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INTEGRAL EQUATION FOR HIGH-ENERGY PION-PION SCATTERING ^{+))}

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ABSTRACT

The structure of the high-energy scattering amplitude is studied in the framework of a model which is based on the choice of the most peripheral graphs. The imaginary part of the scattering amplitude is shown to satisfy an integral equation which has many analogies with the Bethe-Salpeter equation in the ladder approximation. In the high-energy limit it is possible to obtain a solution of our equation which exhibits the energy dependence explicitly.

The analogy of our results with the ones obtained on the basis of the extension of the Regge theory to high-energy scattering is discussed.

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One of the most important features of high energy pion physics is that elastic scattering is almost concentrated in a forward diffraction peak. Diffraction scattering is connected with the absorptive part $A(s,t)$ of the scattering amplitude and the **narrow** width of the diffraction peak is a strong indication that the main contribution to diffraction is due to the absorption of the incoming pions taking place peripherally in a region of the order $1/\mu$.

A theoretical scheme, based on these considerations, has been proposed recently in order to treat high energy pion phenomena ⁽¹⁾ .

This model assumes the dominance of the most peripheral graphs for production processes i.e., the graphs in which all final particles are produced directly either by the colliding pions or by the pion which is exchanged between the two incoming particles. This model allows one to compute all production amplitudes and at the same time, using unitarity, to compute the imaginary part of the elastic amplitude.

A general formulation of the model, together with the discussion of all its results has been made by Amati and al. ⁽²⁾ .

In this paper we wish to discuss in detail the mathematical problems arising from the application of the method to elastic processes. The simple case of forward scattering (related to the total cross section) has been investigated previously by Ceolin and al. ⁽³⁾

This work has been very important to guide us in a search for a solution to our more general problem.

In sect.2 we shall prove that the summation of all graphs of the model can be reduced to the solution of a linear integral equation in three variables.

In sect.3 we shall show that in the high energy limit the solution of the integral equation suggests a very general form for the energy dependence of elastic scattering. This form is the same as the one guessed by many authors ⁽⁴⁾ on the basis of the Regge ⁽⁵⁾ theory of complex angular momentum.

In sect.4 we shall consider a method of solving the integral equation. In sect.5 we shall discuss the relation of our results with the results of Regge in potential theory and with the general Bethe-Salpeter problem.

Finally we devote sect.6 to the discussion of the limits of validity of the model.

2°) THE INTEGRAL EQUATION FOR DIFFRACTION

In this section we shall obtain a linear integral equation for the imaginary part of the $\pi\pi$ scattering amplitude in the limit of very high energy. We start from the assumption that the dominant contribution to diffraction scattering comes from the ladder graphs of fig.1, wherein the particles with four-momenta q_i and q'_i are pions, and the quantities exchanged between these pions are definite groups of low energy particles.

Fig.1

We shall consider only the simple case in which these groups of particles have total angular momentum equal to zero. Moreover we shall neglect all the complications due to the isospin of the pions by considering them to be scalar particles in isospin space.

The following notations will be used (6)

$$Q_i = \frac{q_i + q'_i}{2}; \quad P = \frac{p_i + p'_i}{2}; \quad N = \frac{n_i + n'_i}{2}; \quad \Delta = \frac{q_i - q'_i}{2} = \frac{p_i - p'_i}{2} = \frac{n_i - n'_i}{2}$$

The scattering we are considering is that of pions p_1 and n_1 . Thus the square of the total energy in their center of mass system is $s = (p_1 + n_1)^2$ and the momentum transfer is $t = (p_1 - p'_1)^2 = 4\Delta^2$.

The square of the "mass" of the exchanged quantity is $s_i = (Q_i - Q_{i+1})^2$ with $s_o = (P - Q_1)^2$; $s_n = (Q_n + N)^2$.

The integral equation we shall obtain connects linearly the imaginary part of the scattering amplitude, $A(s, t, u_1, u_2)$ considered as a function of s, t , and of the square of the masses of the particles with four momenta p_1 and p'_1 , with the same imaginary part taken at fixed momentum transfer t . Therefore we shall fix the masses of particles n_1 and n'_1 to their physical value, but we allow the masses of particles p_1 and p'_1 to be unphysical and write

$$n_1^2 = n'^2_1 = \mu^2; \quad p_1^2 = -u_1; \quad p'^2_1 = -u_2$$

The imaginary part of the scattering amplitude at an energy s is the sum of the imaginary parts of all the ladder graphs of fig. I which are compatible with energy conservation.

Then according to ref.¹⁾, it can be written as follows

$$A(P, N, \Delta) = \sum_n A_n(P, N, \Delta) + A_0((P+N)^2) \quad (2.1)$$

where A_0 , which corresponds to the ladder graph with only one rung is the low energy contribution and is a function only of the total energy, whereas A_n , which is the contribution of the ladder graph with n rungs, is given by

$$A_n(P, N, \Delta) = \frac{2^n}{(2\pi)^{4n}} \cdot \frac{d_4 Q_1 \cdot A_0((P-Q_1)^2) d_4 Q_2 \cdot A_0((Q_1-Q_2)^2)}{[(Q_1-\Delta)^2-\mu^2][(Q_1+\Delta)^2-\mu^2][(Q_2-\Delta)^2-\mu^2][(Q_2+\Delta)^2-\mu^2]} \dots$$

$$\dots \frac{d_4 Q_{n-1} \cdot A_0((Q_{n-2}-Q_{n-1})^2) \cdot d_4 Q_n \cdot A_0((Q_{n-1}-Q_n)^2) \cdot A_0((Q_n+N)^2) \theta(\sqrt{s} - \sum_{i=1}^n \sqrt{s_i})}{[(Q_{n-1}-\Delta)^2-\mu^2][(Q_{n-1}+\Delta)^2-\mu^2][(Q_n-\Delta)^2-\mu^2][(Q_n+\Delta)^2-\mu^2]} \quad (2.2)$$

It is easy to verify that A_n satisfies the recursion formula

$$A_n(Q, N, \Delta) = \frac{2}{(2\pi)^4} \int d_4 Q' \frac{A_{n-1}(Q', N, \Delta) A_0((Q-Q')^2) \theta(\sqrt{(Q+N)^2} - \sqrt{(Q'+N)^2} - \sqrt{(Q-Q')^2})}{[(Q'-\Delta)^2-\mu^2][(Q'+\Delta)^2-\mu^2]} \quad (2.3)$$

By inserting formula (2.3) into (2.1) one obtains the following integral equation

$$A(P, N, \Delta) = A_0((P+N)^2) + \frac{2}{(2\pi)^4} \int d_4 Q \frac{A(Q, N, \Delta) A_0((P-Q)^2) \theta(\sqrt{s} - \sqrt{(Q+N)^2} - \sqrt{(P-Q)^2})}{[(Q-\Delta)^2-\mu^2][(Q+\Delta)^2-\mu^2]} \quad (2.4)$$

If one considers this equation in the forward direction for the scattering of particle p_1 on particle n_1 (i.e. $\Delta_\mu = 0$) the imaginary part A depends only from the four vectors P and N and eq. (2.4) reduces to the simpler integral equation, which was studied in ref.(3).

Instead of considering the imaginary part $A(P, N, \Delta)$ as a function of the four vectors P, N, Δ , one can consider it instead as a function of all the scalars one can form with P, N, Δ . The imaginary part is then a function of the invariants $s = (P+N)^2$; $t = (P-N)^2$; $u_1 = -(P+\Delta)^2$; $u_2 = -(P-\Delta)^2$.

Eq. (2.4) can then be transformed to the following form

$$A(s, t, u_1, u_2) = A_0(s) + \int ds_0 ds' du'_1 du'_2 \frac{K(s, s', t, s_0, u_1, u_2, u'_1, u'_2) A(s', t, u'_1, u'_2)}{(u'_1 + \mu^2)(u'_2 + \mu^2)} \quad (2.5)$$

where

$$K(s, s', t, s_0, u_1, u_2, u'_1, u'_2) = \frac{2}{(2\pi)^4} A_0(s_0) \cdot \int d_4 Q \delta((P-Q)^2 - s_0) \delta((Q+\Delta)^2 - u'_1) \cdot \delta((Q-\Delta)^2 - u'_2) \delta((Q+N)^2 - s') = \frac{1}{4(2\pi)^4} A_0(s_0) \frac{\theta(J)}{\sqrt{J}} \quad (2.6)$$

and

$$J = \begin{vmatrix} -t/2 & \frac{u'_1 - u'_2}{2} & \frac{u_1 - u_2}{2} & 0 \\ \frac{u'_1 - u'_2}{2} & u'_1 + u'_2 + t/2 & \frac{u_1 + u_2 + u'_1 + u'_2 + t}{2} + s_0 & s - \mu^2 + \frac{u'_1 + u'_2 + t}{2} \\ \frac{u_1 - u_2}{2} & \frac{u_1 + u_2 + u'_1 + u'_2 + t}{2} + s_0 & u_1 + u_2 + t/2 & s - \mu^2 + \frac{u_1 + u_2 + t}{2} \\ 0 & s - \mu^2 + \frac{u'_1 + u'_2 + t}{2} & s - \mu^2 + \frac{u_1 + u_2 + t}{2} & 2\mu^2 - \frac{t}{2} \end{vmatrix} \quad (2.7)$$

Eq. (2.5) is a linear integral equation in the three variables s, u_1, u_2 ; s is the square of the total energy for the scattering of p_1 on n_1 ; u_1 and u_2 are the squares of the four momenta of the exchanged pions. Thus the values of u_1 and u_2 characterize the virtuality of the exchanged pions. Since the model is based on the idea that only the ladder graphs are important at high energy in the diffraction region, it does of course make sense only if the small values of u_1 and u_2 are dominant.

