the spin and isospin of the particles will not alter the above given relations.

2) Introduction of the isotopic spin

In 1) we have shown that the asymptotic properties of scattering amplitudes for any process are dictated, in the multiperipheral model, by those of the $\pi-\pi$ amplitude. Therefore, the introduction of the particle charges shall be straightforward, once the π isospin has been treated. Accordingly, let us introduce the pion charge in $\pi-\pi$ scattering. Charge independence reduces the problem to only three independent amplitudes as, for instance, the amplitudes for definite total isospin 0, 1 or 2. The procedure followed in the Section VII, in order to obtain integral equations for the absorptive amplitudes, would give rise this time to a system of coupled integral equations. However, if one chooses

^{*)} Similar relations were obtained by M. Gell-Mann, Phys.Rev. Letters 8, 263 (1962); and by Gribov and Pomeranchuk, (preprint), making the assumption of the factorizability of the Regge pole contribution. We note that in the frame of the multiperipheral model, the Regge pole contribution is indeed factorized, as can be seen from Eq. (IX.11).

suitable combinations of these amplitudes (whose physical meaning will appear clear later on), then the integral equations would decouple. In order to understand this fact, we wish to point out that in the integral equation from which we obtained the properties of the scattering amplitudes, the transfer momentum t appeared as a parameter; in other words, there is an integral equation for every value of t .

This result is due to the form of the diagrams considered [Fig. 10] and the fact that t is just the kinematical invariant $(p_1 - p_2)^2$ that is the same for all pairs of pions at any stage of the chain.

The introduction of a new quantum number as the isospin will allow us again independent integral equations if it is the same for every pair of pions in the chain. In order to obtain this, it is evident that one has to diagonalize the isospin in the t channel. In such a way, we shall have three uncoupled integral equations for the three amplitudes A_0, A_1, A_2 where the sub-index indicates the isospin I in the t channel. Using the crossing relations, it is possible to express the amplitudes A^0, A^1, A^2 corresponding to a definite isospin T for the two incident particles (s channel), as a linear combination of A_0, A_1, A_2 :

$$A^T(s, t) = \sum_I F_I^T A_I(s, t) \quad (\text{IX.16})$$

where F_I^T is the $\pi - \pi$ crossing matrix given by:

$$F_I^T \equiv \begin{pmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{pmatrix} \quad (\text{IX.17})$$

where F is equal to its inverse.

Finally, for every A_I we shall obtain an integral equation identical to that of neutral particles [Eq. (VII.4)] with $A^R(s)$ replaced by $A_I^R(s)$, as given by

$$A_I^R = \sum_T F_I^T A^{RT} \quad (\text{IX.18})$$

where F is again given by Eq. (IX.17) and $A^{RT}(s,t)$ is the low energy $\pi\pi$ amplitude in isospin T . The amplitudes A_I have also the property to state a clear distinction between the charge exchange and no-charge exchange amplitudes. In fact A_1 and A_2 contain simple or double charge components in the t channel, implying therefore charge exchange in the s channel. Therefore in the case that charge exchange would vanish at high energies, A_0 is the only remaining amplitude, and the process - in the s channel - is isotopic spin independent. We shall show now that this is indeed the case in our model.

In the way used in Sections IV and VII for obtaining the asymptotic behaviour of the neutral amplitudes, we can obtain the asymptotic properties for the three amplitudes A_I :

$$A_I(s,t) = C_I(t) s^{\alpha_I(t)} \quad (\text{IX.19})$$

where $\alpha_I(t)$ will be given by the eigenvalue condition analogous to Eq. (VII.9'). It is easy to see from such an equation that if A_I^R increases, $\alpha_I(t)$ has also to increase.

Being A^{RT} positive definite in the forward direction, we see from (IX.18) and Eq. (IX.17), that

$$A_0^R(s,0) > A_1^R(s,0), \quad A_2^R(s,0) \quad (\text{IX.20})$$

and therefore, from the preceding argument,

$$\alpha_0(0) > \alpha_1(0), \alpha_2(0) \quad (\text{IX.21})$$

This means that, in the framework of our model, high energy charge exchange shall vanish, as compared to non-charge exchange elastic scattering. Due to the fact that the three numbers of the first column of Eq. (IX.17) are equal, the total cross-sections are isotopic spin independent at high energies, or, in other words, they are independent on pion charge. We want to stress that these results contain more relation than those given by the Pomeranchuk theorem¹⁵⁾, and they hold even if total cross-sections do not behave as constants, but drop slightly with energy.

The fact that $A_0(s,0)$ dominates at high energies allows us to draw conclusions on the relative abundance among the secondaries in multiple processes, of secondaries with isospin 0, 1 or 2. Because in the $I=0$ state the low-energy amplitude A^R appears in the combination

$$A_0^R = \frac{1}{3} A^{R0} + A^{R1} + \frac{5}{3} A^{R2} \quad (\text{IX.22})$$

the relative abundance of secondaries for the three possible (T) values will be given by

$$\int F(k_1^2, k_2^2) \begin{pmatrix} \frac{1}{3} A^{R0}(k^2) \\ A^{R1}(k^2) \\ A^{R2}(k^2) \end{pmatrix} dk^2 \quad (\text{IX.23})$$

where F is the kinematical weighing function we obtained for the neutral case which is given by Eq. (VI.17).

We note that, due to the intrinsic isotopic spin independence, the coefficients in (IX.23) are just the statistical weights of each state. The introduction of the charge of other particles is extremely simple. Once the isospin is diagonalized in the t channel (I), the arguments are analogous to those of the preceding sub-section. Therefore Eq. (IX.11) would still hold with the corresponding α_I . Because, as we discussed before, s^{α_0} will dominate the asymptotic behaviour, the cross-sections will, again, not depend on the charge of the incoming particles (in the s channel).

3) Introduction of the spin

The introduction of the spin of colliding particles will only regard the first and last links of the multiperipheral chain. We shall treat in some detail the πN scattering, the treatment of any other case being clear afterwards. The spinor character of the nucleon implies, in πN scattering, two independent scalar amplitudes, usually defined as

$$T_{(st)}^I = -A_{(st)}^I + i \gamma \cdot Q B_{(st)}^I \quad (\text{IX.24})$$

where the kinematical variables are defined in Section VII, and I refers to the isospin in the t channel ^{*)}.

The addition of the πN link to the pion-pion multiperipheral chain is straightforward. Calling A_a and B_a the absorptive parts of A and B $\overline{A}_a^R, \overline{B}_a^R$ at low energy⁷, then, similarly to Eq. (VII.3) :

$$\begin{aligned} -A_q^I(Q\Delta P) + i \gamma \cdot Q B_q^I(Q\Delta P) = & -A_q^{R I}(Q\Delta P) + i \gamma \cdot Q B_q^{R I}(Q\Delta P) + \\ & + \frac{1}{8\pi^4} \iint \frac{(-A_q^{R I}(s_0) + i \gamma \cdot Q' B_q^{R I}(s_0)) \sqrt{[Q-Q']^2 - s_0} A_{\pi\pi}^I(Q'\Delta P) ds_0 d^4 Q'}{((Q'+\Delta)^2 - \mu^2)((Q'+\Delta)^2 - \mu^2)} \quad (\text{IX.25}) \end{aligned}$$

*) In the usual notations : $T^0 = \sqrt{6} T^{(+)}$, $T^1 = 2T^{(-)}$. G.F. Chew, M. Goldberger, F. Low and Y. Nambu, Phys.Rev. 106, 1337 (1957).

The projection over invariant amplitudes gives now

$$A_a^I(s, t) = A_a^{RI}(s, t) + \iiint \frac{ds' du_1 du_2}{8\pi^4 (u_1 + u_2)} K_t(s, -u_1^2, -u_2^2, s', u_1, u_2) A_{\pi\pi}^I(s', t) \times \\ \times \int ds_0 [A_a^{RI}(s_0, t) - a B_a^{RI}(s_0, t)] \quad (\text{IX.26})$$

$$B_a^I(s, t) = B_a^{RI}(s, t) + \iiint ds' du_1 du_2 K_t(s, -u_1^2, -u_2^2, s', u_1, u_2) A_{\pi\pi}^I(s', t) \times \\ \times \int ds_0 b B_a^{RI}(s_0, t)$$

where

$$a = \frac{(2s+t-2m^2-2u^2)(2s'+t+u_1+u_2-2u^2) + (2s_0+t+u_1+u_2-2m^2)(4u^2-t)}{(2s+t-2m^2-2u^2)^2 - (4u^2-t)(4m^2-t)} \quad (\text{IX.27})$$

$$b = \frac{(2s+t-2m^2-2u^2)(2s_0+t+u_1+u_2-2m^2) + (2s'+t+u_1+u_2-2u^2)(4m^2-t)}{(2s+t-2m^2-2u^2)^2 - (4u^2-t)(4m^2-t)}$$

and K is given by Eq. (VII.5). In the asymptotic limit

$$a = m \times$$

$$b = \frac{1}{5} \left(\frac{\chi}{2} (4m^2-t) + s_0 + \frac{t}{2} - m^2 + \frac{u_1+u_2}{2} \right) \quad (\text{IX.28})$$

where, as before, $x = \frac{s'}{s}$ and $K = \frac{Q}{s}$, where Q is given by (VII.7).

