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ON THE DEUTERON WAVE FUNCTION

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I. Introduction

The problem of the deuteron wave function in non-relativistic theory has been examined many times up to the present time and two possible methods of approach have been adopted to solve it. The first one consists in assuming some static nucleon-nucleon potential, corresponding to more or less justified pictures one can have of this interaction and then in solving the Schrödinger equation with this potential. The point is that for potentials which are sufficiently simple to yield an exact solution of the Schrödinger equation, this solution has only a limited practical value since it does not fit all the most fundamental deuteron data established experimentally. So one has to assume more complicated (and at least to some extent arbitrary) potential functions and then the solution can be determined, but only approximately. The second possible way is to postulate a deuteron wave function in some "phenomenological-like" form with a few free parameters and then to try to fix them so as to get an agreement with the experimental data. In general, one could try to do this by choosing the wave functions "similar" to those given by a "potential method", but sufficiently different from them to be good ones and at the same time sufficiently simple to be of practical use. However, the weakness in this method is that the physical meaning of the parameters introduced into the theory is not always clear, and the whole method reduces to a kind of ansatz for the deuteron problem.

In the present note we wish to adopt a different approach to the deuteron problem. Contrary to the first method described above we do not see any reason to assume an arbitrary potential rather than an arbitrary wave function. We would like, nevertheless, to maintain those features of the non-relativistic potential that follow from the field-theoretical description of the nuclear forces. The second method described above differs here from ours and this fact allows us to keep very clear the physical meaning of all the parameters introduced into the theory. Our method is based on the results of

Martin's paper ^{1),2)} in which he proved that it is possible to solve the problem with Yukawa-like forces step by step and that it is sufficient to know the "outer" part of the potential to find the "outer" part of the wave function. In our case the "outer" part of the potential is merely that given directly by field theory. In the first place, we find the outer part of the wave function corresponding to this potential, then we construct a full wave function such that it reduces in the outer region to the "field theoretical" one. Then, we fix the free parameters left in our theory so that the deuteron data is fitted.

Furthermore, it turns out that the reasonable agreement with the deuteron data can be reached only for the f^2 not very far from the value known from the interpretation of other blocks of experimental data. So, the method used here may be regarded as an independent way of calculating the f^2 value from experimental data (in fact, the first one that makes use of the deuteron data) which yields the results similar to those given by other methods.

The plan of this note is the following. In Section II we clarify the method throughout the paper by an example of the case of central forces; then in Section III we turn to the realistic case with central and tensor forces taken into account. In Section IV the results are presented and discussed.

II. Deuteron wave function with central forces

To make the method clear let us start with a brief sketch of the calculation of the "deuteron" wave function with central forces only. Hereafter we set the pion mass equal to unity.

We have the Schrödinger equation (see Ref. 1))

$$u''(r) - [\kappa^2 + V(r)] u(r) = 0 \quad (2.1)$$

and we look for the solution that has the following form

$$u(r) = f(r) e^{-\kappa r}, \quad (2.2)$$

where $f(r)$ has to fulfil the suitable asymptotic condition

$$\lim_{r \rightarrow \infty} f(r) = 1. \quad (2.2')$$

The Eq. (2.1) can be solved step by step by the method given by Martin ¹⁾.

In this aim we suppose the potential $V(r)$ to be of the Yukawa-like shape

$$V(r) = \int_1^{\infty} c(\alpha) e^{-\alpha r} d\alpha \quad (2.3)$$

so it is natural to look for $f(r)$ in the form

$$f(r) = 1 + \int_1^{+\infty} g(\alpha) e^{-\alpha r} d\alpha. \quad (2.4)$$

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After simple manipulations we get for the $\varphi(\alpha)$ the equation of the form

$$\varphi(\alpha) = \frac{C(\alpha)}{\alpha(\alpha+2\kappa)} + \frac{1}{\alpha(\alpha+2\kappa)} \int_1^{\alpha-1} C(\alpha-\beta) \varphi(\beta) d\beta. \quad (2.5)$$

It is clear that in the region $1 \leq \alpha \leq 2$ the only contribution to the $\varphi(\alpha)$ goes from the first term in (2.5). But this is exactly the region where only the one-pion exchange is responsible for the forces between the proton and the neutron. If we had from field theory the shape of the $V(r)$, we could find $C(\alpha)$ and then $\varphi(\alpha)$. For example, if we suppose $V(r)$ to be of pure Yukawa shape

$$V(r) = C e^{-r}/r, \quad (2.6)$$

then $C(\alpha) = C$ and for $1 \leq \alpha \leq 2$ we have

$$\varphi(\alpha) = \frac{C}{\alpha(\alpha+2\kappa)}. \quad (2.7)$$

Notice that C in (2.6) has the meaning of the coupling constant multiplied by the reduced mass of the neutron-proton system which we can put equal to the mass of the single nucleon.