We are interested in the asymptotic form of the solution of our integral equation when $s \rightarrow \infty$. In order to find this asymptotic solution we shall study a reduced form of the equation, which form is obtained by performing the asymptotic limit of equation (2.5).

The approximations we make in order to obtain the asymptotic limit of the equation are :

- a) we keep only such values of t and s_0 which are small in comparison with s ;
- b) we neglect u_1, u_x, u'_1, u'_2 with respect to s .

The approximation a) depends only on our choice of the value of t (which in the equation is only a parameter) and from the form of $A_0(s_0)$; if it vanishes sufficiently fast when

$s_0 \rightarrow \infty$, this approximation is justified.

The approximation b) can only be justified from the self-consistency of the reduced equation. For approximation b) to be valid it will be necessary that the reduced equation converges and that the solutions of this reduced equation go to zero sufficiently fast when u_1 and u_2 go to infinity.

Using these approximations one can neglect the ratios $u_1/s; u_2/s; u'_1/s; u'_2/s; t/s; m^2/s$ everywhere in the

kernel (2.7); furthermore in eq. (2.5) one can neglect $A_0(s)$ with respect to the integral term, and in the integral term one can replace the upper limit of the integration on s' , which is $(\sqrt{s} - \sqrt{s_0})^2$, with s , and the upper limits of the integrations on u_1' and u_2' with ∞ .

We obtain in this way the asymptotic form of the kernel (2.7) :

$$J = \begin{vmatrix} -\frac{t}{2} & \frac{u_1' - u_2'}{2} & \frac{u_1 - u_2}{2} & 0 \\ \frac{u_1' - u_2'}{2} & u_1' + u_2' + \frac{t}{2} & \frac{u_1 + u_2 + u_1' + u_2' + t}{2} + s_0 & \times \\ \frac{u_1 - u_2}{2} & \frac{u_1 + u_2 + u_1' + u_2' + t}{2} + s_0 & u_1 + u_2 + \frac{t}{2} & 1 \\ 0 & \times & 1 & 0 \end{vmatrix} = s^2 H \quad (2.8)$$

$$H = \left[- \left(\frac{u_1' - u_2'}{2} - \frac{u_1 - u_2}{2} x \right)^2 - t(1-x) \left(\frac{u_1' + u_2' - (u_1 + u_2)x}{2} - \frac{s_0 x}{1-x} + \frac{t}{4}(1-x) \right) \right]$$

where $x = s'/s$.

The asymptotic limit of eq. (2.5) can therefore be written

$$A(s, t, u_1, u_2) = \frac{1}{4(2\pi)^4} \int ds_0 A_0(s_0) \int_0^1 dx \int_0^\infty du_1' du_2' \frac{A(sx, t, u_1', u_2')}{(u_1'^2 + \mu^2)(u_2'^2 + \mu^2)} \frac{\Theta(H)}{\sqrt{H}} \quad (2.9)$$

By introducing the notation

$$B(s, t, u_1, u_2) = \frac{A(s, t, u_1, u_2)}{(u_1 + \mu^2)(u_2 + \mu^2)} \quad (2.10)$$

eq. (2.9) takes the form

$$(u_1 + \mu^2)(u_2 + \mu^2) B(s, t, u_1, u_2) = \frac{1}{4(2\pi)} \int_0^1 ds_0 A_0(s_0) \int_0^1 dx \int_0^\infty du_1' du_2' \cdot B(sx, t, u_1', u_2') \frac{\Theta(H)}{\sqrt{H}} \quad (2.11)$$

It turns out to be useful to perform now the following transformation of variables, which allows a simple geometrical interpretation to the kernel (2.8) to be given. We put

$$\begin{aligned} \mathcal{G} &= -t(1-x) \\ \mathcal{G}_1 &= u_1' - u_1 x - \frac{s_0 x}{1-x} \\ \mathcal{G}_2 &= u_2' - u_2 x - \frac{s_0 x}{1-x} \end{aligned} \quad (2.12)$$

so that

$$H(\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2) = \frac{1}{4} (-\mathcal{G}^2 - \mathcal{G}_1^2 - \mathcal{G}_2^2 + 2\mathcal{G}\mathcal{G}_1 + 2\mathcal{G}\mathcal{G}_2 + 2\mathcal{G}_1\mathcal{G}_2) \quad (2.13)$$

$\frac{1}{2} \sqrt{H}$ represents the area of the plane triangle, whose sides are $\sqrt{\mathcal{G}}$, $\sqrt{\mathcal{G}_1}$, $\sqrt{\mathcal{G}_2}$.

In the following we shall use the notation

$$\frac{\Theta(H(\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2))}{\sqrt{H(\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2)}} = 2 T(\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2) \quad (2.14)$$

In the variables q_1, q_2 eq. (2.11) becomes

$$(u_1 + \mu^2)(u_2 + \mu^2) B(s, t, u_1, u_2) = \frac{1}{4(2\pi)^4} \int ds_0 A_0(s_0) \cdot \int_0^1 dx \int d\varphi_1 d\varphi_2 \cdot 2 T(\varphi_1, \varphi_2) \cdot B(sx, t, q_1 + u_1 x + \frac{s_0 x}{1-x}, q_2 + u_2 x + \frac{s_0 x}{1-x}) \quad (2.15)$$

One can easily verify the following identity, which will often be useful

$$T(\varphi_1, \varphi_2) = 2 \int d_2 q \delta((p+q)^2 - \varphi_1) \delta((p-q)^2 - \varphi_2) \quad (2.16)$$

where p is a two dimensional vector such that $4p^2 = \varphi = -t(1-x)$

Using (2.16), eq.(2.15) can be expressed in the useful form

$$(u_1 + \mu^2)(u_2 + \mu^2) B(s, t, u_1, u_2) = \frac{1}{(2\pi)^4} \int ds_0 A_0(s_0) \cdot \int_0^1 dx \int d_2 q B(sx, t, (p+q)^2 + u_1 x + \frac{s_0 x}{1-x}, (p-q)^2 + u_2 x + \frac{s_0 x}{1-x}) \quad (2.17)$$

An interesting point concerning the asymptotic limit is to consider what happens in the forward direction where we know that the kinematical conditions impose that $V = u_1 - u_2 = 0$. If we consider the kinematics of the problem more closely, we see that $|V| < \sqrt{ts}$. This means that if we set $s \rightarrow \infty$ before $t \rightarrow 0$ the integral equation we are considering no longer contains this limitation.

In fact if we take the limit for $t = 0$ of our asymptotic equation (2.17) we obtain

$$(u_1 + \mu^2)(u_2 + \mu^2) B(s, u_1, u_2, t=0) = \frac{\pi}{(2\pi)^4} \int ds_0 A_0(s_0) \int_0^1 dx \int d_2 q B(sx, 0, q^2 + u_1 x + \frac{s_0 x}{1-x}, q^2 + u_2 x + \frac{s_0 x}{1-x}) \quad (2.18)$$

in which u_1 can be different from u_2 . Therefore the limitation $V = 0$ for $t = 0$ has to be imposed from outside as a supplementary condition.

The consistency of this procedure is that eq. (2.18) connects the amplitude for $V = 0$ only with itself; in other words we can say that the operator kernel of eq. (2.17) "commutes" with the supplementary condition. Setting now $B(s, u, u) = \tilde{B}(s, u)$ we obtain

$$(u + \mu^2)^2 \tilde{B}(s, u) = \frac{\pi}{(2\pi)^4} \int ds_0 A_0(s_0) \int_0^1 dx \int_0^\infty dq^2 \tilde{B}(sx, q^2 + ux + \frac{s_0 x}{1-x}) \quad (2.19)$$

which coincides with the one-dimensional equation studied in ref. 3).

3°) THE ENERGY DEPENDENCE OF THE SCATTERING AMPLITUDE

In the present section we discuss the general structure of the scattering amplitude as given by eq. (2.12).

First of all we notice that eq. (2.9) is a linear homogeneous integral equation, in which the momentum transfer t is only a fixed parameter. Thus this equation allows one to determine $A(s, t, u_1, u_2)$ only apart from an arbitrary multiplicative function $N(t)$ of the momentum transfer. Hence from the solution of eq. (2.9) we cannot predict the form of the diffraction peak, but only the dependence of the form of this peak on the total energy s . The function $N(t)$ can be determined only by using a non linear integral equation of the kind of the one considered in ref. (2); this problem will be considered in a subsequent paper.

Let us study now the s dependence of $A(s, t, u_1, u_2)$.

From eq. (2.9) one clearly sees that this dependence is factorable. If one inserts the ansatz

$$A(s, t, u_1, u_2) = s^\alpha \varphi(u_1, u_2, t) \quad (3.1)$$

into (2.9), this form reproduces itself, and the function $\varphi(u_1, u_2, t)$ satisfies the following homogeneous linear integral equation

$$\varphi(u_1, u_2, t) = \frac{1}{2(2\pi)^4} \int ds_0 A_0(s_0) \int_0^1 dx \int_0^\infty \frac{du'_1 du'_2 x^\alpha \varphi(u'_1, u'_2, t)}{(u'_1 + \mu^2)(u'_2 + \mu^2)} T(y_1, y_2, y_3) \quad (3.2)$$

The combination of the form (3.1) for the amplitude and the homogeneous eq. (3.2) constitutes the fundamental point of our paper.