Introducing in Eq. (IX.26) the asymptotic expressions for $A_{\pi\pi}$, a and b , we obtain for the πN asymptotic absorptive amplitudes :

$$A_a^I(s, t) = C_I(t) \varphi_A^I(t) s^{\alpha_I(t)} \\ B_a^I(s, t) = C_I(t) \varphi_B^I(t) s^{\alpha_I(t)-1} \quad (\text{IX.29})$$

where

$$\varphi_A^I(t) = \int dx x^{\alpha_I(t)} \int \frac{du_1 du_2 Q_t(-u_1^2, -u_2^2, u_1, u_2, x)}{8\pi^4 (u_1 + u^2)(u_2 + u^2)} \varphi_{\pi\pi}^I(u_1, u_2, t) \times \left(A_q^{RI}(s_0 t) + m \times B_q^{RI}(s_0 t) \right) ds_0 \quad (\text{IX.30})$$

$$\varphi_B^I(t) = \int dx x^{\alpha_I(t)} \int \frac{du_1 du_2 Q_t(-u_1^2, -u_2^2, u_1, u_2, x)}{8\pi^4 (u_1 + u^2)(u_2 + u^2)} \varphi_{\pi\pi}^I(u_1, u_2, t) \times \left[\frac{x}{2} (4u_1^2 - t) + s_0 + \frac{t}{2} - u_1^2 + \frac{u_1 + u_2}{2} \right] B_q^{RI}(s_0 t) ds_0$$

Eq. (IX.30) is analogous to Eq. (IX.4) obtained for scalar particles.

Therefore the asymptotic behaviour of πN scattering is completely determined from the $\pi-\pi$ one. Using fixed momentum transfer dispersion relations for A and B, we obtain

$$A(s, t) = -C_I(t) \varphi_A^I(t) s^{\alpha_I(t)} \left[\frac{\cos \pi \alpha_I(t) + (-1)^I}{\sin \pi \alpha_I(t)} \right]$$

$$B(s, t) = -C_I(t) \varphi_B^I(t) s^{(\alpha_I(t)-1)} \left[\frac{\cos \pi \alpha_I(t) + (-1)^I}{\sin \pi \alpha_I(t)} \right] \quad (\text{IX.31})$$

From (IX.31) we can obtain the asymptotic behaviour of the non-spin flip and spin flip amplitudes ^{*})

$$\varphi_{NF}^I(s, t) \sim \frac{-m}{4\pi} s^{[\alpha_I(t)-\frac{1}{2}]} C_I(t) \left(\varphi_A^I(t) + \frac{\varphi_B^I(t)}{2m} \right) \frac{e^{-i\pi \alpha_I(t)} + (-1)^I}{2 \sin \pi \alpha_I(t)}$$

$$\varphi_F^I(s, t) \sim \frac{s^{\alpha_I(t)}}{16\pi} C_I(t) \varphi_A^I(t) \frac{e^{-i\pi \alpha_I(t)} + (-1)^I}{\sin \pi \alpha_I(t)} \quad (\text{IX.32})$$

^{*}) Similar expressions were obtained in Ref. ⁹⁾, generalizing to spin particles the Regge pole conjecture.

as well as total and differential cross-sections

$$\begin{aligned}\sigma_{tot}^I(s) &= 2m s^{\alpha_I(s)-1} C_I(s) \left(\varphi_A^I + \frac{\varphi_B^I}{2m} \right) = \\ &= \sigma_{\pi\pi}^{(s)} \left(\varphi_B^I + 2m \varphi_A^I \right)\end{aligned}\quad (\text{IX.33})$$

$$\begin{aligned}\frac{d\sigma^I(s,t)}{dt} &= C_I^2(s) s^{2\alpha_I(s)-1} \frac{m^2}{16\pi^2} \left[\left(\varphi_A^I + \frac{\varphi_B^I}{2m} \right)^2 + \frac{t}{16m^2} \left(\varphi_A^I \right)^2 \right] \times \\ &\times \frac{|e^{-i\pi\alpha_I(t)} + (-1)^I|^2}{4m^2\pi\alpha_I(t)} = \frac{d\sigma_{\pi\pi}^{(s,t)}}{dt} \frac{\left(\varphi_B^I + 2m \varphi_A^I \right)^2 + \frac{t}{4} \left(\varphi_A^I \right)^2}{\left(\varphi_A^I \right)^2}\end{aligned}\quad (\text{IX.34})$$

Remembering that $\alpha_0 > \alpha_1$, we have from

$$\begin{aligned}\sigma^0 &= \sqrt{\frac{3}{2}} (\sigma_{\pi^+p} + \sigma_{\pi^-p}) \\ \sigma^1 &= \sigma_{\pi^-p} - \sigma_{\pi^+p}\end{aligned}\quad (\text{IX.35})$$

that, asymptotically, $\sigma_{\pi^+p} \sim \sigma_{\pi^-p}$, and, therefore, isospin independence. Contrarily to the isospin case, the spin flip contribution to the asymptotic differential cross-section is of the same order in s as the non spin flip one. This fact can be easily understood, noting that two different isospin amplitudes differ in all the links of the multiperipheral chain, having therefore two different asymptotic behaviours. On the contrary, two different spin amplitudes differ only in one link (or, at most, two as in N-N scattering) and therefore their contributions must have the same asymptotic behaviour, differing only by coefficients related to low-energy physics.

PART D

USE OF THE UNITARITY IN THE s CHANNEL

X. UNITARITY IN THE s CHANNEL AND GENERALIZATION OF THE MULTIPERIPHERAL MODEL

The model we have discussed until now has the important limitation of not being unitary in the s physical channel. We expect the effect of unitarity to be important, since we know that on its basis Froissart showed that all powers $\alpha(t) > 1$, for $t \leq 0$, are inconsistent. Now, our linear equations do not show any such limitations for s . In this Part we wish to give some rough arguments on which corrections to the simple formulae $s^{\alpha(t)}$ can be obtained by using unitarity. The solution of the problem of including unitarity in a satisfactory manner is very far from our present possibilities, and we shall depend heavily on intuitive physical arguments based on graphs.

As we already discussed in Section II, we have considered - in the model as presented up to here - the pure inelastic process [represented in Fig. 2], to be the most important at high energies. As compared to it, the pure elastic one shall be less important, but the mixed elastic and inelastic ones, represented in Fig. 4, can be competingly large. We also discussed briefly in Section II how the inclusion of these effects can be done consistently in our model. In fact, when we calculate the diffraction through unitarity [Eq. (I.2)], not restricting to pure inelastic intermediate states, we shall find that it will be given by the following diagrams :

Fig. 15-a

Fig. 15-b

Fig. 15-c

and diagrams containing diffraction in a more complicated way, as, e.g.,

Fig. 15-d

Fig. 15-a ("pure inelastic" intermediate state) is the one already considered before, whose solution was asymptotically ^{*)}

$$T(s,t) = C(t) s^{\alpha(t)} \left(-\cotg \pi \frac{\alpha(t)}{2} + i \right) \quad (\text{X.1})$$

i.e., the Regge pole behaviour. Fig. 15-b ("pure elastic" intermediate state), Figs. 15-c and 15-d ("mixed" intermediate states) contain themselves the diffraction we are calculating. The self-consistent procedure, therefore, is given by a non-linear integral equation for the diffractive amplitude, whose inhomogeneous term is given by (X.1). We shall not attempt now the solution of such a problem; we will investigate it, however, by using an iterative procedure. This means to use Eq. (X.1) for diffraction to calculate Figs. 15-b, 15-c and 15-d, in order to obtain a first iterated solution, and so on.

We shall show that such contributions (corresponding to the exchange of at least 4 pions) will provide essentially new features, i.e.,

- asymptotic properties showing a continuous superposition of powers of s (or a continuous distribution of singularities in the angular momentum variable);
- a change in the discrete power behaviour (or a change of the form of the Regge pole trajectory).