Now, it is possible to write down the "deuteron" wave function in the outer region in the form

$$u(r) = C e^{-\kappa r} \int_1^2 \frac{e^{-\alpha r} d\alpha}{\alpha(\alpha+2\kappa)} + e^{-\kappa r}. \quad (2.8)$$

We wish now to complete this function by some phenomenological terms that could be chosen so as to fit some "experimental" data. Let us state that these data are the regularity of the wave function $u(r)$ at the origin

$$u(0) = 0, \quad (2.9)$$

and the good "triplet effective range"

$$r_{tr} = \frac{1}{\kappa} - 2 \int_0^{\infty} u^2(r) dr. \quad (2.10)$$

To fulfil it we could try to write the wave function in a form with two free parameters. It seems to be very natural to assume that the inner part of the wave function is dominated mainly by two contributions, i.e., by a two-pion correlated pair and by a nucleon repulsive core. So we can try to fit the data with the following simple wave function

$$u(r) = e^{-\kappa r} \left[1 + C \int_1^2 \frac{e^{-\alpha r} d\alpha}{\alpha(\alpha+2\kappa)} + \gamma_1 e^{-\xi_1 r} + \gamma_2 e^{-\xi_2 r} \right] \quad (2.11)$$

where γ_1, γ_2 are just the free parameters we mentioned above, whereas $\xi_1 \sim 4-5$, and ξ_2 are the order of the nucleon mass ($m_N = 6.72$).

Notice that this wave function corresponds to

$$\rho(\alpha) = \frac{C}{\alpha(\alpha+2\kappa)} + \gamma_1 \delta(\xi_1 - \alpha) + \gamma_2 \delta(\xi_2 - \alpha). \quad (2.12)$$

It is easy to recalculate from (2.12) the potential, that is, the function $C(\alpha)$, using the equation (2.5). We have

$$C(\alpha) = g(\alpha) \alpha (\alpha + 2\kappa) - \int_1^{\alpha-1} C(\gamma) g_{\kappa}(\alpha - \gamma) d\gamma. \quad (2.5')$$

We see that we get

$$C(\alpha) = C + \gamma_1 \xi_1 (\xi_1 + 2\kappa) \delta(\xi_1 - \alpha) + \\ + \gamma_2 \xi_2 (\xi_2 + 2\kappa) \delta(\xi_2 - \alpha) + \text{iteration terms}, \quad (2.13)$$

that corresponds to

$$V(r) = C \frac{e^{-r}}{r} + B_1 e^{-\xi_1 r} + B_2 e^{-\xi_2 r} + \dots, \quad (2.14)$$

where we put

$$B_i = \gamma_i \xi_i (\xi_i + 2\kappa). \quad (2.14')$$

Although the additional terms in (2.14) are not Yukawian, we could hope that after having performed numerical calculations we should get $B_1 < 0$ and $B_2 > 0$ according to their meaning analogous to the coupling constants. We shall see that, constructed along these lines, the realistic deuteron wave function will also fulfil this condition.

III. Deuteron wave functions with tensor forces

The starting point for our calculations is the Schrödinger equation for the deuteron radial wave functions ³⁾

$$\begin{aligned} u''(r) - [\kappa^2 + V_c(r)] u(r) &= 2\sqrt{2} V_T(r) w(r) \\ w''(r) - \left[\kappa^2 + \frac{6}{r^2} + V_c(r) - 2V_T(r)\right] w(r) &= 2\sqrt{2} V_T(r) u(r) \end{aligned} \quad (3.1)$$

where $V_c(r)$ and $V_T(r)$ represent the potentials of the central and tensor forces. We look for a solution of this equation having the following form

$$\begin{pmatrix} u(r) \\ w(r) \end{pmatrix} = a \begin{pmatrix} f_1(r) \\ g_1(r) \end{pmatrix} e^{-\kappa r} + b \begin{pmatrix} f_2(r) \\ g_2(r) \end{pmatrix} e^{-\kappa r}. \quad (3.2)$$