Eq. (3.2) is an homogeneous linear integral equation of the Fredholm kind and thus gives rise to an eigenvalue problem (7). If the weighting function $A_0(s_0)$ is known then, at a fixed value of the parameter t the equation is satisfied only in correspondence with well defined values of α . In particular we are interested in the largest possible α this giving the leading term at high energy. In this manner eq. (3.1) determines the exponent α as a function of the momentum transfer.

The general properties of eq. (3.2) will be discussed in sect. 4), where also an approximate derivation of $\alpha(t)$ will be given.

Eq. (3.1) shows that diffraction scattering does not have a purely energy independent form, but that its energy dependence is controlled by the fundamental function $\alpha(t)$. The value $\alpha(0)$ fixes the high energy behaviour of the total cross section, and the t dependence of $\alpha(t)$ gives the dependence

of the width of the diffraction peak on the energy. Indeed if $\frac{d\alpha}{dt}$ is positive ($\alpha(t) < \alpha(0)$ in the physical region in which $t < 0$) the diffraction peak shrinks logarithmically when s increases.

The approximate solution we will obtain in sect.4) shows that $\frac{d\alpha}{dt}$ is positive. This property can actually be proven rigorously (see app.A) in the case of $t = 0$.

We now turn to the question of determining the asymptotic form of the real part of the scattering amplitude $D(s,t)$. The real part $D(s,t)$ can be obtained from the imaginary part $A(s,t)$ by means of a fixed t dispersion relation.

In this dispersion relation we must take the contributions of the singularities in the s -channel as well as in the crossed channel ($s \leftrightarrow \bar{s}$). If the two colliding particles are two neutral pions, as in our simplified model, the real part $D(s,t)$ must be symmetric with respect to the exchange $s \leftrightarrow \bar{s}$, and therefore the crossed term must be added to the uncrossed one.

If the asymptotic form of $A(s,t)$ for $s \rightarrow \infty$ is $\varphi(t) s^{\alpha(t)}$, ref. 8), as in formula (3.1), we must write for $A(s,t)$ a dispersion relation with m subtractions, where m is the minimum integer greater than $\alpha(t)$.

Therefore

$$D(s,t) = \frac{s^{-m}}{\pi} P \int_{-\mu^2}^{\infty} \frac{A(s',t) ds'}{s'^m (s'-s)} + \frac{\bar{s}^{-m}}{\pi} \int_{-\mu^2}^{\infty} \frac{A(\bar{s}',t) d\bar{s}'}{\bar{s}'^m (\bar{s}'-\bar{s})} +$$

(3.3)

+ polynomials in s and $\bar{s}$,

where the maximum power of s and $\bar{s}$ appearing in the polynomials is $m-1$. From (3.3) we obtain the asymptotic behaviour of $D(s,t)$ by using the asymptotic form (3.1) for $A(s,t)$, by making the following approximations in (3.3): first of all, we substitute for $A(s,t)$ its asymptotic form (3.1) arguing that terms whose asymptotic behaviour is smaller than (3.1) cannot contribute to the asymptotic form of $D(s,t)$. Then we extend the integration ranges in (3.3) from 0 to ∞ , since the contribution of the integration between 0 to $4\mu^2$ is negligible, owing to the fact that the integrals do not contain any infra-red divergence (i.e., for $s' \rightarrow 0$). Finally, since the asymptotic behaviour of the integrals is $s^{\alpha(t)}$, we neglect in (3.3) the subtraction polynomials (whose maximum power is $m-1$).

Since in the asymptotic limit $s = -\bar{s}$, the asymptotic form of $D(s,t)$ becomes

$$D(s,t) = s^m \frac{\varphi(t)}{\pi} p \int_0^\infty \frac{ds'}{s'^{m-\alpha}(s'-s)} + (-1)^m s^m \frac{\varphi(t)}{\pi} \int_0^\infty \frac{ds'}{s'^{m-\alpha}(s'+s)} \quad (3.4)$$

The crossed term can be easily calculated, and gives as a result

$$D(s,t)_{\text{cross}} = (-1)^m \frac{\varphi(t)}{\pi} s^{\alpha(t)} \int_0^\infty dx \frac{x^{\alpha-m}}{1+x} = (-1)^m \frac{\varphi(t) s^{\alpha(t)}}{\sin \pi(\alpha+1-m)} = - \frac{\varphi(t) s^{\alpha(t)}}{\sin \pi \alpha(t)} \quad (3.5)$$

The uncrossed term can similarly be evaluated after some calculations, and gives

$$D(s,t)_{\text{uncross}} = - \varphi(t) s^{\alpha(t)} \cot \pi \alpha(t) \quad (3.6)$$

Therefore

$$D(s,t) = -\varphi(t) s^{\alpha(t)} \left(\frac{1 + \cos \pi \alpha(t)}{\sin \pi \alpha(t)} \right) = -\varphi(t) s^{\alpha(t)} \cotg \frac{\pi \alpha(t)}{2} \quad (3.7)$$

In this way we are led to the conclusion that the real part of the scattering amplitude has poles for any even integer value of $\alpha(t)$, and vanishes when $\alpha(t)$ is odd integer. This last property is nothing other than the Pauli principle for the scattering of equal particles in the t -channel. It is interesting to notice that if in the dispersion relation (3.4) we had not taken into account the crossed channel, we would have obtained a real part proportional to $\cotg \pi \alpha$, with poles for each $\alpha(t)$ integer.

Finally then, we obtain from our model the asymptotic expression for the scattering amplitude

$$F(s,t) = -\varphi(t) s^{\alpha(t)} \frac{e^{-i\pi \frac{\alpha(t)}{2}}}{\sin \frac{\pi \alpha(t)}{2}} \quad (3.8)$$

In the real case, in which the isospin of the pions is also taken into account, we must also consider scattering amplitudes antisymmetric under the exchange $s \longleftrightarrow \bar{s}$. In that case the crossed term must be subtracted from the uncrossed one and formulae (3.7) and (3.8) become

$$D(s,t) = \varphi(t) s^{\alpha(t)} \left(\frac{1 - \cos \pi \alpha(t)}{\sin \pi \alpha(t)} \right) = \varphi(t) s^{\alpha(t)} \tg \frac{\pi \alpha(t)}{2} \quad (3.9)$$

$$F(s,t) = i \varphi(t) s^{\alpha(t)} \frac{e^{-i\pi \frac{\alpha(t)}{2}}}{\cos \frac{\pi \alpha(t)}{2}} \quad (3.10)$$

Therefore this scattering amplitude has poles for any odd value of $\alpha(t)$, and its real part has zeros for $\alpha(t)$ even.

Formulae (3.8) and (3.10) are identical to those conjectured by many authors ⁽⁴⁾, under the assumption that the results of Regge (5) in potential theory may be extended to field theory.

It is interesting to note that our formulae have been obtained as asymptotic limits of solutions of an ordinary integral equation and without having to use any analytic continuation procedure. In other words we have checked on a well defined model that the asymptotic behaviour is really the one of eqs. (3.8) and (3.10).

Our result is therefore a strong argument in favour of the validity of the hypothesis that the results of Regge may be safely transferred from potential scattering to relativistic theory.

In sect.5 we will discuss in more detail the relationship between our results and the results of Regge.

Using the form (3.1) for the imaginary part of the scattering amplitude, we can now calculate by means of the optical theorem the total inelastic cross section

$$\sigma_{\text{in}}(s) = \frac{A(s,0)}{s} = \varphi(0) s^{\alpha(0)-1} \quad (3.11)$$

This is only the inelastic contribution to the total cross section since we have used only the ladder graphs of fig.1 in obtaining (3.1). We can get a better expression for the total cross section by adding the inelastic contribution to the elastic one, the latter being obtained from the form (3.8) for the scattering amplitude.

The elastic cross section is given by $\sigma_e = \frac{1}{32\pi} \int_{-1}^{+1} \frac{|F|^2}{s} d(\cos \theta)$

Using (3.8) we have then

$$\sigma_{el} = \frac{1}{16\pi} \int_{-t_{max}}^0 \frac{2(\alpha(t)-1) \phi^2(t)}{\sin^2 \frac{\pi \alpha(t)}{2}} dt \quad (3.12)$$

where t_{max} is the maximum value of the momentum transfer i.e., corresponding to $\cos \theta = -1$. The asymptotic behaviour of the elastic total cross section is therefore

$$\sigma_{el}(s) \propto \frac{s^{2(\alpha(0)-1)}}{\log s} \quad (3.13)$$

where we have made use of the fact that $\alpha(t)$ is an increasing function so that $\alpha(0)$ is the maximum value of $\alpha(t)$ in the range of integration. We can see from (3.13) that if $\alpha(0) < 1$, then the elastic total cross section decreases asymptotically faster than the inelastic total cross section; if $\alpha(0) = 1$, however, there is only a logarithmic difference in the two asymptotic behaviours.

The experimental evidence for the high energy total cross section indicates that the actual value of $\alpha(0)$ is not very different from 1.

4°) THE EIGENVALUE PROBLEM

In order to find the function $\alpha(t)$ it is necessary to solve the eigenvalue equation (3.2).

As we have previously discussed, $\alpha(t)$ can be determined by the eigenvalue condition from the knowledge of the function

$A_0(s_0)$. It will be useful now to express $A_0(s_0)$ as a product of a coupling constant g^2 times a known function $f(s_0)$ normalized to 1

$$\frac{A_0(s_0)}{2(2\pi)^4} = g^2 f(s_0)$$

Then the eigenvalue condition for $t = 0$ determines the coupling constant g^2 from the value of $\alpha(0)$, which is assumed to be known from the asymptotic behaviour of the total cross section, as in ref. (3).