^{*)} We concentrate now on the true elastic scattering (no exchange of quantum numbers).

Let us concentrate first our attention on Fig. 15-b; the application to it of the **iterative** approach discussed before, will give rise to a series of Regge pole iterations represented in Fig. 16 :

Fig. 16

We shall study in the next Section the properties of such an iteration of Regge poles.

XI. ITERATION OF REGGE POLES : CUTS IN THE ANGULAR MOMENTUM VARIABLE

Let us call $T_n(s, t)$ the scattering amplitude due to the n^{th} iteration of a Regge pole [and call $A_n(t)$ the absorptive amplitude]. T_0 is a Regge pole itself, and is given by (X.1). Let us now start with the first iteration :

$$A_1(s, t) = \frac{1}{(8\pi)^2} \int T_0^+(s, t') T_0(s, t'') d\Omega \quad (\text{XI.1})$$

where the notations are trivial : t' is the square transfer momentum carried by the first Regge pole, Ω is the c.m. angle between the incident and the intermediate states, and t'' is the square transfer momentum carried by the second Regge pole. From (XI.1) we can find, with some algebra :

$$A_1(s, t) = \frac{2}{(8\pi)^2 s} \int_{-\infty}^0 dt' \int_{-\infty}^0 dt'' T_0^+(s, t') T_0(s, t'') \tilde{T}(t, t', t'') \quad (\text{XI.2})$$

where $\tilde{T}(t, t', t'')$ is the usual triangle function defined in (VI.14). Using Eq. (X.1) for $T_0(s, t)$ we find

$$A_1(s, t) = \int d\zeta s^\zeta g(t, \zeta) \quad (\text{XI.3})$$

where

$$g(t, \zeta) = \frac{2}{(8\pi)^2} \int_{-\infty}^0 dt' \int_{-\infty}^0 dt'' \delta(\zeta + 1 - \alpha(t') - \alpha(t'')) \tilde{T}(t, t', t'') \times \\ \times (\alpha(t') \alpha(t'')) \left[\cotg \frac{\pi \alpha(t')}{2} \cotg \frac{\pi \alpha(t'')}{2} + 1 \right] \quad (\text{XI.4})$$

Therefore, from (XI.3) and fixed t dispersion relation - as done in Section VIII - we find the asymptotic behaviour of T_1 to be given by

$$T_1(s, t) = \int_{\zeta_M(t)}^{\zeta_m(t)} s^\zeta g(t, \zeta) \left[-\cotg \frac{\pi \zeta}{2} + i \right] d\zeta \quad (\text{XI.5})$$

where $g(t, \zeta)$ is again given by (XI.4). $\zeta_M(t)$ $\left[\zeta_m(t) \right]$ is just the upper (lower) limit of the support of $g(t, \zeta)$. Actually, $\zeta_M(t)$ determines the asymptotic behaviour of $T_1(s, t)$. In fact, without need of the exact computation of $g(t, \zeta)$, through (XI.4), we can say that the asymptotic behaviour of T_1 will be given by

$$T_1 \propto \frac{s^{\frac{1}{2}\alpha_M(t)}}{\log s} f(t, \log s) \quad (\text{XI.6})$$

where $f(t, \log s)$ is a rational function of $\log s$.

Before trying to obtain some properties of $\frac{1}{2}\alpha_M(t)$, we want to discuss some very general characteristics of T_1 as given by (XI.5). We see that while T_0 was showing an asymptotic behaviour given by s to a power $\alpha(t)$, so T_1 shows a superposition of powers up to a maximum value $\frac{1}{2}\alpha_M(t)$. Or, in a more sophisticated language, while $T_0(s, t)$ was associated with a pole (Regge pole) in the angular momentum variable at $\alpha(t)$, so T_1 can be interpreted as a continuous superposition of Regge poles in the angular momentum variable running up to $\frac{1}{2}\alpha_M(t)$. We shall call this superposition of poles a cut where $\frac{1}{2}\alpha_M(t)$ is the branch point. This is a very important result whose physical meaning we shall explore in the next Section.

In Appendix 2 we will show that

$$\frac{1}{2}\alpha_M(0) \geq \frac{1}{2}\alpha_M(t) > \alpha(t) + \alpha(0) - 1 \quad \text{for } t < 0 \quad (\text{XI.7})$$

while

$$\frac{1}{2}\alpha_M(0) = 2\alpha(0) - 1 \quad (\text{XI.8})$$

In Eqs. (XI.7) and (XI.8) we have an important result. We note at first that values of $\alpha(0)$ bigger than 1 would be inconsistent with our model. In fact, any further iteration - imposed by unitarity in the s channel - would increase the power of s of the asymptotic behaviour. This would mean that the high-energy amplitude would increase faster than any power law in contradiction with the standpoint of the model (as well as physics!). This result is in agreement with the one obtained by Froissart¹⁶⁾, that if the amplitude is bounded by a power law, then the maximum power compatible with the Mandelstam representation and unitarity is just one.

Let us discuss next the case $\alpha(0) = 1$ (i.e., constant total cross-sections). In this case Eqs. (XI.7) and (XI.8) read

$$\zeta_M(t) \geq \alpha(t) \quad (t \leq 0) \quad (\text{XI.9})$$

where the equal sign holds only for $t=0$. This means that for $t < 0$ the maximum power of s appearing in T_1 shall always be bigger than $\alpha(t)$, except for $t=0$ when both coincide. I.e., being up to the second iteration

$$T = T_0 + T_1,$$

and, making use of (XI.6), (XI.9) implies that - except for the forward direction - the asymptotic behaviour of T_1 would prevail upon that of T_0 (Regge pole contribution).

The situation would be different if $\alpha(0) < 1$. If, in fact,

$$\alpha(0) = 1 - \varepsilon \quad (\text{XI.10})$$

then $\alpha(t)$ and $\zeta_M(t)$ would show the following behaviour

Fig. 17

which means that there will be a region of transfer momentum for which T_0 (the pole) dominates over T_1 (the cut), while for sufficiently big values (in magnitude) of the transfer momentum, T_1 will again prevail. The region ($t \leq 0$), for which the pole prevails, is smaller for smaller values of ξ and shrinks to the point $t=0$ when $\xi=0$. We shall postpone the discussion of the physical meaning of such cuts to Section XII. We want to stress here that we could obtain the n^{th} iteration of the Regge pole (T_n) in the same way as we have obtained T_1 . In fact, starting from T_0 of Eq. (X.1) and T_1 of Eq. (XI.5), we can obtain T_2 and so on, with the recurrence formula

$$A_n(s, t) = \frac{2}{(8\pi)^2} \sum_{l=0}^n \iint T_l^+(s, t') T_{n-l-1}(s, t'') \widetilde{T}(t, t', t'') dt' dt'' \quad (\text{XI.11})$$

and the fixed t dispersion relations. From (XI.11) and the previous discussion, we can easily understand that any successive iteration gives rise to a new superposition of powers of s going up to $\sum_M^{(n)}(t)$ for which

$$\sum_M^{(n)}(t) \geq \sum_M^{(n-1)}(t) + \alpha(0) - 1$$

with the equal sign holding for $t=0$. Therefore, in the case $\alpha(0) = 1$, any successive iteration should dominate over the preceding one, except for $t=0$.

For the previously discussed case $\alpha(0) = 1 - \xi$

$$\sum_M^{(n)}(0) = 1 - n\xi \quad (\text{XI.12})$$

so that, as discussed for the first iteration, there shall be a region of small transfer momenta for which T_0 would dominate over T_1 ; T_1 over T_2 ; and so on.

The other remark is that the properties we established for the pure elastic amplitude are trivially extended to other amplitudes ^{*)} (we concentrate, as an example, on charge exchange). In fact, being $\alpha_{el}(t) > \alpha_{c.e.}(t)$, then the largest contribution due to the Regge pole iteration is when the charge exchange amplitude T_0 (Regge pole with isospin different from zero) is not iterated with itself, but with the absolute elastic amplitude. Eq. (XI.7) will now read

$$\sum_M^{c.e.}(t) \geq \sum_M^{c.e.}(t) + \alpha_{el}(0) - 1$$

and therefore all the discussions of the relative importance of T_1 over T_0 (and also for other iterations) as function of the value of $\alpha(0)$ we did before, apply equally well for the case of charge exchange (or exchange of any quantum number).