For $r \rightarrow \infty$

$$\lim f_1 = \lim f_2 = \lim g_1 = -\lim g_2 = 1. \quad (3.2')$$

Then, we solve our equations (3.1) by a method given by Vinh Mau and Martin ²⁾. After having put (3.2) into (3.1) one obtains the following equations for $f_\lambda(r)$ and $g_\lambda(r)$:

$$\begin{aligned} f_\lambda'' - 2\kappa f_\lambda' - U_c f_\lambda &= U_T g_\lambda \\ g_\lambda'' - 2\kappa g_\lambda' - \left(\frac{6}{r^2} + U_M\right) g_\lambda &= U_T f_\lambda \end{aligned} \quad (3.3)$$

where $\lambda = 1, 2$, and

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$$U_c = V_c ,$$

$$U_T = 2\sqrt{2} V_T , \quad (3.4)$$

$$U_M = V_c - 2V_T = U_c - \frac{U_T}{\sqrt{2}} .$$

In the following we shall assume the Yukawa-like shape for U_c , U_T and U_M according to the field-theoretical result. It means that ($m_\pi = 1$) we have

$$U_{c,T,M} = \int_1^\infty C_{c,T,M}(\alpha) e^{-\alpha r} d\alpha . \quad (3.5)$$

We look then for the solutions of Eqs. (3.3) that have the form [see (3.2a)]

$$\begin{aligned} f_\lambda &= 1 + \int_0^\infty \rho_\lambda(\alpha) e^{-\alpha r} d\alpha , \\ g_\lambda &= \eta_\lambda + \int_0^\infty \sigma_\lambda(\alpha) e^{-\alpha r} d\alpha , \end{aligned} \quad (3.6)$$

where $\eta_\lambda = \delta_{\lambda 1} - \delta_{\lambda 2}$. We then get after some calculations (see Ref. ²⁾) the following expressions for the ρ and σ functions

$$\begin{aligned} \rho_\lambda(\alpha) &= \frac{C_c(\alpha) + \eta_\lambda C_T(\alpha)}{\alpha(\alpha + 2\kappa)} + \frac{1}{\alpha(\alpha + 2\kappa)} \int_0^{\alpha-1} d\beta \left[C_c(\alpha-\beta) \rho_\lambda(\beta) + \right. \\ &\quad \left. + C_T(\alpha-\beta) \sigma_\lambda(\beta) \right] , \end{aligned} \quad (3.7)$$

$$\sigma_\lambda(\alpha) = \eta_\lambda \frac{dP_2(1 + \frac{\alpha}{\kappa})}{d\alpha} + \frac{F_\lambda(\alpha)}{\alpha(\alpha + 2\kappa)} - \frac{1}{\kappa} \int_0^\alpha F_\lambda(\beta) K(\alpha, \beta) d\beta ,$$

where

$$F_{\lambda}(\alpha) = \eta_{\lambda} C_M(\alpha) + C_T(\alpha) + \int_0^{\alpha-1} d\beta [C_M(\alpha-\beta) \sigma_{\lambda}(\beta) + C_T(\alpha-\beta) \vartheta_{\lambda}(\beta)] , \quad (3.8)$$

and

$$K(\alpha, \beta) = \frac{dP_2(1+\frac{\alpha}{\kappa})}{d\alpha} \frac{dQ_2(1+\frac{\beta}{\kappa})}{d\beta} - (\beta \leftrightarrow \alpha) \quad (3.9)$$

Now we see that for $0 \leq \alpha \leq 1$

$$\vartheta_{\lambda} \equiv 0, \quad \sigma_{\lambda}(\alpha) = \eta_{\lambda} \frac{dP_2}{d\alpha} \quad (3.7')$$

$$F_{\lambda}(\alpha) \equiv 0 \quad (3.8')$$

and that for $1 \leq \alpha \leq 2$

$$\vartheta_{\lambda}(\alpha) = \frac{C_C(\alpha) + \eta_{\lambda} C_T(\alpha)}{\alpha(\alpha+2\kappa)} + \frac{\eta_{\lambda}}{\alpha(\alpha+2\kappa)} \int_0^{\alpha-1} d\beta C_T(\alpha-\beta) \frac{dP_2}{d\beta} \quad (3.7'')$$

$$\sigma_{\lambda}(\alpha) = \eta_{\lambda} \frac{dP_2}{d\alpha} + \frac{F_{\lambda}(\alpha)}{\alpha(\alpha+2\kappa)} - \frac{1}{\kappa} \int_0^{\alpha} F_{\lambda}(\beta) K(\alpha, \beta) d\beta ,$$

where now

$$F_{\lambda}(\alpha) = \eta_{\lambda} C_M(\alpha) + C_T(\alpha) + \eta_{\lambda} \int_0^{\alpha-1} d\beta C_M(\alpha-\beta) \frac{dP_2}{d\beta} . \quad (3.8'')$$