Our eigenvalue condition expresses therefore $\alpha(t)$ as a function of the coupling constant g^2 .

In this section it will be useful to define a function

$$\psi(u_1, u_2, t) = \frac{\varphi(u_1, u_2, t)}{(u_1 + \mu^2)(u_2 + \mu^2)} \quad (4.1)$$

which satisfies the equation

$$(u_1 + \mu^2)(u_2 + \mu^2) \psi(u_1, u_2, t) = g^2 \int_0^1 ds_0 f(s_0) \int_0^1 dx x^{\alpha(t)} \cdot \int_0^\infty du' du'_2 \psi(u'_1, u'_2, t) T(\varphi, \varphi_1, \varphi_2) \quad (4.2)$$

The method we shall use is analogous to the one used in refs.(3) and (9), and is based on trying to solve the equation by using the following representation

$$\psi(u_1, u_2, t) = \int da db \frac{g(a, b, t)}{(u_1 + a)(u_2 + b)} \quad (4.3)$$

In this way we are led to an inhomogeneous integral equation for the weight function $g(a, b, t)$, supplemented with

3 integral conditions, which exhibit explicitly the eigenvalue condition contained in eq. (4.2).

In fact, substituting (4.3) into (4.2), we obtain

$$\begin{aligned} & \int da db g(a, b, t) + \int da db \frac{(\mu^2 - a)}{(u_1 + a)} g(a, b, t) + \\ & + \int da db \frac{(\mu^2 - b)}{(u_2 + b)} g(a, b, t) + \int da db \frac{(\mu^2 - a)(\mu^2 - b)}{(u_1 + a)(u_2 + b)} g(a, b, t) = \\ & = g^2 \int ds_0 f(s_0) \int_0^1 dx x^{\alpha(t)} \int da' db' \frac{K(a, b, a', b', t, x, s_0)}{(u_1 + a)(u_2 + b)} g(a', b', t) \end{aligned} \quad (4.4)$$

where

$$K(a, b, a', b', t, x, s_0) = T\left(ax - a' - \frac{s_0 x}{1-x}, bx - b' - \frac{s_0 x}{1-x}, -t(1-x)\right)$$

Eq. (4.4) is equivalent to the following equation for the weight function $g(a, b, t)$

$$\begin{aligned} g(a, b, t) = & N \delta(a - \mu^2) \delta(b - \mu^2) + \varphi(a) \delta(b - \mu^2) + \varphi(b) \delta(a - \mu^2) + \\ & + \frac{g^2 \int ds_0 f(s_0) \cdot \int_0^1 dx x^{\alpha(t)} \int da' db' K(a, b, a', b', t, x, s_0) g(a', b', t)}{(a - \mu^2)(b - \mu^2)} \end{aligned} \quad (4.5)$$

with the conditions

$$(a - \mu^2) \int db g(a, b, t) = 0 \quad (4.6')$$

$$(b - \mu^2) \int da g(a, b, t) = 0 \quad (4.6'')$$

$$\int da db g(a, b, t) = 0 \quad (4.6''')$$

Eq. (4.5) is obtained by equating in the two sides of (4.4) the discontinuity across the singularities in the (u_1, u_2) plane. The conditions (4.6) must be imposed on the solution

of eq. (4.5) in order that it will be equivalent to eq. (4.4), and they imply that $\psi(u_1, u_2, t) \rightarrow 0$ faster than $1/u_1$ and $1/u_2$ when u_1 and $u_2 \rightarrow \infty$.

Let us now consider for a moment the general structure of eq. (4.5).

The kernel $K(a, b, a', b', t, x, s_0)$ is of the Volterra kind and possesses the spectral property of connecting the function $g(a', b', t)$ with the function $g(a, b, t)$ at the points

$$a \geq (\sqrt{a'} + \sqrt{s_0})^2; \quad b \geq (\sqrt{b'} + \sqrt{s_0})^2$$

where s_0 is the least possible value of s .

If we consider as given the constant N and the functions $\varphi(a)$, $\varphi(b)$, then by solving iteratively eq. (4.5) we would determine $g(a, b, t)$ in any finite point of the (a, b) plane by means of a finite number of iterations; the Volterra nature of the kernel would then assure the convergence of the solution.

To this solution one must then impose to satisfy the conditions (4.6); conditions (4.6') and (4.6'') read

$$\begin{aligned} \varphi(b) &= -g^2 \int ds_0 f(s_0) \int_0^1 dx x^{\alpha(t)} \int da da' db' K(a, b, a', b', t, x, s_0) \frac{g(a', b', t)}{(a - \mu^2)(b - \mu^2)} \\ \varphi(a) &= -g^2 \int ds_0 f(s_0) \int_0^1 dx x^{\alpha(t)} \int db da' db' K(a, b, a', b', t, x, s_0) \frac{g(a', b', t)}{(a - \mu^2)(b - \mu^2)} \end{aligned} \quad (4.7)$$

When in (4.7) one inserts for $g(a, b, t)$ the iteration solution obtained starting with N , $\varphi(a)$, $\varphi(b)$, these equations become inhomogeneous integral equations for the functions $\varphi(a)$, $\varphi(b)$.

The problem of solving eq. (4.2) is thus reduced to the solution of a one-dimensional integral equation. Unfortunately it is not possible to solve this equation with a method similar to the previous one since this new equation is not of the Volterra kind.

Therefore, in order to have an idea of the general features of the solution and of the form of the eigenvalue condition, we shall use a provisional method, which consists in incorporating the conditions (4.7) in eq.(4.5). That is, we write

$$g(a, b, t) = N \delta(a - \mu^2) \delta(b - \mu^2) + g^2 \int ds_0 f(s_0) \int dx x^{\alpha(t)} \cdot \int da' db' \bar{K}(a, b, a', b', t, x, s_0) g(a', b', t). \quad (4.8)$$

where the kernel $\bar{K}$ is given by

$$\bar{K}(a, b, a', b', t, x, s_0) = \frac{K(a, b, a', b', t, x, s_0)}{(a - \mu^2)(b - \mu^2)} - \delta(b - \mu^2) \int dc \frac{K(a, c, a', b', t, x, s_0)}{(a - \mu^2)(c - \mu^2)} - \delta(a - \mu^2) \int dc \frac{K(c, b, a', b', t, x, s_0)}{(c - \mu^2)(b - \mu^2)} \quad (4.9)$$

In eq. (4.8) we have only one inhomogeneous term

$N \delta(a - \mu^2) \delta(b - \mu^2)$ and the solution must satisfy the condition (4.6').

We try now to solve this equation with an iteration method, even though the kernel is no more of the Volterra kind,

and therefore the convergence of the series is not assured.

In this way we obtain (10)

$$\begin{aligned}
 g(a, b, t) &= \sum_0^{\infty} g_n(a, b, t) \\
 g_0(a, b, t) &= N \delta(a - \mu^2) \delta(b - \mu^2) \\
 g_1(a, b, t) &= N g^2 \int_0^1 ds_0 f(s_0) \int_0^1 dx x^{\alpha(t)} \bar{K}(a, b, \mu^2, \mu^2, t, x, s_0) \\
 g_2(a, b, t) &= N g^4 \int_0^1 ds_0 f(s_0) \int_0^1 dx x^{\alpha(t)} \int_0^1 da_1 db_1 \bar{K}(a, b, a_1, b_1, t, x, s_0) \cdot \\
 &\quad \cdot \int_0^1 ds'_0 f(s'_0) \int_0^1 dx_1 x_1^{\alpha(t)} \bar{K}(a_1, b_1, \mu^2, \mu^2, t, x_1, s'_0) \cdot \\
 &\quad \vdots \\
 g_n(a, b, t) &= N g^{2n} \int_0^1 ds_1 ds_2 \dots ds_n f(s_1) f(s_2) \dots f(s_n) \cdot \int_0^1 dx_1 dx_2 \dots dx_n \cdot \\
 &\quad \cdot (x_1 x_2 \dots x_n)^{\alpha(t)} \cdot \int_0^1 da_1 db_1 da_2 db_2 \dots da_{n-1} db_{n-1} \cdot \\
 &\quad \cdot \bar{K}(a, b, a_1, b_1, t, x_1, s_1) \cdot \bar{K}(a_1, b_1, a_2, b_2, t, x_2, s_2) \dots \bar{K}(a_{n-1}, b_{n-1}, \mu^2, \mu^2, t, x_n, s_n).
 \end{aligned} \tag{4.10}$$

The third condition, (4.6'''), becomes now the following condition, to be satisfied for each value of t

$$N = g^2 \int_0^1 ds_0 f(s_0) \int_0^1 dx x^{\alpha(t)} \int_0^1 da db da' db' \frac{K(a, b, a', b', t, x, s_0) g(a', b', t)}{(a - \mu^2)(b - \mu^2)} \tag{4.11}$$

Using the property (2.16) of the function $T(\varphi, \varphi_1, \varphi_2)$

$$\begin{aligned}
 K(a, b, a', b', t, x, s_0) &= T(ax - a' - \frac{s_0 x}{1-x}, bx - b' - \frac{s_0 x}{1-x}, -t(1-x)) = \\
 &= 2 \int d_2 q \delta((p+q)^2 - (ax - a' - \frac{s_0 x}{1-x})) \delta((p-q)^2 - (bx - b' - \frac{s_0 x}{1-x}))
 \end{aligned}$$

$$4p^2 = -t(1-x)$$

using Feynman parametrization, and integrating on $d_2 q$ one can rewrite eq. (4.11) in the following form

$$N = 2\pi g^2 \int_0^1 dx x^{\alpha(t)} \int ds_0 f(s_0) \int da' db' g(a', b', t) \cdot \int_{-1}^{+1} dz \frac{1}{\left[-\frac{t}{4}(1-x)(1-z^2) + (a'-\mu^2)\frac{(1+z)}{2} + (b'-\mu^2)\frac{(1-z)}{2} + \frac{s_0 x}{1-x} + \mu^2(1-x) \right]} \quad (4.12)$$

This is the eigenvalue condition, and relates the function $\alpha(t)$ to the coupling constant g^2 .