Let us now discuss the diagrams that contain peripheral, as well as diffractive, parts in them, i.e., Figs. 15-c and 15-d. To have a first orientation, let us concentrate on the simplest one, i.e., Fig. 15-c, in which diffraction appears only once. We have therefore to compound peripherically a ladder diagram (pole) with an iterated Regge pole amplitude. This shall be given by an equation of the type :

$$A(s, t) = \frac{1}{8\pi^4} \iint ds' ds'' du_1 du_2 \frac{A_0(s', u_1, u_2, t) A_1(s'', u_1, u_2, t) K(s, s', s'', u_1, u_2)}{(u_1 + \mu^2)(u_2 + \mu^2)} \quad (XI.13)$$

where the asymptotic behaviours of A_0 and A_1 are given by (X.1) and (XI.5), respectively ^{*)} and K is given by (VII.5).

^{*)} If the amplitude in question, instead of being even under crossing as the absolute elastic one, happens to be odd, then the cotangent must be replaced by the tangent in Eqs. (X.1) and (XI.5).

The asymptotic property of Eq. (XI.13) is examined in Appendix 1, where we show that A has two terms : one indicating the asymptotic properties of A_0 and the other one that of A_1 ; which means that A has both the pole and the cut behaviours.

To calculate the other mixed contributions [Fig. 15-d], the problem is to sum all graphs in which diffraction appears many times. This could be done by making use of the same techniques we have employed in the preceding sections, in order to sum over multiperipheral graphs. This time we shall obtain a new integral equation whose kernel shall contain A^R , as well as the diffractive amplitude previously calculated (pole plus cut). Due to the combination properties of the kernel, studied in Appendix 1, the equation could have a solution s^β with a (discrete) exponent β different, however, from the preceding discrete exponent α . This is a typical "renormalization" effect of the pole ($\alpha(t)$). We shall not attempt now to solve this very difficult problem. What can be expected is that the "renormalized" pole trajectory $\beta(t)$ shall lie over the branch point trajectory $\zeta_M(t)$. Therefore, the modification on the pole, due to the mixed contributions, will be quite important, at least where $\zeta_M(t) > \alpha(t)$ and hence (in the light of the results obtained in this Section) they will be very important for nearly any momentum transfer.

We see that this more ambitious programme in which also states with 4 pions can be exchanged is indeed quite complicated and presents features by no means elementary. In this sense, we consider the iterative method developed in this Section only as a preliminary rough attempt.

XII. GENERAL CONSIDERATIONS ON THE CUTS IN THE ANGULAR MOMENTUM VARIABLE

In Section VIII we discussed, in the framework of our model, the relation between the asymptotic properties of the scattering amplitudes and the bound state problem. Indeed, we have seen that the sum on all multiperipheral graphs represents the "exchange of a Regge pole" or, more precisely, the accumulative effect of the exchanges of all particles of a same family. In the previous Section we have seen, however, that an iterative extension of our relativistic approach gives rise also to contributions that seem to be important in high-energy scattering, and which can be interpreted as cuts in the angular momentum variable. One of the main features of these cuts was to introduce in our model the limit $\alpha(t) \leq 1$.

We do not want to associate much importance to the dominance of the cut over the pole from a certain value of t onwards, as obtained in the preceding Section, because it is clear that our entire iteration procedure does not make much sense in such a situation. Moreover, we have also discussed that another effect of the cut could be to modify the pole trajectory (or to renormalize it) in such a way to change it from the zeroth iteration. In any case - and even if their contributions are less important than those of the poles - these cuts seem to us to be a new interesting feature; they are suggested by a simple extension of the multiperipheral model, and whose physical meaning is worthwhile investigating. Indeed, it seems to us that the presence of the cuts is more general than the model we are discussing. In fact, the cut contribution was brought already by the iteration of Regge poles and, in our point of view, in any relativistic theory, the exchange of two or more bound states (Regge poles) has its own right to exist independently on the dynamical origin of the bound state itself. This fact, as well as the reason why the cut contributions were not present in the conjectures made by extending to the S-matrix theory, the potential result of Regge, will appear even more clearly by arguing in the t channel.

The multiperipheral diagrams of Fig. 10 that gave rise to the Regge poles in our model, have the particularity that - at any stage of the chain - there are only two intermediate particles. This is a common feature between the relativistic Bethe-Salpeter equation and the Schroedinger potential problem. But the characteristic of the field theory is that particles can be created as soon as the thresholds are overcome; therefore, two incoming particles (in the t channel) can give rise not only to one bound state, but also to two bound states, and therefore to a whole continuum. This possibility of producing particles is beyond the range of the potential picture studied by Regge, and it is therefore clear why in his treatment, the cuts are not present. In the light of the well-known relation between poles and bound states, branch points and production thresholds, the results we obtained seem indeed very natural.

On the basis of the preceding discussion, we can also understand that the branch point $\sum_M(t)$ curve has a slope smaller than the pole one ($\alpha(t)$). As the cut contributions are generated by the presence of at least two ladder exchanges, their singularities in t (at fixed s) are further away; actually they lie outside the strip. Therefore, their t dependence (for negative t) is weaker than that of the "Regge pole" contribution, whose t singularities lie in the strip.

The relations between these cuts and the angular momentum variable are less clear. To explore it, we would need to be able to compose non-integer angular momenta. A special case for which the meaning of the composition is simple is the case $t=0$. In this case we must analyze the spin of a system of zero mass, $\alpha(0)$ being the spin of the one zero-energy bound state. Having no kinetic energy at our disposal (no orbital momentum), a system composed by two bound states would have a spin composed by twice $\alpha(0)$: which means a maximum of $2\alpha(0)$. This is just the beginning of our continuum (a factor $1/s$ being purely kinematical).

We want to note, finally, that - in our model - even if cuts are present, both σ_{tot} and σ_{el} are dominated by the Regge pole term. However, we want to point out that, in order to extract from the experiments the dominant pole behaviour, much care would probably be needed, because the corrections to it could be of the logarithmic type (being $\alpha(0) \sim 1$).

XIII. CONCLUSIONS

We wish now to summarize briefly the different results achieved, and their physical meaning. In this paper we have obtained definite predictions for the different high-energy processes on the basis of the multiperipheral model. We have been able to sum the whole series of multiperipheral graphs, by means of a linear integral equation, for the off-mass shell absorptive amplitude $A(s, u_1, u_2, t)$. The kernel of this integral equation depends on the low-energy amplitude A^R . The knowledge of this amplitude is sufficient to allow the computation, both of the elastic scattering, and of multiple production. The on-mass shell amplitude $A(s, -\mu^2, \mu^2, t)$ leads to the elastic diffraction cross-sections, whereas on the basis of the forward off-mass shell amplitude $A(s, u, u, 0)$ we can evaluate the average distributions of particles in multiple production.

The asymptotic behaviour of the amplitude is obtained by considering the high-energy limit of the integral equation. In this limit, the integral equation shows a very remarkable feature, which is independent of the specific form of the amplitude A^R . The kernel depends only on the ratio $\frac{s'}{s}$, so that the equation is invariant under the transformation $s \rightarrow cs$, $s' \rightarrow cs'$. This allows us to factorize the s dependence of the amplitude in the simple form

$$A(s, u_1, u_2, t) = s^{\alpha(t)} C(t) \varphi(u_1, u_2, t) \quad (\text{XIII.1})$$

The problem is then reduced to the solution of a homogeneous integral equation for $\varphi(u_1, u_2, t)$, whose solution determines both the exponent $\alpha(t)$ and the eigenfunction $\varphi(u_1, u_2, t)$. As already pointed out, both eigenvalues and eigenfunctions have a physical meaning: the eigenvalue gives the well-known shrinking of the diffraction peak, whereas the eigenfunction is connected with the average properties of multiple production. A form of the scattering amplitude analogous to Eq. (XIII.1) has been obtained by many people by adapting to high-energy scattering the results of Regge in potential theory. This analogy can be understood by considering that our multiperipheral graphs, observed in the crossed channel, are the relativistic analogous of the different iterations of the potential model used by Regge.

The predictions obtained by means of the model can be divided into two categories :

- a) many general trends of the high-energy collisions do only depend on the transformation property of the integral equation, which is a consequence of the topology of the multiperipheral graphs. These general trends do not depend, indeed, on any special choice of the low-energy amplitude A^R .
- b) the specific numerical answers (like, for example, the value of the total cross-sections) do, of course, depend on the choice of A^R and on the manner in which A^R is continued off the mass shell. As already pointed out in Section III, it has not yet been possible to find a completely satisfactory way of performing such a continuation, especially in the case of higher waves.

In this paper, we have therefore concentrated our attention on the general model independent predictions, which we shall now try to summarize.