Then we have

$$\begin{aligned}
 u(r) &= e^{-\kappa r} [a f_1 + b f_2] = \\
 &= A e^{-\kappa r} \left[1 + \int_1^2 e^{-\alpha r} \vartheta^+(\alpha) d\alpha + H \int_1^2 e^{-\alpha r} \vartheta^-(\alpha) d\alpha \right]
 \end{aligned} \tag{3.10}$$

$$\begin{aligned}
 w(r) &= e^{-\kappa r} [a g_1 + b g_2] = \\
 &= A e^{-\kappa r} \left[H + \int_1^2 e^{-\alpha r} \sigma^+(\alpha) d\alpha + H \int_0^2 e^{-\alpha r} \sigma^-(\alpha) d\alpha \right],
 \end{aligned}$$

where

$$\begin{aligned}
 A &= a+b, \quad H = \frac{a-b}{a+b}, \quad \vartheta^\pm = \frac{1}{2} (\vartheta_1 \pm \vartheta_2), \\
 \sigma^\pm &= \frac{1}{2} (\sigma_1 \pm \sigma_2).
 \end{aligned}$$

Now, it is easy to see that

$$\vartheta^+ = \frac{C_c(\alpha)}{\alpha(\alpha+2\kappa)}, \quad \vartheta^- = \frac{C_T(\alpha)}{\alpha(\alpha+2\kappa)} + \frac{1}{\alpha(\alpha+2\kappa)} \int_0^{\alpha-1} d\beta C_T(\alpha-\beta) \frac{dP_2}{d\beta} \tag{3.11}$$

$$\sigma^+ = \frac{C_T(\alpha)}{\alpha(\alpha+2\kappa)} - \frac{1}{\kappa} \int_0^\alpha C_T(\beta) K(\alpha, \beta) d\beta, \tag{3.12}$$

$$\sigma^- = \frac{dP_2}{d\alpha} + \frac{F^-(\alpha)}{\alpha(\alpha+2\kappa)} - \frac{1}{\kappa} \int_0^\alpha F^-(\beta) K(\alpha, \beta) d\beta,$$

where

$$F^-(\alpha) = C_M(\alpha) + \int_0^{\alpha-1} d\beta C_M(\alpha-\beta) \frac{dP_2}{d\beta}. \tag{3.13}$$

As it was said before, to have the functions $\rho^{\pm}(\alpha)$ and $\sigma^{\pm}(\alpha)$ for $\alpha \leq 2$ it is sufficient to have $\rho_C(\alpha)$, $\rho_T(\alpha)$ and $\rho_M(\alpha)$ up to $\alpha = 2$. We assume in the following that the one-pion exchange potential derived from field theory describes well this part of nuclear interaction. This potential has the well-known form

$$V = m f^2 \left\{ (\vec{\tau}_\alpha \vec{\tau}_\beta) (\vec{\sigma}_\alpha \vec{\sigma}_\beta) \frac{e^{-r}}{3r} + \frac{1}{3} \vec{\tau}_\alpha \vec{\tau}_\beta \left(1 + \frac{3}{r} + \frac{3}{r^2} \right) S_{\alpha\beta} \frac{e^{-r}}{r} \right\} \quad (3.14)$$

We are interested in the deuteron state, that is, in the state $T = 0$, $J = 1$; for it we have $\vec{\sigma}_\alpha \vec{\sigma}_\beta = 1$, $\vec{\tau}_\alpha \vec{\tau}_\beta = -3$, so that

$$V = C \left\{ \frac{e^{-r}}{r} + S_{\alpha\beta} \left(1 + \frac{3}{r} + \frac{3}{r^2} \right) \frac{e^{-r}}{r} \right\}, \quad (3.15)$$

where

$$C = -m f^2 \sim -0.55$$

(m denotes here the nucleon mass and $f^2 \sim 0.08$.) We see that

$$\begin{aligned} V_C(r) &= C \frac{e^{-r}}{r}, \\ V_T(r) &= C \left(1 + \frac{3}{r} + \frac{3}{r^2} \right) \frac{e^{-r}}{r}. \end{aligned} \quad (3.16)$$