If we insert the iterative solution (4.10) into (4.12), we obtain as first approximation for the eigenvalue condition

$$1 = 2\pi g^2 \int_0^1 dx \frac{x^{\alpha(t)}}{(1-x)} \int ds_0 f(s_0) \int_{-1}^{+1} dz \frac{1}{\left[-\frac{t}{4}(1-z^2) + \mu^2 + \frac{s_0 x}{(1-x)^2} \right]} \quad (4.13)$$

This approximate expression is convergent for $\alpha(t) > -1$, and this fact suggests that the complete eigenvalue condition should also converge for $\alpha(t) > -1$.

Furthermore from this expression one can easily see that $\alpha(t)$ is indeed an increasing function not only for $t = 0$ but also for each value of $t < 0$.

5) RELATIVISTIC POTENTIAL MODEL AND REGGE POLES

In this paper we have studied the application to diffraction scattering of a model which assumes that the high energy phenomena are dominated by the graphs of fig.1.

The result we have obtained is that at the asymptotic limit $s \rightarrow \infty$ the dependence of the elastic scattering amplitude on the total energy is

$$F(s, t) = -\varphi(t) \int \frac{e^{\alpha(t) - i\pi \frac{\alpha(t)}{2}}}{\sin \pi \frac{\alpha(t)}{2}} \quad (5.1)$$

Moreover our approach allows one to calculate explicitly the function $\alpha(t)$, which is determined as the solution of an eigenvalue problem as discussed in sect.4.

We want now to discuss the relevance and the physical meaning of our results. We have already noticed the striking analogy between the expression (5.1) and the results of Regge ⁵⁾ in potential theory. The same formula has in fact been obtained by Gell-Mann and al. ⁽⁴⁾ by postulating the extension of the Regge results to field theory in the crossed channel.

In fact, in the non-relativistic theory, the contribution to the scattering amplitude of the poles in the complex angular momentum (Regge poles) can be considered as dominant at low energy and very large scattering angles. If we look to our scattering amplitude in the crossed t-channel, this means small positive values of t and large negative values of s. The postulate is the extension of the dominance of Regge poles also to asymptotic diffraction scattering, in which t is small and negative, and s large and positive.

What we have so achieved as a result of our theory of diffraction is essentially the extension of the Regge theory to a relativistic case. In fact if we consider the graphs of fig.I in the t-channel, they represent a ladder approximation for the elastic scattering of the two pions n_1 and n'_1 , in which the covariant potential acting between them is given by the exchange of well defined groups of low energy particles. The eigenvalue condition (4.1) then allows one to evaluate explicitly the trajectory of the Regge pole as a function of low energy parameters.

Therefore, although our results have been obtained only for space-like t ($t < 0$) (a region which has no analogous in the nonrelativistic theory), one is tempted to extend our results also to time-like values of t .

We notice that if $\alpha(t)$ assumes even integer values for some values t_b of t in the interval $0 \leq t \leq 4\mu^2$, then for such values of t $\sin \pi \frac{\alpha(t)}{2}$ vanishes. Therefore for t near to t_b , $\sin \pi \frac{\alpha(t)}{2} \approx (t - t_b)$ and the scattering amplitude has a pole for $t = t_b$. The meaning of these poles is obvious: they represent the possible two pion bound states, in which the binding force is given by the exchange between the two pions of groups of low energy particles. These bound states have mass $\sqrt{t_b}$, spin $\alpha(t_b)$ and coupling constant $\phi(t_b)$ with the two pion system. Furthermore if we consider the eigenvalue condition (4.13) in the positive t interval, we see that for $t \geq 4\mu^2$ the denominator of the eigenvalue integral can vanish, and this integral therefore develops an imaginary part. As a consequence of this fact $\alpha(t)$ becomes complex in the region $t \geq 4\mu^2$. This result is also in accordance with the nonrelativistic theory of Regge, in which in the bound state region $\alpha(t)$ is purely real, and in the scattering region $\alpha(t)$ becomes complex. In the region $t \geq 4\mu^2$ the values t_n of t such that $\text{Re } \alpha(t_n)$ is even integer correspond to two pion resonances, with mass $\sqrt{t_n}$, spin $\text{Re } \alpha(t_n)$, and width related to $\text{Im } \alpha(t_n)$.

From the previous discussion it follows that our method, continued to positive values of t , allows one (at least in principle) to calculate the masses of the two pion resonances, arising from the ladder graphs we have taken into account for calculating diffraction scattering. Therefore, if this extension is possible, there must be a strict connection between our equation for diffraction and the Bethe-Salpeter equation

for a bound state, with an equivalent ladder potential. This turns out to be indeed the case. For $t=0$ and $\alpha(t)$ integer Ceolin and al.⁽³⁾ have in fact shown that the asymptotic equation (2.19) is identical to the Bethe-Salpeter equation in the Thirring⁽¹¹⁾ limit (mass of the bound state equal to zero). For $t \neq 0$ it is shown in app.B that the Bethe-Salpeter equation, continued to $t < 0$, is identical to the asymptotic eq. (4.2) taken for integer values of the function $\alpha(t)$, and gives rise to an eigenvalue condition which is identical to the eigenvalue condition (4.12).

This equivalence between the Bethe-Salpeter equation and eq. (4.2) is by no means a trivial one. In fact one obtains both equations by summing an infinite sum of ladder graphs, but the B.S. equation is an homogeneous equation for a bound state problem, whereas eq.(4.2) is the asymptotic limit of an inhomogeneous equation for a scattering problem. The equivalence we have found is due to the fact that a scattering amplitude has the same poles in t (which do correspond to bound states or resonances) for any values of s , and hence also in the asymptotic limit $s \rightarrow \infty$. Therefore this equivalence is a proof that the approximations we have used in sect.2 in order to obtain the asymptotic equation lead to the correct high energy limit.

The equivalence between the asymptotic diffraction equation and the Bethe-Salpeter equation shows that the analogy of our results with the Regge results in potential theory is very strict and that $\alpha(t)$ has really the meaning of the analytic continuation of the angular momentum.

We will now discuss briefly a critical point of our model, namely the possible appearance of a ghost. From the form (3.8) of the scattering amplitude we see that if $\alpha(t)$ assumes even integer values for values of t in the physical region ($t < 0$), the scattering amplitude has a pole for a physical value of

the momentum transfer. This is clearly a result which is physically absurd, since it gives an infinite cross section and corresponds to the existence of a particle (the ghost) which travels with a speed greater than the velocity of light.

If now we look at our eigenvalue condition (4.13), we see that for $t \rightarrow -\infty$, $\alpha(t) \rightarrow -1$. Therefore if we require that $\alpha(0)$ must be a positive number near to 1, $\alpha(t)$ must reach the value $\alpha(t) = 0$ for some value of t between $-\infty$ and 0.

We think that this difficulty is not connected with the model we have studied but with the assumption we made on the behaviour of the potential $A_0(s_0)$ when $s_0 \rightarrow \infty$, i.e., with the assumption that $A_0(s_0) \rightarrow 0$ at least as $1/s_0$. Probably this difficulty can be avoided by using more singular "potentials", which could allow one to obtain a function $\alpha(t)$ which remains positive when t goes from 0 to $-\infty$ and which varies from 1 to 0 in this interval. The only conclusion we can say now is that our model in the present form makes sense only for values of t which are far from the value for which the ghost appears.

6°) THE LIMITS OF VALIDITY OF THE MODEL

As we have extensively discussed, the basic assumption of our model is that diffraction scattering is given by the ladder of fig.I. Thus it can be considered as a covariant generalization of a potential model for the scattering of two pions in the t -channel, in which the groups of low energy particles, which are exchanged between the pions, act as a potential.

We will now discuss critically the validity of our model from the point of view of field theory. In field theory (in contrast to potential theory) the scattering amplitude by vir-

tue of the substitution rule, must describe at the same time three physical processes, in which the total energy is represented respectively by s , $\bar{s}$, and t . Therefore the scattering amplitude must simultaneously satisfy the unitarity condition in all the three channels.

Let us now consider our result for what concerns unitarity. In the t -channel (corresponding to potential scattering) the unitarity condition is partially satisfied, at least for what concerns the elastic phenomena. On the contrary in the s -channel unitarity is completely disregarded and our approximation is non unitary in the same way as the Born approximation. In a certain sense our ladder can be considered as the analogous of a Born approximation in the s -channel in which what is exchanged between the two scattering particle is a Regge pole. This is not surprising since for a potential model unitarity has a meaning only in the t -channel.

Thus in order to have an idea of the size and form of the corrections we shall try to satisfy, at least partially, the unitarity condition also in the s -channel. The first correction to the imaginary part of the scattering amplitude is obtained by considering those contributions of the two pion intermediate states which correspond to the insertion of the amplitude (3.9) in the unitarity condition. Then using a fixed momentum transfer dispersion relation, as in sect.3, it is possible to obtain the first correction to the scattering amplitude. This procedure can be iterated by inserting again in the unitarity condition the so obtained amplitude. These corrections correspond to taking into account only those scattering amplitudes which consist in the simple iteration in the s -channel of the ladder graphs of fig.1, represented in fig.2.