1) Elastic amplitude

The high-energy behaviour of the scattering amplitude $T(s,t)$ is

$$T_j(s,t) = s^{\alpha_j(t)} C_j(t) \left[-\cotg \frac{\pi \alpha_j(t)}{2} + i \right] \quad (\text{XIII.2})$$

for symmetric amplitudes under crossing $s \leftrightarrow \bar{s}$, as, for instance, absolute elastic scattering and

$$T_j(s,t) = s^{\alpha_j(t)} C_j(t) \left[\tan \frac{\pi \alpha_j(t)}{2} + i \right] \quad (\text{XIII.3})$$

for antisymmetric amplitudes under the crossing. We obtain $\frac{d\alpha}{dt} > 0$. The exponent for the charge exchange amplitude is always smaller than the one for the purely elastic one. Eqs. (XIII.2) and (XIII.3) turn out to be independent of the scattering particles, apart from the value of $C(t)$. The $C(t)$ can be factorized in such a manner that the relation between different amplitudes (dominated by the same pole) is the following :

$$\frac{T_{xy}(s,t)}{T_{zy}(s,t)} = \frac{T_{xw}(s,t)}{T_{zw}(s,t)}$$

where x, y, z and w represent any kind of particles.

2) Inelastic scattering

The average properties of inelastic scattering are also very simple and depend only on the general form of Eq. (XIII.1), and not explicitly on A^R . The multiplicity grows with the logarithm of the energy, and the inelasticity is energy independent. The spectra of the secondary particles are given by

$$N(k) d^4k = F(k^2, \vec{k}_T^2) d\vec{k}_T^2 dk^2 \frac{dk_L}{k_L}$$

for $k_L \ll$ the initial energy. k_L (k_T and k_L being the transverse and longitudinal momenta), where $F(k^2, k_T^2)$ is a universal function independent both of s and k_L , and strongly peaked for $k_T^2 \leq 4\mu^2$.

These results, especially the separability of the transverse and longitudinal spectra, are not inconsistent with the present experimental data ¹⁷⁾.

We have seen that the multiperipheral model gives simple and nice predictions for all general trends of high-energy interactions, and forms a basis for understanding the meaning of the guesses made on the analogy with the Regge poles. Our scheme allows to discuss critically the validity of the model and to obtain an idea of the general form of its most important conditions. The weakest point of the whole approach is that unitarity in the s channel is not taken into account. A similar difficulty appears in all approaches based on the Regge theory, since in the potential model s means only momentum transfer, and therefore unitarity in this channel has no meaning. Now, Froissart has shown that unitarity in the s channel, together with the Mandelstam representation, do not allow cross-sections increasing more rapidly than $(\lg s)^2$ ^{*}). So this means that Eq. (XIII.1) is only acceptable for $\chi(t) \leq 1$, for $t \leq 0$. Now both our equation and the potential model do not give any such limitation for $\chi(t)$.

In order to have an idea on the effect of unitarity, we have tried to take into account the corrections due to elastic unitarity corresponding to effects of multiple scattering. These new terms show a new interesting feature : they give rise to continuous distributions of powers (cuts in the angular momentum) :

^{*}) Greenberg and Low - Phys.Rev. 124, 2047 (1961) - using only the Lehmann ellipse, obtain $s \lg^2 s$.

$$A_{\text{mult.}}(st) = \int_{\xi_{\text{min}}(t)}^{\xi_{\text{H}}(t)} g(t) s^{\xi} d\xi$$

The position of the upper limit of integration for $t=0$ is simply given by

$$\xi_{\text{H}}(0) = 2\alpha(0) - 1$$

So the dominance of the pole on the cut depends on whether $\alpha(0)$ is larger, or less than 1. For $\alpha(0) > 1$ the cut dominates on the pole, so the whole theory breaks down, as predicted by the Froissart theorem. In the case $\alpha(0) = 1$, which seems to be suggested by experiment, the cut and the pole coincide.

We wish to emphasize that we have not proved the existence of cuts in the angular momentum, but have just given arguments in favour of their existence. It is, however, interesting to note that these cuts enter in the theory in a non-trivial manner, and that they are a possible mechanism by which unitarity leads to the Froissart limitation in the cross-section.

There is an amusing analogy between the properties of the singularities, both in angular momentum and in the usual s, t variables. In both case there exist poles and cuts corresponding to one and more particle states. The relative positions of singularities in the s, t representation depend on the masses of the exchanged particles. For real mass, the pole is separated from the cut; for unphysical imaginary mass, the opposite situation would be met, i.e., the pole would be embedded in the cut. In the zero-mass case (the photon), the pole coincides with the cut. In the angular momentum case, the situation is quite analogous: the dominance of the cut on the pole indicates the breakdown of the theory. $\alpha = 1$ is a limiting case, analogous to the zero photon mass.

A self-consistent evaluation of the effect of unitarity in the s channel constitutes a very difficult non-linear mathematical problem, whose solution we are not attempting in this paper. We expect, however, that the result of this more ambitious programme will give not only the presence of new singularities (the cuts), but also the renormalization of the old ones, in such a manner that the old leading pole trajectory might be essentially modified.

ACKNOWLEDGEMENTS

We wish to express our gratitude to Professors S. Drell, T. Regge and L. Van Hove for their interest in our work and many illuminating discussions. We are also very much indebted to our colleagues L. Bertocchi, C. Ceolin, F. Duimio, R. Stroppolini and M. Tonin for their friendly collaboration in the investigation of many points of this work.

APPENDIX 1

NON-LINEAR EQUATION FOR THE AMPLITUDE

1) Properties of a bilinear integral

In the first part of this Appendix we want to study some properties of our basic equation in which two amplitudes A_1 and A_2 are combined peripherally [cf. Fig. 18] :

Fig. 18

$$J = \iiint ds' ds'' du_1 du_2 \frac{K(s'u_1^2, s'u, u, s'') A_1(s'u, u_2) A_2(s''u, u_2)}{(u_1 + u^2)(u_2 + u^2)} \quad (A1.1)$$

where K is given by (VII.5), A_1 and A_2 , asymptotically, are given by :

$$\begin{aligned}
 A_1(s'u, u_2 t) &\sim C_1(t) \varphi_1(u, u_2 t) s^{\alpha_1(t)} \\
 A_2(s''u, u_2 t) &\sim C_2(t) \varphi_2(u, u_2 t) s^{\alpha_2(t)}
 \end{aligned}
 \tag{A1.2}$$

We shall show later that many interesting physical quantities are related to expressions similar to those of Eq. (A1.1). Let us therefore study the asymptotic properties of η as given by Eqs. (A1.1) and (A1.2). As a result :

$$\eta \sim \begin{cases} a_1(t) s^{\alpha_1(t)} + a_2(t) s^{\alpha_2(t)} & \text{if } \alpha_1 \neq \alpha_2 \\ & (\alpha_1 \text{ and } \alpha_2 > -1) \\ a(t) s^{\alpha(t)} \log s & \text{if } \alpha_1 = \alpha_2 > -1 \\ \frac{a(t)}{s} \log^2 s & \text{if } \alpha_1 = \alpha_2 = -1 \\ a_1(t) s^{\alpha_1(t)} + \frac{a_2(t)}{s} & \text{if } \alpha_2 < -1 \end{cases}$$

(A1.3a)

(A1.3b)

(A1.3c)

(A1.3d)

Because t is a parameter in Eq. (A1.1), we shall restrict the details of the derivation of Eq. (A1.3) to the case $t=0$, for which the formulae look slightly simpler.

For the case $t=0$, K is given by Eq. (III.11) under the name of Q ; let us then use its asymptotic expression for s, s' and s'' big, and introduce it, together with Eq. (A1.2), into (A1.1) - leaving until later the discussion of the error introduced by using the asymptotic expressions in the whole s', s'' range of integration in Eq. (A1.1).

We obtain :

$$J = \frac{C_1 C_2 \pi}{2s} \int \frac{du ds' ds'' \varphi_1(u) \varphi_2(u) s'^{x_1} s''^{x_2}}{(u + u')^2} \Theta(u's'^2 - ss's'' - \mu^2(s'^2 + s''^2)) \times \Theta(\sqrt{s} - \sqrt{s'} - \sqrt{s''}) \quad (A1.4)$$

remembering [cf. Section VII] that, for $t=0$, φ depends only on one mass variable. Introducing the variables

$$x = \frac{s's''}{s} \quad y = \frac{s'}{s''} \quad (A1.5)$$

$$J = s^{\frac{x_1+x_2}{2}} C_1 C_2 \frac{\pi}{4} \int_0^{\frac{s}{m^2}} \frac{du'}{y} \int_{\frac{m^2}{s}}^{\frac{s}{m^2}} \frac{dy}{y} y^{\frac{x_1-x_2}{2}} \int_{\frac{m^2}{s}}^{\frac{s}{m^2}} dx \Theta(u' - \frac{\mu^2 x}{s} (y + \frac{1}{y}) - x) \times \Theta\left(\left(\frac{s}{x}\right)^{1/4} - y^{1/4} - y^{-1/4}\right) \quad (A1.6)$$

where m^2 represents the minimum value s' or s'' can have.