It is clear that

$$\begin{aligned} V_C(r) &= C \int_1^\infty e^{-\alpha r} d\alpha, \\ V_T(r) &= \frac{1}{2} C \int_1^\infty (3\alpha^2 - 1) e^{-\alpha r} d\alpha. \end{aligned} \quad (3.17)$$

By comparison of the Eq. (3.17) with (3.4) and (3.5) we see that

$$\begin{aligned} C_c(\alpha) &= C, \\ C_T(\alpha) &= C\sqrt{2}(3\alpha^2 - 1), \\ C_M(\alpha) &= C(2 - 3\alpha^2). \end{aligned} \quad (3.18)$$

Having introduced the notation x for the dimensionless variable

$$x = 1 + \frac{\alpha}{\kappa} \quad (3.19)$$

we get finally (with $\vartheta^{\pm}(x)_{\text{df}} = \vartheta^{\pm}(\alpha(x))$, $\sigma^{\pm}(x)_{\text{df}} = \sigma^{\pm}(\alpha(x))$)

$$\vartheta^+(x) = \frac{C}{\kappa^2(x^2 - 1)}, \quad (3.20)$$

$$\vartheta^-(x) = \frac{3C\sqrt{2}}{4}(x^2 - 1) - \frac{3C\sqrt{2}}{2\kappa^2} + \frac{C\sqrt{2}(3 - 4\kappa^2)}{4\kappa^4(x^2 - 1)},$$

$$\begin{aligned} \sigma^+(x) &= \frac{C\sqrt{2}[3\kappa^2(x-1)^2 - 1]}{\kappa^2(x^2 - 1)} - \frac{3C\sqrt{2}}{\kappa^2} \left\{ \frac{3\kappa^2}{8}(3\kappa^5 + \right. \\ &+ x) \ln \left| \frac{1+x}{1-x} \right| - \frac{3\kappa^2}{4}(3\kappa^4 + x^2) + (3\kappa^2 - 1) \left[\frac{1}{4}(3\kappa^3 - x) \ln \left| \frac{1+x}{1-x} \right| - \right. \\ &- \frac{3}{2}\kappa^2] - 6\kappa^2(-x^3 + \frac{x^4}{2} \ln \left| \frac{1+x}{1-x} \right|) - \mathcal{K}_1^+ x + \mathcal{K}_2^+ [\frac{1}{4}(3\kappa^2 - \\ &- 1) \ln \left| \frac{1+x}{1-x} \right| - \frac{3}{2}x] - [3\kappa^2(\frac{1}{4}\kappa^4 - \frac{2}{3}\kappa^3 + \frac{1}{2}\kappa^2) - \frac{1}{2}\kappa^2] \cdot [\\ &\cdot [\frac{1}{4}(3\kappa^2 - 1) \ln \left| \frac{1+x}{1-x} \right| - \frac{3}{2}x] \} \end{aligned} \quad (3.21)$$

$$\begin{aligned}
G^-(x) &= \frac{3}{x}x - \frac{3C}{4}(x^2-1) + \frac{3C}{x^2} - \frac{3C}{x^3(x+1)} + \\
&+ \frac{C(3-12x+8x^2)}{4x^4(x^2-1)} - \frac{3C}{4x^3} \left\{ -3x^4 \left[\frac{1}{4}x(x^2-1)^3 \ln \left| \frac{1+x}{1-x} \right| - \right. \right. \\
&- \frac{1}{2}x^6 + \frac{4}{3}x^4 - \frac{3}{2}x^2 \left. \right] + x^2 \left[\left(\frac{9}{2}x^5 - 3x^3 + \frac{5}{2}x \right) \ln \left| \frac{1+x}{1-x} \right| - \right. \\
&- 9x^4 + 3x^2 \left. \right] - 6x \left[x^4 \ln \left| \frac{1+x}{1-x} \right| - 2x^3 \right] + 3 \left[\frac{1}{4}(3x^3 - \right. \\
&- x) \ln \left| \frac{1+x}{1-x} \right| - \frac{3}{2}x^2 \left. \right] - K_1^- x + K_2^- \left[\frac{1}{4}(3x^2-1) \ln \left| \frac{1+x}{1-x} \right| - \right. \\
&- \frac{3}{2}x \left. \right] + \left[3x^4 \left(\frac{1}{6}x^6 - \frac{1}{2}x^4 + \frac{1}{2}x^2 \right) - x^2(3x^4 - 2x^2) + 4xx^3 - \right. \\
&- \frac{3}{2}x^2 \left. \right] \cdot \left[\frac{1}{4}(3x^2-1) \ln \left| \frac{1+x}{1-x} \right| - \frac{3}{2}x \right] \left. \right\}, \tag{3.21}
\end{aligned}$$