Fig.2

Although in this way the unitarity condition is verified only in an approximate way, one can find the limits of validity of the model and verify whether the improvements obtained in this way do not contradict the assumptions of the model itself.

The first correction in the forward direction amounts to evaluating the total elastic cross section. In sect.3 we have already seen that the elastic imaginary part in the forward direction is given by

$$A_{el}(s,0) = \frac{1}{16\pi} \int_{-t_{max}}^0 dt s^{\alpha(t)-1} \frac{\varphi^2(t)}{\sin^2 \pi \frac{\alpha(t)}{2}} \quad (6.1)$$

whereas the inelastic imaginary part is

$$A_{in}(s,0) = \varphi(0) s^{\alpha(0)} \quad (6.2)$$

The improved form is therefore

$$A(s,0) = A_{in}(s,0) + A_{el}(s,0) = \varphi(0) s^{\alpha(0)} + \frac{1}{16\pi} \int_{-t_{max}}^0 dt s^{\alpha(t)-1} \frac{\varphi^2(t)}{\sin^2 \pi \frac{\alpha(t)}{2}} \quad (6.3)$$

The s dependence of the elastic correction is similar to that of the leading term but with the important difference

that we have now to deal with a continuous distribution of powers in s .

To obtain the first correction at non-zero angles we insert now the scattering amplitude (3.8) in the unitarity condition and we obtain in the asymptotic limit

$$A_u(s, t) = \frac{1}{32\pi} \int dt_1 dt_2 s^{\alpha(t_1) + \alpha(t_2) - 1} T(-t, -t_1, -t_2) \frac{\varphi(t_1) \varphi(t_2) \cos \frac{\pi}{2} (\alpha(t_1) - \alpha(t_2))}{\sin \frac{\pi}{2} \alpha(t_1) \sin \frac{\pi}{2} \alpha(t_2)} \quad (6.4)$$

which corresponds to the imaginary part of the graph of fig.3

Fig.3

We can rewrite formula (6.4) in the following form

$$A_u(s, t) = \int d\beta s^\beta g(\beta, t) \quad (6.5)$$

where

$$g(\beta, t) = \frac{1}{32\pi^2} \int dt_1 dt_2 T(-t, -t_1, -t_2) \frac{\varphi(t_1) \varphi(t_2) \cos \frac{\pi}{2} (\alpha(t_1) - \alpha(t_2))}{\sin \frac{\pi}{2} \alpha(t_1) \sin \frac{\pi}{2} \alpha(t_2)} \int (\beta - \alpha(t_1) - \alpha(t_2) + 1) \quad (6.6)$$

We notice that by virtue of the definition of the function $T(-t, -t_1, -t_2)$ the integration in (6.6) runs over the area of a plane triangle whose sides are $\sqrt{-t}$, $\sqrt{-t_1}$, $\sqrt{-t_2}$.

Using a fixed t dispersion relation we can now calculate the first correction to the scattering amplitude, and we obtain

$$F_{el}(s, t) = - \int d\beta s^\beta g(\beta, t) \frac{e^{-i\frac{\pi}{2}\beta}}{\sin \frac{\pi}{2}\beta} \quad (6.7)$$

which also here consists of a continuous distribution of powers of s .

Since we have interpreted the form (3.8) as the contribution of a Regge pole, we shall interpret (6.7), which is a continuous superposition of Regge poles, as a Regge branch line, i.e., a cut in the complex angular momentum plane, which at fixed t starts at the maximum value $\alpha_c(t)$ of β given by (6.6).

Let us now study the location of these cuts. Since we know that the diffraction peak is concentrated in a limited region of small values of t , we shall consider formula (6.6) only for small values of t . We can therefore expand $\alpha(t)$, writing (12)

$$\alpha(t) = \alpha(0) + \dot{\alpha}(0)t \quad (6.8)$$

Owing to the triangular inequality which must be satisfied by $\sqrt{-t}$, $\sqrt{-t_1}$, $\sqrt{-t_2}$, the maximum value of $\beta = 2\alpha(0) + \dot{\alpha}(0)(t_1 + t_2) - 1$ is reached in the physical region ($t < 0$) for $t_1 + t_2 = t/2$. The value of this maximum is

$$\alpha_c(t) = 2\alpha(0) + \dot{\alpha}(0)\frac{t}{2} - 1 \quad (6.9)$$

So we can conclude that the inelastic part of the amplitude gives rise to the pole line $\alpha(t) = \alpha(0) + \dot{\alpha}(0)t$, and the elastic part to the cut line $\alpha_c(t) = 2\alpha(0) + \dot{\alpha}(0)\frac{t}{2} - 1$.

It is easy to verify that the n -th iteration of the graph

of fig.1, represented in fig.3, gives rise to the cut line

$$\alpha_n(t) = n \alpha(0) + \frac{t}{n} - (n-1) \quad (6.10)$$

These results considerably limit the range of validity of the model. First of all it follows that we must limit ourselves to values of $\alpha(0)$ which are not greater than 1. If $\alpha(0) > 1$ each iteration gives an imaginary part which increases faster with the energy, so that not only can we no longer consider the contribution of the inelastic processes to be dominant, but also we cannot give any upper bound to the maximum power of s . In other words if $\alpha(0) > 1$ the amplitude increases faster than any power law. There is a strict connection between this limitation on the value of $\alpha(0)$ and the results of Froissart⁽¹³⁾ on the asymptotic behaviour of the total cross section. Froissart has shown on the basis of the Mandelstam representation and applying the unitarity condition in the s -channel that if the total cross section behaves asymptotically as a power of the total energy, this power must be zero. On the other hand our limitation means that if we start with a power of the total cross section which is greater than zero, we shall finally end up with no power behaviour. In other words in the Froissart theorem the unitarity condition puts an upper limit on the possible values of $\alpha(0)$; and in our model the use of the elastic unitarity condition gives rise to effects, which make the form $s^{\alpha(t)}$ for the imaginary part of the scattering amplitude inconsistent just for those values of $\alpha(0)$ which violate the Froissart theorem.

If $\alpha(0)=1$, then in the region $t \leq 0$ the cuts dominate the pole, except for $t = 0$, where, as we have seen, all the iterations have the same power 1 in s , and they differ only by inverse powers of $\log(s)$. If $\alpha(0) < 1$, then the n -th cut

line starts for $t=0$ at the point $\alpha_n(0) = n(\alpha(0)-1) + 1 < \alpha(0)$ and crosses the pole line at the value $t_n = \frac{n(\alpha(0)-1)}{\alpha(0)}$.

Therefore in this case there exists an interval of t between $t=0$ and $t_2 = \frac{2(\alpha(0)-1)}{\alpha(0)}$ in which the pole dominates all the cuts. Thus there is a hierarchy of effects, i.e., the imaginary part coming from the shadow of the inelastic processes is the most important contribution to the asymptotic scattering, and the next important contributions come from the elastic effects of fig.3.

We shall now discuss the physical meaning of the cuts we have found and of the limitations they give rise to.

We must remember that our model is a relativistic generalization of a potential scattering of two pions in the t -channel. We have seen that the ladder approximation we have made takes into account the exchange in the t -channel of all the possible $\pi\pi$ bound states or resonances, with all possible values of the angular momentum $\alpha(t)$. This effect is interpolated by the form

$$F(s,t) = -\psi(t) s^{\alpha(t)} \frac{e^{-i\frac{\pi\alpha(t)}{2}}}{\sin \frac{\pi\alpha(t)}{2}}$$

of the asymptotic scattering amplitude. In the potential model this is obtained in the Regge theory as a residuum of a pole in the complex angular momentum $\alpha(t)$.

The new effects which give rise to the form (6.7) are related to diagrams of the kind of that in fig.3, where two ladders are exchanged in the t -channel. Therefore they must be interpreted as containing the effects of the exchange in the t -channel of any pair of $\pi\pi$ bound states or resonances, where each of the two bound states can have any value of its angular momentum $\alpha(t_1)$ and $\alpha(t_2)$. It appears reasonable to think that because of the existence of this two dimensional

infinity of states it is necessary that there be singularities in the angular momentum plane which are more complicated than poles, e.g., cuts or a continuous distribution of poles in $\alpha(t)$.

In conclusion we can say that the exchange of a single particle gives rise to a pole in the variable t and the exchange of a family of particles gives rise to a pole in the variable $\alpha(t)$. In the same way the exchange of two particles gives a cut in t and the exchange of two families gives a cut in $\alpha(t)$.

We turn now to discuss the meaning of the relative positions of poles and cuts. The key to the understanding comes from considering the range of the different effects.

It is clear from inspection of fig.2 that the n -th order iterated term must have a range of the order of $\frac{1}{2^n r}$. This is confirmed by the shape of the different trajectories. In fact we have seen that the higher terms have a smaller slope, and this is connected with a shorter range.

Therefore the fact that the pole line dominates in the forward direction, and the cut lines start to dominate at large angles can be explained by the simple idea that the long range forces influence the small angle scattering and the central forces influence the large angle scattering. As we have seen, the interval of momentum transfer in which the pole line is the leading term is controlled by the range of the forces, (on which the derivative $\dot{\alpha}(0)$ depends) and by the mass of the lowest $\bar{\pi}\pi$ bound state or resonance (the value of t for which $\alpha(t) = 1$) which is related to the value of $\alpha(0)$.

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APPENDIX A

In this appendix we want to show that the derivative

$\frac{d\alpha}{dt} \big|_{t=0}$ is a positive quantity.