Let us perform the integration over x : its result will be

$$\frac{X_{\max}}{1 + \frac{x_1+x_2}{2}}$$

where $x_{\max}$ is given by the first value of s that makes one of the arguments of the Θ functions vanish. $x_{\max}$ shall be a function of y ,

but it is easy to see that it will be equal to u (up to terms in $1/s$), except for y very near its upper limits, where its value shall change slightly. The limits of y correspond to s' and s'' very small, and there, anyhow, our integrand is badly evaluated. Forgetting for the moment this fact, for α_1 and $\alpha_2 > -1$, we have

$$J = \frac{\pi C_1 C_2}{(1 + \frac{\alpha_1 + \alpha_2}{2})(1 + \frac{\alpha_1 - \alpha_2}{2})} \left[\int du \gamma(u) \gamma_d(u) u^{1 + \frac{\alpha_1 + \alpha_2}{2}} \right] \left(\frac{s^{\alpha_1}}{m^{\alpha_1 - \alpha_2}} - \frac{s^{\alpha_2}}{m^{\alpha_2 - \alpha_1}} \right) \quad (A1.7a)$$

if $\alpha_1 \neq \alpha_2$

$$J = \frac{\pi C_1 C_2}{1 + \alpha} \left[\int du \gamma(u) \gamma_d(u) u^{1 + \alpha} \right] s^{\alpha} (\log s - \log m) \quad (A1.7b)$$

if $\alpha_1 = \alpha_2$

Let us now discuss the error made by considering the asymptotic behaviour of A_1 , A_2 and K in (A1.1). The contribution when both s' and s'' are small is obviously of the order $1/s$, giving therefore a negligible contribution, as compared to s^{α_1} and s^{α_2} (α_1 or $\alpha_2 > -1$). The main error shall come when one variable will be small, but the other one arbitrarily big. This region of integration is just the one studied in Section III, and we know that it reproduces the asymptotic behaviour of the amplitude, whose energy variable is allowed to be large. Therefore, the region of integration badly evaluated shall not change the behaviour $a_1 s^{\alpha_1} + a_2 s^{\alpha_2}$ (for $\alpha_1 \neq \alpha_2$), but shall contribute to it. This means that for $\alpha_1 \neq \alpha_2$ we cannot trust the coefficients of the asymptotic behaviour (a_1 and a_2) as given by (A1.7a). For the case $\alpha_1 = \alpha_2$ the error made will be smaller by a $\frac{1}{\log s}$ factor, as compared to the result of Eq. (A1.7b). This means that for $\alpha_1 = \alpha_2$ we can also trust the coefficient of the asymptotic behaviour $s^{\alpha} \log s$, as given by (A1.7b). We shall see later how the knowledge of such a coefficient will allow us to obtain an equation for the normalization of the amplitude $A(s, u)$.

Eqs. (A1.7a) and (A1.7b) correspond to the results (A1.3a) and (A1.3b) we wished to illustrate. The case of α'_1 and/or α'_2 smaller than -1 is trivial: the low values of s' and s'' give the dominant contributions of the integral and, therefore, the $1/s$ behaviour given in Eq. (A1.3d). The result (A1.3c) is also easy to obtain from (A1.6).

2) On the value of asymptotic total cross-sections

The integral equation we studied in Section IV was linear and asymptotically homogeneous. We were able to obtain from it the functional form of $A(s,u)$ [Eqs. (IV.4) and (IV.5)], but clearly not its normalized value. Or, in other words, having understood that our model can give us constant high energy total cross-section, it would be interesting to ask ourselves which magnitude these total cross-sections would have. In order to answer this question, we must obtain an asymptotic equation that would contain $A(s,u)$ in a non-linear form. We obtained a linear equation - (IV.3) - by using a recurrence relation - Eq. (III.12) - that separates the last link of the chain from the rest. We shall now find a new separation that can give us A in terms of an expression containing A itself more than once.

Let us use again expressions (III.1) and (III.2) for A . We note that G_n satisfies

$$G_n(q_1 \dots q_n; p p') = \frac{1}{8\pi^4} \int \frac{d^4 q_i G_{i-1}(q_1 \dots q_{i-1}; q' p) G_{n-i}(q_{i+1} \dots q_n; q' p')}{(q_i^2 - \mu^2)^2} \quad (\text{A1.8})$$

and therefore

$$\begin{aligned} n G_n(q_1 \dots q_n; p p') &= \\ &= \frac{1}{8\pi^4} \int \frac{d^4 q_i}{(q_i^2 - \mu^2)^2} \sum_{i=1}^n G_{i-1}(q_1 \dots q_{i-1}; q' p) G_{n-i}(q_{i+1} \dots q_n; q' p') \end{aligned} \quad (\text{A1.9})$$

or

$$\sum_n A_n(p, p') = \frac{1}{8\pi^4} \int \frac{d^4 q'}{(q'^2 - \mu^2)^2} \sum_{i=1}^n A_{i-1}(q', p) A_{n-i}(-q', p') \quad (\text{A1.10})$$

By summing over n and noting

$$\sum_{n=1}^{\infty} \sum_{i=1}^n A_{i-1} A_{n-i} = \sum_{i=0}^{\infty} \sum_{n=i+1}^{\infty} A_{i-1} A_{n-i} = A A \quad (\text{A1.11})$$

we find

$$\sum_n A_n(p, p') = \frac{1}{8\pi^4} \int \frac{d^4 q'}{(q'^2 - \mu^2)^2} A(q', p) A(-q', p') \quad (\text{A1.12})$$

and, using (III.5), (III.7) and (V.2)

$$\begin{aligned} \sum_0^{\infty} (n+1) A_n(s, u) &= \int \frac{\delta A(s, u)}{\delta A(s_0^R)} A(s_0^R) ds_0 = \\ &= A(s, u) + \frac{1}{8\pi^4} \iiint ds' ds'' du' \frac{Q(s, u, s', s'') A(s', u') A(s'', u')}{(u' + \mu^2)^2} \end{aligned} \quad (\text{A1.13})$$

where Q is given by (III.11).

Eq. (A1.12) can be represented graphically by Fig. 18, where the whole amplitude A is subdivided in any two groups of peripherisms.

We note the similarity of Eq. (A1.13) and the formula of Salzman: there are, however, two big differences. One is that in the left-hand side, we have not the amplitude itself, but essentially the amplitude times the

multiplicity $\langle N \rangle$. This is due to the fact that in the multiperipheral chain there are $\langle N \rangle$ possible ways of selecting the intermediate pion in the simple peripheral language. The next difference is that the $A(s', u')$ and $A(s'', u')$, in the integrand of (A1.13), are not related to cross-sections but have a non-negligible u' dependence, as we learnt in Section IV.

This is a theoretical analogous to the phenomenological cut-off procedure that was sometimes attempted¹⁸⁾. We note, finally, that equations of the type of (A1.13), relating the derivative of the amplitude with respect to a coupling constant to a quadratic expression of the amplitude itself, were already found and used in the literature. We shall not use now Eq. (A1.13) as an integral equation; we shall instead use it in order to find the constant C that normalizes our forward amplitude

$$A(s, u) = C s^\alpha \varphi(u) \quad (\text{A1.14})$$

with the convention $\varphi(-\mu^2) = 1$. For high-energy constant cross-sections ($\alpha=1$), C is just the actual value of the asymptotic $\pi-\pi$ total cross-section. We shall use (A1.13) for the value $u = -\mu^2$.