where

$$\begin{aligned}
K_1^+ &= \frac{3}{8}x^2(3x_1^4+1) \ln \left| \frac{1+x_1}{1-x_1} \right| - \frac{3}{4}x^2(3x_1^3+x_1) + \\
&+ (3x^2-1) \left[\frac{1}{4}(3x_1^2-1) \ln \left| \frac{1+x_1}{1-x_1} \right| - \frac{3}{2}x_1 \right] - 6x^2(-x_1^2 + \\
&+ \frac{1}{2}x_1^3 \ln \left| \frac{1+x_1}{1-x_1} \right|), \\
K_1^- &= 3x^2 \left(\frac{1}{4}x_1^4 - \frac{2}{3}x_1^3 + \frac{1}{2}x_1^2 \right) - \frac{1}{2}x_1^2, \tag{3.22} \\
K_2^+ &= -3x^4 \left[\frac{1}{4}(x_1^2-1)^3 \ln \left| \frac{1+x_1}{1-x_1} \right| - \frac{1}{2}x_1^5 + \frac{4}{3}x_1^3 - \frac{3}{2}x_1 \right] + \\
&+ x^2 \left[\left(\frac{9}{2}x_1^4 - 3x_1^2 + \frac{5}{2} \right) \ln \left| \frac{1+x_1}{1-x_1} \right| - 9x_1^3 + 3x_1 \right] - \\
&- 6x \left[x_1^3 \ln \left| \frac{1+x_1}{1-x_1} \right| - 2x_1^2 \right] + 3 \left[\frac{1}{4}(3x_1^2-1) \ln \left| \frac{1+x_1}{1-x_1} \right| - \frac{3}{2}x_1 \right], \\
K_2^- &= -3x^4 \left[\frac{1}{6}x_1^6 - \frac{1}{2}x_1^4 + \frac{1}{2}x_1^2 \right] + x^2(3x_1^4 - 2x_1^2) - 4xx_1^3 + \frac{3}{2}x_1^2.
\end{aligned}$$

Notice that in the region $0 \leq \alpha \leq 1$ we have only the first term in $\bar{\sigma}^-(x)$ (independent of C) and that all the terms which depend on C start at $\alpha = 1$; $x_1 = 1 + \alpha^{-1}$.

Now, we have assumed these functions to describe our problem for $\alpha \leq 2$ only. The "inner" part of the N-N potential is very complicated in view of existing strong many-pion correlations, so that we do not know of any sufficiently exact formula to describe it. Instead of trying to fit the data with an arbitrarily chosen potential function we prefer rather to introduce some arbitrariness into our wave functions in such a way, however, that it reduce to the "field-theoretical" form for r big enough. In this aim, we will try to take into account the "inner" part of the interaction in the wave functions themselves by adding two terms corresponding (the lower one) to the attractive forces coming from the two-pion exchange and (the higher one) to the repulsive nucleon core. We make the simplest choice, assuming that

$$\begin{aligned} \varrho^\pm(\alpha) &= \bar{\varrho}^\pm(\alpha) + \gamma_1^\pm \delta(\xi_1 - \alpha) + \gamma_2^\pm \delta(\xi_2 - \alpha) \\ \sigma^\pm(\alpha) &= \bar{\sigma}^\pm(\alpha) + \gamma_3^\pm \delta(\xi_1 - \alpha) + \gamma_4^\pm \delta(\xi_2 - \alpha) \end{aligned} \quad (3.23)$$

where $\bar{\varrho}$ and $\bar{\sigma}$ are those parts of ϱ and σ that are given by the formulae (3.20) and (3.21) with $\gamma_{1,2}^\pm$ given by (3.22); the position of ξ_1 and ξ_2 is to be fixed at