By differentiating (3.1) with respect to t one has

$$\frac{dA}{dt}(s, t, u_1, u_2) = \dot{\varphi}(t, u_1, u_2) s^{\alpha(t)} + \dot{\alpha}(t) \log s \cdot s^{\alpha(t)} \cdot \varphi(t, u_1, u_2) \quad (\text{A.1})$$

For $t=0$ and $u_1 = -\mu^2$, $u_2 = -\mu^2$ (A.1) becomes

$$\frac{dA}{dt} \big|_{t=0}(s, t) = \dot{\varphi}(0) s^{\alpha(0)} + \dot{\alpha}(0) \log s \cdot s^{\alpha(0)} \varphi(0) \quad (\text{A.2})$$

It is possible also to obtain the function $\frac{dA}{dt} \big|_{t=0}$ by differentiating directly eq. (2.1). In this way we obtain

$\frac{dA}{dt} \big|_{t=0}$ as a quadratic integral on $\tilde{A}(s, u) = A(s, 0, u, u)$ and by taking then the asymptotic limit of this integral one can isolate the part of $\frac{dA}{dt} \big|_{t=0}$ which is proportional to $\log s \cdot s^{\alpha(0)}$ identifying in this way the expression for $\dot{\alpha}(0)$.

We can rewrite eq. (2.2) in the following form

$$A_n(p, N, \Delta) = \frac{2^n}{(2\pi)^{4n}} \frac{\left(d_1 Q_1 A_0((p-Q_1)^2) d_1 Q_2 A_0((Q_1-Q_2)^2) \cdots d_1 Q_n \right)}{\left[(Q_1^2 + \Delta^2 - \mu^2)^2 - 4(Q_1 \Delta)^2 \right] \left[(Q_2^2 + \Delta^2 - \mu^2)^2 - 4(Q_2 \Delta)^2 \right] \cdots} \cdot \frac{A_0((Q_{n-1}-Q_n)^2) A_0((Q_n+M)^2)}{\left[(Q_n^2 + \Delta^2 - \mu^2)^2 - 4(Q_n \Delta)^2 \right]} \theta(\sqrt{s} - \sum_0^n \sqrt{s_i}) \quad (\text{A.3})$$

By differentiating now (A.3) with respect to Δ_μ and Δ_ν for $\Delta_\mu = 0$ we obtain ⁽¹⁴⁾.

$$\begin{aligned} \frac{\partial^2 A_n(P, N, \Delta)}{\partial \Delta_\mu \partial \Delta_\nu} (\Delta_\mu = 0) &= \sum_{m=1}^n \frac{2^n}{(2\pi)^{4n}} \left\{ d_4 Q_1 \frac{A_0((P-Q_1)^2)}{(Q_1^2 - \mu^2)^2} \cdot d_4 Q_2 \frac{A_0((Q_1-Q_2)^2)}{(Q_2^2 - \mu^2)} \cdots d_4 Q_m \frac{A_0((Q_{m-1}-Q_m)^2)}{(Q_m^2 - \mu^2)} \right. \\ &\quad \cdot \frac{O_{\mu\nu}}{(Q_m^2 - \mu^2)^4} \cdot d_4 Q_{m+1} \frac{A_0((Q_m-Q_{m+1})^2)}{(Q_{m+1}^2 - \mu^2)^2} \cdots d_4 Q_n \frac{A_0((Q_{n-1}-Q_n)^2)}{(Q_n^2 - \mu^2)^2} A_0((Q_n+N)^2) \theta(\sqrt{s} - \sum_0^n \sqrt{s_i}) \Big\} \\ &= \sum_{m=1}^n \frac{2^n}{(2\pi)^{4n}} \left\{ d_4 Q A_m(P, Q) \frac{O_{\mu\nu} A_{n-m}(Q, N) \theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2})}{(Q^2 - \mu^2)^4} \right\} \quad (A.4) \end{aligned}$$

where

$$O_{\mu\nu} = 8Q_\mu Q_\nu - 4(Q^2 - \mu^2)g_{\mu\nu} ; s_1 = (P-Q)^2 ; s_2 = (Q+N)^2 \quad (A.5)$$

Therefore we have

$$\frac{\partial^2 A(P, N, \Delta)}{\partial \Delta_\mu \partial \Delta_\nu} (\Delta_\mu = 0) = \sum_{n=1} \frac{\partial^2 A_n(P, N, \Delta)}{\partial \Delta_\mu \partial \Delta_\nu} (\Delta_\mu = 0) = \int d_4 Q A(P, Q) \frac{O_{\mu\nu} A(Q, N) \theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2})}{(Q^2 - \mu^2)^4} \quad (A.6)$$

We can now introduce the four vectors $C = \frac{P+N}{2}$; $K = \frac{N-P}{2}$, (in the forward direction and on the mass shell $C \cdot K = 0$). The function $A(P, N, \Delta)$ can be considered as dependent upon the fourvector Δ_μ through the scalars Δ^2 ; ΔC ; ΔK and we can therefore write

$$\begin{aligned} \int d_4 Q A(P, Q) \frac{O_{\mu\nu} A(Q, N) \theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2})}{(Q^2 - \mu^2)^4} &= \left(2 \frac{\partial^2 A(P, N, \Delta)}{\partial \Delta^2} g_{\mu\nu} + \frac{\partial^2 A(P, N, \Delta)}{\partial (\Delta C)^2} C_\mu C_\nu + \right. \\ &\quad \left. + \frac{\partial^2 A(P, N, \Delta)}{\partial (\Delta K)^2} K_\mu K_\nu + \frac{\partial^2 A(P, N, \Delta)}{\partial (\Delta C) \partial (\Delta K)} (C_\mu K_\nu + C_\nu K_\mu) \right) \quad (A.7) \end{aligned}$$

($\Delta_\mu = 0$)

In order to obtain the derivative $\frac{dA(s,t)}{dt}$ at $t=0$ on the mass shell we multiply eq. (A.7) in turn by $g_{\mu\nu}, C_\mu C_\nu, K_\mu K_\nu$, we add the equations so obtained and use the orthogonality $C \cdot K = 0$ to get

$$\frac{dA}{dt} \Big|_{t=0} = \frac{1}{4} \frac{\partial^2 A}{\partial \Delta^2} \Big|_{\Delta_\mu=0} = \int d_1 Q A(p, Q) \frac{(\mu^2 - \frac{(CQ)^2}{c^2} - \frac{(KQ)^2}{k^2})}{(Q^2 - \mu^2)^4} A(Q, N) \theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2}) \quad (A.8)$$

In the high energy limit $\frac{(CQ)^2}{c^2} + \frac{(KQ)^2}{k^2} = -\frac{s_1 s_2}{s}$. We can choose now as integration variables $u = -Q^2$, s_1 and s_2 and we get

$$\frac{dA}{dt} \Big|_{t=0} = \frac{\pi}{2} \int_0^\infty ds_1 ds_2 \int_0^\infty du \frac{\tilde{A}(s_1, u) \tilde{A}(s_2, u) (\mu^2 + \frac{s_1 s_2}{s})}{s (u + \mu^2)^4} \theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2}) \quad (A.9)$$

Then substituting for $\tilde{A}(s_1, u)$ and $\tilde{A}(s_2, u)$ their asymptotic expressions $\tilde{A}(s_1, u) = s_1^{\alpha(10)} \tilde{\varphi}(u)$, $\tilde{A}(s_2, u) = s_2^{\alpha(10)} \tilde{\varphi}(u)$, and introducing the variables $y = s_1/s_2$; $z = s_1 s_2/s$, we obtain

$$\frac{dA}{dt} \Big|_{t=0} = \frac{\pi}{4} s^\alpha \int \frac{dy}{y} \int dz \int_0^\infty du \frac{\tilde{\varphi}(u)^2}{(u + \mu^2)^4} (\mu^2 + z) z^\alpha \theta(u - z) \theta(\sqrt{s} - \sqrt{s_1} - \sqrt{s_2}) \quad (A.10)$$

We can now discuss the integration limits on z and y . If we call s_0 the lowest possible value of s_1 and s_2 , the upper and lower limits of y are $y_{\max} = \frac{(\sqrt{s} - \sqrt{s_0})^2}{s_0}$; $y_{\min} = \frac{s_0}{(\sqrt{s} - \sqrt{s_0})^2}$;

and in the high energy limit those can be replaced by s/s_0 and s_0/s , respectively.

The lower limit of integration on z , which is $z_{\min} = \frac{s_1 s_2}{s}$, gives rise to terms which depend on inverse powers of s , and hence give no contribution to the leading term of (A.10).

We can therefore replace this lower limit with 0. The same reason justifies the use of the asymptotic expression of $\tilde{A}(s_1, u)$ and $\tilde{A}(s_2, u)$ in the whole integral. We can therefore write

$$\frac{dA}{dt}_{t=0} = \frac{\pi}{4} s^{\alpha(0)} \int_{s_0/s}^{s/s_0} \frac{dy}{y} \int_0^\infty \frac{du \tilde{\varphi}^2(u)}{(u+\mu^2)^4} \int_0^u dz (\mu^2+z) z^\alpha \quad (\text{A.11})$$

Integrating now over y and z and taking only the term proportional to $\log s \cdot s^{\alpha(0)}$, we finally obtain

$$\frac{dA}{dt}_{t=0} = \frac{\pi}{2} \log s \cdot s^{\alpha(0)} \int_0^\infty \frac{du \tilde{\varphi}^2(u)}{(u+\mu^2)^4} \left(\mu^2 \frac{u^{\alpha+1}}{\alpha+1} + \frac{u^{\alpha+2}}{\alpha+2} \right) \quad (\text{A.12})$$

Comparing now (A.12) with (A.2) we get the result

$$\dot{\alpha}(0) = \frac{\pi}{2} \frac{\int_0^\infty \frac{\tilde{\varphi}^2(u)}{(u+\mu^2)^4} \left(\mu^2 \frac{u^{\alpha+1}}{\alpha+1} + \frac{u^{\alpha+2}}{\alpha+2} \right)}{\varphi(0)} \quad (\text{A.13})$$

which shows that $\dot{\alpha}(0)$ is indeed a positive quantity.