The introduction of (A1.14) into the first integral of (A1.13) was already studied in Section V; asymptotically, the dominant term was

$$s^\alpha \log s \left(\int A_R(s_0) \frac{\partial \alpha}{\partial A_R(s_0)} ds_0 \right)$$

We have already discussed in detail the evaluation of the integral appearing in the r.h.s. of (A1.13); its evaluation is given in (A1.7b). By equating the coefficient of the dominant $s \log s$ terms [cf. the discussion after Eq. (A1.7)], we obtain

$$\int \frac{\partial \alpha}{\partial A_R(s_0)} A_R(s_0) ds_0 = \frac{C}{(1+\alpha) 16\pi^3} \int_0^\infty \frac{u'^{1+\alpha} \varphi^2(u') du'}{(u' + \mu^2)^2} \quad (\text{A1.15})$$

Therefore, for $\alpha = 1$, i.e., constant high-energy cross-section, $C = \sigma_{\text{tot}}$ and from (A1.15) :

$$\sigma_{\pi\pi_{\text{tot}}} = 32\pi^3 \left[\int \frac{\partial \alpha}{\partial A_{(s_0)}^R} A_{(s_0)}^R ds_0 \right] \frac{1}{\int_0^\infty \left(\frac{u'}{u'+\mu^2} \right)^2 \varphi(u')^2 du'} \quad (\text{A1.16})$$

Taking for $\alpha(A^R)$ and $\varphi(u')$ the first approximations [Eqs. (IV.11) and (IV.12)], we find

$$\sigma_{\pi\pi_{\text{tot}}} \sim \frac{64\pi^3}{N} \sim \frac{2 \cdot 10^4 \text{ mb}}{N} \quad (\text{A1.17})$$

where, in the right-hand side of (A1.17), πN is expressed in units of μ . It is clear from (A1.17) that in order to obtain a reasonable $\pi-\pi$ cross-section of the order of some tenths mb, [cf. Eq. (IX.15)] N must be very big, i.e., of the order of $\sim 30\mu$, this is a too big number when compared with a rough evaluation of it, that can be done by the approximate way of solving the integral equation¹⁰⁾. Or, in other words, the solution of the integral equation would give in our model constant high energy total cross-sections too big in magnitude by a factor of the order of 10. This is a clear indication of the limitation of the model in this simple form as far as regards its possibility of predicting actual numbers.

Obviously, the model, as it stands, includes just the most peripheral processes. By asking only these processes to give constant high-energy cross-sections, it is clear that we shall obtain a big value for such cross-sections. This difficulty is intimately related to the limitations of our model, discussed in Part D of this paper. Because the elastic intermediate processes are less peripheral, inclusion of them among the processes whose sum gives the (constant) high-energy cross-section, would lower the actual value of such a cross-section.

The equation for the function $C(t)$ [Eq. (VII.8)] of the diffraction amplitude - similar to Eq. (A1.15) even if rather more complicated - is under investigation.

Another interesting application of the properties of an expression of the form of $\int$, as given by (A1.1), is the value of $\frac{d\alpha(t)}{dt}$ for $t=0$. In fact, it is possible to see ¹³⁾ that

$$\left(\frac{dA(s,t)}{dt}\right)_{t=0} = \iint \frac{ds' ds'' du}{(u+\mu')^4} A(s',u) A(s'',u) \left(\mu^2 + \frac{s's''}{2}\right) Q(s, \mu', s' u s'') \quad (\text{A1.18})$$

With a method similar to that used in paragraph 1) of this Appendix, one could obtain from (A1.18) :

$$\left(\frac{d\alpha(t)}{dt}\right)_{t=0} = \frac{\pi}{2} C \int du \frac{\varphi^2(u)}{(u+\mu')^2} \left(\mu^2 \frac{u^{1+\alpha}}{1+\alpha} + \frac{u^{2+\alpha}}{2+\alpha} \right) \quad (\text{A1.19})$$

which shows, by the way, that

$$\left(\frac{d\alpha(t)}{dt}\right)_{t=0} > 0$$

APPENDIX 2

In this Appendix we shall discuss the value of $\} _M$ as a function of t (i.e., the branch point in the angular momentum variable). We shall find it useful to introduce two symmetrical variables

$$\begin{cases} \xi = t' + t'' \\ \eta = t' - t'' \end{cases} \quad (\text{A2.1})$$

Let us investigate now the region in the ξ, η plane (or in the t', t'' plane) allowed to the integration in Eq. (X.4) by virtue of the Θ function present in the definition of $\widetilde{T}(t, t', t'')$. This region is represented in Fig. 19, where both the t', t'' and ξ, η axes are indicated

Fig. 19

and is bounded by the curve $\xi_{\max} = \frac{\eta^2 + t^2}{2t}$ if $t \neq 0$, while it degenerates on the negative ξ axis ($\eta = 0$) for $t=0$.

Due to the fact that $\alpha(t)$ is a monotonous decreasing function when t decreases from zero, it is easy to see that the maximum value of

$$\alpha(t') + \alpha(t'') = \alpha\left(\frac{\xi + \eta}{2}\right) + \alpha\left(\frac{\xi - \eta}{2}\right)$$

will be acquired on the boundary of the allowed region. It is then clear that

$$\xi_M(t) = \max_{\eta} F(t, \eta) \quad (\text{A2.2})$$

where

$$F(t, \eta) = \alpha\left(\frac{\eta - t}{2t}\right) + \alpha\left(\frac{\eta + t}{2t}\right) - 1 \quad (\text{A2.3})$$

and $\max_{\eta} F(t, \eta)$ means the maximum value of $F(t, \eta)$ for any η at a fixed value of t . Let us investigate $F(t, \eta)$. First it is clear that

$$\begin{aligned} F(t, 0) &= 2\alpha\left(\frac{t}{2}\right) - 1 \\ F(t, t) &= F(t, -t) = \alpha(t) + \alpha(0) - 1 \end{aligned} \quad (\text{A2.4})$$

On the boundary, $\eta = t$ ($\eta = -t$) corresponds to $t''=0$, $t'=t$ ($t'=0$, $t''=t$), while $\eta = 0$ corresponds to $t' = t'' = \frac{t}{2}$. Besides

$$\left(\frac{\partial F(t, \eta)}{\partial \eta}\right)_{\eta=0} = 0 \quad (\text{A2.5})$$

and if $t \neq 0$

$$\left(\frac{\partial F(t, \eta)}{\partial \eta} \right)_{\eta=t} = - \left(\frac{\partial F(t, \eta)}{\partial \eta} \right)_{\eta=-t} = \frac{d\alpha(t)}{dt} > 0 \quad (\text{A2.6})$$

From (A2.6) and (A2.4)

$$\zeta_M(0) \geq \zeta_M(t) > \alpha(t) + \alpha(0) - 1 \quad \text{for } t < 0 \quad (\text{A2.7})$$

while

$$\zeta_M(0) = 2\alpha(0) - 1 \quad (\text{A2.8})$$

The function $F(t, \eta)$ can be rather complicated. It is not even necessary that at the point $\eta = 0$ ($t' = t'' = \frac{t}{2}$), it has its maximum. Indeed, this point can even be a minimum.

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CM-P00056802

INVITED PAPER TO THE 1962 INTERNATIONAL
CONFERENCE ON HIGH-ENERGY PHYSICS, CERN

THEORY OF HIGH-ENERGY SCATTERING AND MULTIPLE PRODUCTION

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We wish to report about the different consequences of a model for high-energy interactions. This model has been suggested to us by the structure of the strip approximation to the Mandelstam representation and can be simply understood as a generalization to very high energy of the peripheral model. The basic idea is that the main contributions to multiple production are given by a combination of a large number of low-energy processes. The graphs we are considering are shown in Fig. 1 :

Fig. 1

Each bubble represents a low-energy two-body process. The number of multiperipheral graphs does, of course, increase with increasing energy. We wish to show that the sum of all multiperipheral effects exhibits, in the high-energy limit, particularly simple features, both for elastic scattering and for multiple production.

It is clear that the knowledge of the production amplitude allows to compute not only the production cross-section, but also the imaginary part of the elastic scattering amplitude through the unitarity relation [see Fig. 2]

$$A(p_1 p_1', p_2 p_2') = \langle p_1' p_2' | \mathcal{L}_n T | p_1 p_2 \rangle = \sum_n \langle p_1' p_2' | T^\dagger | n \rangle \langle n | T | p_1 p_2 \rangle \quad (1)$$

2.