$$2 \lesssim \xi_2 \lesssim 6, \quad \xi_2 \sim m. \quad (3.24)$$

After these assumptions have been made we get for $u(r)$ and $w(r)$ the following form (the constant A is omitted; it clearly fixes the normalization of the functions)

$$\begin{aligned}
u(r) &= e^{-\alpha r} \left[1 + \int_1^2 e^{-\alpha r} \varrho^+(\alpha) d\alpha + H \int_1^2 e^{-\alpha r} \varrho^-(\alpha) d\alpha + \right. \\
&\quad \left. + \gamma_1 e^{-\xi_1 r} + \gamma_2 e^{-\xi_2 r} \right] \\
w(r) &= e^{-\alpha r} \left[H + \int_1^2 e^{-\alpha r} \sigma^+(\alpha) d\alpha + H \int_0^2 e^{-\alpha r} \sigma^-(\alpha) d\alpha + \right. \\
&\quad \left. + \gamma_3 e^{-\xi_1 r} + \gamma_4 e^{-\xi_2 r} \right], \quad (3.25)
\end{aligned}$$

where $\varrho^\pm$ and $\sigma^\pm$ are given by (3.20) and (3.21) but

$$\gamma_k = \gamma_k^+ + H \gamma_k^- \quad (3.26)$$

for $k = 1, 2, 3, 4$.

The formulae (3.25) contain five parameters to be fixed (γ_k and H). We calculate them demanding both the u and w functions to be regular at the origin

$$u(0) = 0, \quad w(0) = 0, \quad w'(0) = 0, \quad (3.27)$$

and the following conditions to hold

$$r_{tr} = 1.225, \quad Q = 0.137 \quad (3.27')$$

where

$$Q = \frac{\sqrt{2}}{10} \int_0^\infty u w r^2 dr \quad (3.28)$$

where r_{tr} is the triplet effective range and Q is the deuteron quadrupole moment. Then the crucial point is to check whether the functions (3.25) will also provide us with a good p_d value too (we mean $p_d \sim 0.04$); to get this value we could if necessary change the position of the poles ξ_1 and ξ_2 according to (3.24).

IV. Results and discussion

The numerical calculations were made by use of the computer IBM 709 at CERN. They showed that functions of the form (3.25) and fulfilling all the conditions stated above do exist. Now we wish to present and to discuss the results obtained.

First, it should be noted that only the equations (3.27) are linear in γ_i , H , whereas (3.27') are both quadratic. Therefore, one can in general expect to get more than one solution for this system of equations, namely, at most four that might be given by the two intersecting second order curves. It turned out, however, that in the range of the realistic (from the physical point of view) values of the parameters, only two solutions exist. In general, they give p_d values not too far apart (from each other). When the parameters involved are changed, one of the solutions (let us call it the solution I) gives a smaller p_d value than the other one (say, solution II). In general, both values are of the order of 0.05 and do not strongly depend on the actual values of the parameters. From the experimental point of view we would like to have

$$0.02 \lesssim p_d \lesssim 0.06. \quad (4.1)$$

Since both values are rather near the upper limit for p_d we would prefer to use solution I rather than solution II in applications.

The main features of both solutions are quite similar, however, as will be seen from the further discussion.

The next question is how the solutions depend on the values of the parameters ξ_1 , ξ_2 and C (read - f^2) that are more or less fixed numerically by their physical significance.

One may show that with fixed ξ_2 and f^2 the solutions do not depend strongly on ξ_1 . Nevertheless, one can observe that the dependence of the solution I is weaker and of another type than that of the solution II. Namely, it can be seen that $p_d^I(\xi_1)$ has a minimum that corresponds roughly to $\xi_1 = 4$ (when $\xi_2 \sim m$ and $f^2 \sim 0.08$). Going to lower values of ξ_1 , we get still higher and higher p_d up to the least acceptable value of ξ_1 , that is $\xi_1 = 2$. On the other hand, when we go to bigger ξ_1 values, we reach the point (approximately $\xi_1 = 4.7$) for which the curve $p_d^I(\xi_1)$ ends; for still higher ξ_1 values there is no solution. This is due to the fact that just for this value of ξ_1 the two second order curves come together and both solutions are identical. The behaviour of the second solution with ξ_1 changed is similar to that of the first one, but its dependence on ξ_1 is more pronounced and there is no minimum of $p_d(\xi_1)$; $p_d^{II}(\xi_1)$ is a decreasing function of its argument. So the value of p_d reached for the first solution and for ξ_1 at a minimum is the lowest possible value of p_d with fixed ξ_2 and f^2 .