APPENDIX B

In order to obtain the identity of our equation with the continuation of the Bethe-Salpeter equation for $t < 0$, we shall consider formally the B.S. equation for negative values of the square of the mass of the bound state.

We will sketch the proof for the simple case of a bound state of spin 0, which means $\alpha(t) = 0$. A more complete and detailed discussion on this topic will be the subject of a future investigation.

Let us consider the Bethe-Salpeter equation

$$(u_1 + p^2)(u_2 + p^2)\varphi(k) = \frac{i}{(2\pi)^4} \int ds_0 \frac{A_0(s_0)}{\pi} \int \frac{\varphi(k') d_4 k'}{s_0 - (k' - k)^2} \quad (\text{B.1})$$

where the "potential" is due to the exchange of groups of particles of mass s_0 with weight function $A_0(s_0)$, $u_1 = -(p+k)^2$, $u_2 = -(p-k)^2$ and $p^2 = t/4 < 0$.

We now assume that $\varphi(k)$ satisfies the following integral representation ¹⁵⁾

$$\varphi(k) = \frac{\int d\alpha d\beta g(\alpha, \beta, t)}{[\alpha - (p+k)^2][\beta - (p-k)^2]} \quad (\text{B.2})$$

By using Feynman parametrization, and defining

$$q_z = k + pz \quad ; \quad \alpha_z = \alpha \frac{(1+z)}{2} + \beta \frac{(1-z)}{2} \quad (\text{B.3})$$

one has

$$\varphi(k) = 2 \int d\alpha d\beta \int_{-1}^{+1} dz \frac{g(\alpha, \beta, t)}{[\alpha_z - q_z^2 - p^2(1-z^2)]^2} \quad (\text{B.4})$$

Inserting this formula in (B.1) and using again Feynman parametrization we obtain, with

$$\begin{aligned} (u_1 + \mu^2)(u_2 + \mu^2) \varphi(k) &= \frac{2i}{(2\pi)^4} \int ds_0 \frac{A_0(s_0)}{\pi} \int_0^1 d\alpha d\beta \int_0^1 dz \frac{d_4 k' g(\alpha, \beta, t)}{[s_0 - (k' - k)^2][\alpha_2 - q_2'^2 - p^2(1-z^2)]} = \\ &= \frac{4i}{(2\pi)^4} \int ds_0 \frac{A_0(s_0)}{\pi} \int_0^1 dx(1-x) \int_0^1 dz \frac{d_4 k' g(\alpha, \beta, t)}{[-(q_2' - q_2 x)^2 - q_2^2 x(1-x) + s_0 x + \alpha_2(1-x) - p^2(1-z^2)(1-x)]^3} \end{aligned} \quad (B.5)$$

where $q_2' - q_2 x = k' - Kx + pz(1-x)$.

Shifting now the integration from k' to $q_2'' = q_2' - qx$ and integrating we get

$$(u_1 + \mu^2)(u_2 + \mu^2) \varphi(k) = \frac{1}{(2\pi)^3} \int ds_0 A_0(s_0) \int_0^1 d\alpha d\beta \int_0^1 dx \int_0^1 dz \frac{g(\alpha, \beta, t)}{A_2 - p^2(1-x)(1-z^2)} \quad (B.6)$$

where

$$\begin{aligned} A_2 &= A_1 \frac{(1+z)}{2} + A_2 \frac{(1-z)}{2} \\ A_1 &= \alpha - (p+k)^2 x + \frac{s_0 x}{1-x} = \alpha + u_1 x + \frac{s_0 x}{1-x} \\ A_2 &= \beta - (p-k)^2 x + \frac{s_0 x}{1-x} = \beta + u_2 x + \frac{s_0 x}{1-x} \end{aligned} \quad (B.7)$$

The next step is the most important in our derivation; we notice the identity

$$\int_0^1 dz \frac{1}{A_2 - p^2(1-x)(1-z^2)} = \frac{1}{2\pi} \int \frac{d_2 q}{[(p'+q)^2 + A_1][(p'-q)^2 + A_2]} \quad (B.8)$$

where p' and q are two dimensional euclidean vectors, such that $p'^2 = -p^2(1-x)$. This identity can be easily checked by using again Feynman parametrization.

Then using the representation (B.2) one can write

$$\begin{aligned} (u_1 + \mu^2)(u_2 + \mu^2) \varphi(u_1, u_2, t) &= \frac{1}{(2\pi)^4} \int_0^1 ds_0 A_0(s_0) \cdot \int_0^1 dx \int d\alpha d\beta \cdot \\ &\cdot \frac{\int d_2 q \, g(\alpha, \beta, t)}{[(p'+q)^2 + A_1][(p'-q)^2 + A_2]} = \\ &= \frac{1}{(2\pi)^4} \int_0^1 ds_0 A_0(s_0) \int_0^1 dx \int d_2 q \, \varphi\left((p'+q)^2 + u_1 x + \frac{s_0 x}{1-x}, (p'-q)^2 + u_2 x + \frac{s_0 x}{1-x}\right) \quad (B.9) \end{aligned}$$

and from the identity (2.16) one finally obtains

$$\begin{aligned} (u_1 + \mu^2)(u_2 + \mu^2) \varphi(u_1, u_2, t) &= \\ &= \frac{1}{2(2\pi)^4} \int_0^1 ds_0 A_0(s_0) \int_0^1 dx \int_0^\infty du'_1 du'_2 \varphi(u'_1, u'_2, t) T(\varphi, \varphi_1, \varphi_2) \quad (B.10) \end{aligned}$$

which is identical with the equation (4.2) (with $\alpha(t)=0$).

Then the Bethe-Salpeter equation for $t < 0$ (B.10) can be solved with the same method as the asymptotic equation (4.2) and leads to the same eigenvalue condition (4.12).

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- 5) - T.REGGE, Nuovo Cimento 14, 951 (1959); Nuovo Cimento 8 947 (1960); S.BOTTINO, A.M.LONGONI, T.REGGE, to be published.
- 6) - The metric is $g_{\mu\nu} = 1, -1, -1, -1$.
- 7) - Actually we have been able to demonstrate that eq.(3.2) satisfies the Fredholm conditions only when $A_0(s_0) \rightarrow 0$ for $s_0 \rightarrow \infty$ at least as $1/s_0$ and when $\alpha(t) > 0$. The solution we shall find however will suggest that this equation is an eigenvalue equation also for $\alpha(t)$ negative but greater than -1 .
- 8) - Since we are considering now the scattering amplitude on the mass shell, we shall write $\mathcal{Q}(t)$ for $\mathcal{Q}(-\mu^2, -\mu^2, t)$.
- 9) - L.BERTOCCHI, C.CEOLIN, M.TONIN, Nuovo Cimento 18, 770 1960.
- 10) - It is possible to show that the support of the weight function $g(a, b, t)$ consists of the point $a = \mu^2$, $b = \mu^2$ of the two lines $a = \mu^2$, $b \geq (\mu + \sqrt{s_0})^2$ and $b = \mu^2$,

$a \geq (\mu + \sqrt{s_0})^2$, and of the region $a \geq (\mu + \sqrt{s_0})^2 ; b \geq (\mu + \sqrt{s_0})^2$.

Now whereas the next iterations give contributions in the plane which are different from zero in regions which are always more external, each iteration covers again the whole lines $a = \mu^2$, $b \geq (\mu + \sqrt{s_0})^2$, $b = \mu^2$,

$a \geq (\mu + \sqrt{s_0})^2$. In other words, in any point of the region $a \geq (\mu + \sqrt{s_0})^2$, $b \geq (\mu + \sqrt{s_0})^2$ the method we have described allows to determine $g(a,b,t)$ after a finite number of iterations, whereas on the lines $a = \mu^2$, $b \geq (\mu + \sqrt{s_0})^2$; $b = \mu^2$, $a \geq (\mu + \sqrt{s_0})^2$ it is obtained by means of a perturbation series in the coupling constant g^2 .

11)- K.BAUMANN and W. THIRRING, Nuovo Cimento 18, 357 (1960).

12)- We have seen in sect.3 that the derivative $\alpha'(t)$ is definite positive, and moreover that it is never infinite, since it is given by a convergent integral. Therefore we are not allowed to perform an expansion of $\alpha(t)$ in powers of $\sqrt{-t}$, since this would imply an infinite value for $\alpha'(0)$ (and a branch point of $\alpha(t)$ for $t=0$, which is not present in our eigenvalue condition (4.13)).

13)- M.FROISSART, Phys.Rev. 123, 1053 (1961).

14)- We notice that when we perform this differentiation we must only differentiate twice the same denominator, since the products of the first derivatives of two different denominators vanish for $\Delta_\mu = 0$.

15)- A spectral representation for the Bethe-Salpeter amplitude which has some analogy to ours, has been proposed by G. Wanders, Phys.Rev. 104, 1782 (1956); Helv.Phys.Acta 30, 417 (1957), on the basis of the method of Wick. Work is now in progress to understand the relation between the two methods.