Fig. 2

Let us consider the sum of all multiperipheral effects giving rise to A . This sum can be performed by making use of a recurrence relation which allows to compute the $(n+1)$ contribution, once the n^{th} is known. This recurrence relation is visualized in Fig. 3 :

Fig. 3

$$A_{n+1}(p, p', p_2, p_2') = \int d_4 q d_4 q' A_n(p, p', q, q') K(p, p', q, q') \quad (2)$$

Eq. (2) shows that A_{n+1} can be computed only when A_n is known not only on the mass shell, but in correspondence to all space-like values of the four momenta q, q' . The recurrence formula does indeed allow to calculate all multi-peripheral contributions in terms of the low-energy one A_0 . This procedure can be summarized by means of the integral equation

$$A(s, u, v, t) = A_0(s, u, v, t) + \frac{1}{s} \iiint ds' du' dv' Q_t(ss'uu'vv') A(s', u', v', t) \quad (3)$$

where

$$s = (p_1 + p_2)^2$$

$$u = p_2'^2 \quad v = p_2'^2$$

$$t = (p_1 - p_1')^2 = (q - q')^2 = (p_2 - p_2')^2$$

$$s' = (q + p_1)^2$$

$$u' = q^2 \quad v' = q'^2$$

The integral equation

The knowledge of the solution $A(s, u, v, t)$ of the integral equation is the fundamental problem of our work. Indeed, the on-mass-shell amplitude $A(s, -\mu^2, -\mu^2, t)$ leads to the elastic diffraction cross-section, whereas it will be shown that the forward off-mass shell amplitude $A(s, u, u, 0)$ leads to predictions concerning the average asymptotic properties of energy multiple production. In the asymptotic limit (high multiplicities), the integral equation reduces considerably. First,

the term A_0 can be dropped and the kernel turns out to depend only on the ratio s'/s , so that the equation is invariant under the transformation $s \rightarrow cs$, $s' \rightarrow cs'$. This allows us to factorize the s dependence of the amplitude in the simple form

$$A(s, u, v, t) = s^{\alpha(t)} \varphi(u, v, t) \quad (4)$$

The problem is then reduced to the solution of a homogeneous integral equation for $\varphi(u_1, u_2, t)$, whose solution determines both the exponent $\alpha(t)$ and the eigenfunction $\varphi(u_1, u_2, t)$. Both eigenvalues and eigenfunctions have a physical meaning: the eigenvalue gives the well-known shrinking of the diffraction peak, whereas the eigenfunction - as already pointed out - is connected with the average properties of multiple production. A form of the scattering amplitude analogous to Eq. (4) has been obtained by many people by adapting to high-energy scattering the results of Regge in potential theory. This analogy can be understood by considering that our multiperipheral graphs, observed in the crossed channel, are the relativistic analogous of the different iterations of the potential model used by Regge.

The predictions obtained by means of the model can be divided into two categories:

- a) many general trends of the high-energy collisions do only depend on the transformation property of the integral equation, which is a consequence only of the topology of the multiperipheral graphs.

- b) the specific numerical answers (like, for example, the value of the total cross-sections) do depend, of course, on the choice of A_0 and on the manner in which A_0 is continued off the mass shell. As is generally known, it has not yet been possible to find a completely satisfactory way of performing such a continuation, especially in the case of higher waves.

We have therefore concentrated our attention on the general model independent predictions, which we shall try to summarize now.

1) Elastic amplitude

The high-energy behaviour of the scattering amplitude $T(s,t)$ is given by

$$T(s,t) = -C(t) s^{\alpha(t)} \frac{e^{-i\pi\alpha(t)} \pm 1}{\sin \pi\alpha(t)} \quad (5)$$

where the first sign stands for amplitudes symmetric under crossing ($s \leftrightarrow \bar{s}$) as, for instance, absolute elastic scattering, while the second sign stands for antisymmetric amplitudes under the crossing. We obtain $\frac{d\alpha}{dt} > 0$. The exponent for the charge exchange amplitude is always smaller than the one for the purely elastic one. Eq. (5) turns out to be independent of the scattering particles, apart from the value of $C(t)$. The $C(t)$ can be factorized in such a manner that the relation between different amplitudes (dominated by the same pole) is the following :

$$\frac{T_{xy}(s,t)}{T_{zy}(s,t)} = \frac{T_{xw}(s,t)}{T_{zw}(s,t)}$$

where x, y, z and w represent any kind of particles.

2) Inelastic scattering

The average properties of multiple production are also easy to obtain. To find an average value, for instance, we must perform the sum over all multiperipheral processes with suitable weighing factors. The multiplicity, for instance, shall be given by

$$\langle N \rangle = \frac{\sum (n+1) A_n(s, 0)}{\sum A_n(s, 0)}$$

due to the fact that the n th peripherism gives just $n+1$ final states. The expression for $\langle N \rangle$ is indeed extremely simple to obtain :

$$\langle N \rangle = \frac{\int \frac{\delta A(s, 0)}{\delta A_0(s, 0)} A_0(s, 0) ds}{A(s, 0)} = c_1 \log s + c_2$$

If we wish to obtain the average spectra of secondaries, i.e., which is the number $dN_s(k)$ of secondaries with 4 momentum k : what must be done is to compute how many secondaries in any multiperipherism can have such a momentum and then sum over all multiperipherisms. The procedure is indeed simple and can be visualized in Eq. (6) and Fig. 4 :

$$N_s(k) = \iint A(s_1, u_1) A(s_2, u_2) \bar{I}(s, s_1, s_2, u_1, u_2, f) \quad (6)$$

Fig. 4

The actual evaluation leads to

$$dN_s(k) = F(M^2, \vec{k}_T^2) dM^2 d\vec{k}_T^2 \frac{dE_{lab}}{E_{lab}} \quad (7)$$

where M^2 represents the mass, $\vec{k}_T$ the transverse momentum, and E_{lab} the lab. energy of the secondary in question.

Eq. (7) has the feature that the spectrum of mass and transverse momentum is independent of both the incident and secondary energies (as suggested by experiments). The energy spectrum (for not too high or too low energies) is given by $\frac{1}{E_{lab}}$. The transverse spectrum is strongly peaked in the forward direction, and its actual shape $F(k_T^2)$ is model dependent.

3) Relation between asymptotic properties and bound states

All our equations, established for $t \leq 0$, can be continued to positive t . At $t = 4\mu^2$, α develops an imaginary part. Eq. (7) shows already that, for the value $t = t_B$ for which $\alpha(t_B)$ is an integer, $\text{Re}T$ develops a pole.

This means that the particles exchange a bound state (or resonance if $t_B > 4\mu^2$) of mass t_B and of spin $\alpha(t_B)$. This relation between asymptotic properties of scattering and bound states can be understood without making appeal to pole theory. It is possible to show, in fact, that our fundamental integral equation for α integer, written for $t > 0$, coincides with the Bethe-Salpeter ladder equation for a bound state of angular momentum α .

4) Cuts in the angular momentum variable

The weakest point of the whole approach is that unitarity in the s channel is not taken into account. A similar difficulty appears in all approaches based on the Regge theory, since in the potential model s means only momentum transfer, and therefore unitarity in this channel has no meaning. Now, Froissart has shown that unitarity in the s channel, together with the Mandelstam representation, do not allow cross-sections increasing more rapidly than $(\lg s)^2$. So this means that Eq. (4) is only acceptable for $\alpha(t) > 1$, for $t \leq 0$. Now both our equation and the potential model do not give any such limitation for $\alpha(t)$.

In order to have an idea of the effect of unitarity, we have tried to take into account the corrections due to elastic unitarity corresponding to effects of multiple scattering. These new terms show a new interesting feature : they give rise to continuous distributions of powers (cuts in the angular momentum) :

$$A_{mult.}(st) = \int_{s_{Min}}^{s_{Max}(t)} s^{\alpha(s,t)} \rho(s,t) ds$$

The position of the upper limit of integration for $t=0$ is simply given by :

$$s_{Max}(0) = 2\alpha(0) - 1$$

So the dominance of the pole on the cut depends on whether $\alpha(0)$ is larger or smaller than 1. For $\alpha(0) > 1$ the cut dominates on the pole, so the whole theory breaks down, as predicted by the Froissart theorem. In the $\alpha(0)=1$ case, which seems to be suggested by experiment, the cut and the pole coincide.

We wish to emphasize that we have not proved the existence of cuts in the angular momentum, but have just given arguments in favour of their existence. It is, however, interesting to note that these cuts enter in the theory in a non-trivial manner, and that they are a possible mechanism by which unitarity leads to the Froissart limitation in the cross-section.

In conclusion, we have seen that in the framework of our work, many characteristics of high-energy physics show simple regular features. One of such features concerns the Regge pole behaviour of elastic scattering. We want to stress, however, that many other characteristics as, for instance, multiplicities and spectra of production processes, show analogous regularities which - in our opinion - have exactly the same physical basis.

A very interesting confirmation of the results concerning relativistic elastic scattering has been obtained independently by B. Lee and R. Sawyer. These authors have succeeded in extending to the Bethe-Salpeter equation the methods of Regge and Blankenbecler-Goldberger for Yukawa scattering and have proven that the amplitude is meromorphic in the complex angular momentum half-plane $\text{Re } \ell > -\frac{3}{2}$.

An expression for $\alpha(t)$, in the case of weak coupling, is obtained, and coincides - in that limit - with the one obtained by Bertocchi, Fubini and Tonin.