The dependence on ξ_2 was also examined in the range of the ξ_2 values from $\xi_2 = 5$ to $\xi_2 = m = 6.72$. If one interprets the ξ_2 value as a parameter describing the influence of the nucleon core on the nucleon-nucleon interaction, one notes that these values of ξ_2 correspond to the radii of the core in the range

$$0.2 \lesssim r_c \lesssim 0.3.$$

It is found that $p_d(\xi_2)$ is an increasing function of its argument but the dependence on it is not very strong. Both solutions behave in a very similar way. Note that the position of the minimum mentioned above is shifted to the left when ξ_2 decreases.

The next point which may have a particular meaning is the problem of how the p_d value depends on f^2 . So far we have used the f^2 value taken from field theory and the results were reasonable. Now there is the question for what value of f^2 the results will no longer be reasonable; the answer can be treated as a statement concerning the upper and lower limits of f^2 compatible with the low energy deuteron data. A priori, one can expect that p_d will depend on f^2 as a function of the type

$$p_d = \frac{A f^4 + B f^2 + C}{D f^4 + E f^2 + F} \quad (4.2)$$

since both the functions $u(r)$ and $w(r)$ depend on f^2 linearly.

The function $p_d(f^2)$, as given by (4.2), was examined in a wide range of f^2 values for a few sets of ξ_1 and ξ_2 . It turned out that for f^2 not too small and not too large

$$0.03 \lesssim f^2 \lesssim 3,$$

the p_d has a minimum corresponding to approximately $f^2 = 0.14$, where this function takes on the value of about 0.02. The p_d values are not smaller than 0.02 and not bigger than 0.06, if f^2 belongs to the interval

$$0.07 \lesssim f^2 \lesssim 0.3 .$$

These limits are certainly very broad ones, but this can be understood, since the only criterion we used was the p_d value ^{*)}. It is hoped, nevertheless,

*) It should be noted, moreover, that the limits in question do not depend strongly on the ξ 's values. Generally speaking, the smaller values of the ξ 's, the more both the limits are shifted to the left, but the magnitude of this correction is only a small one.

that the upper limit especially could be reasonably lowered when other experimental information is used. One such piece of information is, e.g., the triplet shape parameter. It is defined as

$$P_t = \frac{1}{r_0^3} \int_0^\infty \left\{ u(r) \frac{du(r)}{dr^2} - e^{-\kappa r} \frac{de^{-\kappa r}}{dr^2} \right\} dr. \quad (4.3)$$

We see that the terms independent of f^2 cancel each other, and P_t is a function of f^2 of the type: $P_t = Af^2 + Bf^4$. We can then expect that to have as small a P_t value as observed experimentally ($P_t \sim 0.017$) we must keep f^2 sufficiently small. The corresponding calculations are not very easy to be performed as the dependence of the γ 's on κ^2 is not given explicitly. To gain insight into the problem we have examined it neglecting the tensor forces. Then the functions $u(r)$ of the form $u(r) = e^{-\kappa r} f(r)$ were found as described in Part II of this paper for a few values of κ^2 . Then the derivatives $d\gamma_i/d\kappa^2$ were calculated numerically and put into (4.3). It turned out, then, that (a) the main contribution to the integral in (4.3) came from the part $de^{-\kappa r}/d\kappa^2 \times f(r)$ and not from $e^{-\kappa r} \left(\frac{df(r)}{d\kappa^2} \right)$; (b) in the "big" part of the $du/d\kappa^2$ the most important term is

$$\frac{\gamma_1}{(\xi_1 + 2\kappa)^2}.$$

Had these two features been valid also for the full solution with tensor forces taken into account, we would be able to put as an upper limit for f^2 the value $f^2 \sim 0.15$. This limit, however, would depend on the values of ξ_1 and ξ_2 rather strongly, and the estimate we have given (0.15) should be "the worst one", in other words, for other choices of ξ_i , this limit would be lowered. Moreover, the criterion of the P_t value would also give a sharper lower limit for f^2 . If we accept the points (a) and (b) we should get $f^2 \gtrsim 0.09$, but it is also "the worst" limit, in the sense that for other ξ 's this limit

would be lower. To draw a conclusion we could say that it is very probable that in our model the upper limit of f^2 could be determined by a good value of the triplet shape parameter, and that this limit is not bigger than $f^2 \lesssim 0.15$.

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