ON THE RELATIVISTIC EFFECTS IN LOW ENERGY PION-NUCLEON SCATTERING

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The parts of the S and P amplitudes for the pion-nucleon scattering, which in the Cini-Fubini approach do not depend on the pion-pion interaction, are evaluated in a relativistic way. The self-consistency of the approximate theory used here is discussed on the basis of the integral equation for the resonance. This discussion shows that the theoretical results are affected by some uncertainty which, however, does not prevent one to make definite predictions on the contribution of the pion-pion interaction, especially in the case of the small P waves. This contribution, although somewhat reduced with respect to the one expected in the non-relativistic approximation, is nevertheless required in order to fit the experimental data, and can be calculated, with some uncertainty, at about 20%.

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RIASSUNTO--

Sono state calcolate in modo relativistico le parti delle ampiezze delle onde S e P dello scattering pione-nucleone che nella formulazione di Cini e Fubini sono indipendenti del contributo dell'interazione pione-pione. E' stata discussa l'autoconsistenza della teoria approssimata di cui si é fatto uso basandosi sulla equazione integrale per la risonanza. Questa discussione mostra che i risultati teorici sono affetti da una certa indeterminazione la quale tuttavia non impedisce, specialmente nel caso delle onde P piccole, di fare significative previsioni sul contributo dell'interazione pione-pione. Questo contributo, sebbene alquanto ridotto rispetto a quello previsto nella approssimazione non relativistica, é sempre necessario per ottenere l'accordo con i dati sperimentali, e può essere valutato con una incertezza di circa il 20%.

1. INTRODUCTION

Over the last few years several authors have tackled the problem of calculating the phase shifts in pion-nucleon scattering¹⁻⁸⁾. Substantial progress has been made using Cini and Fubini's unidimensional approximation⁹⁾ to the Mandelstam representation. This approximation allows one to separate what, in the scattering amplitude, is due to the processes with one pion and one nucleon in the intermediate state ("elastic" processes) from what is due to the processes where the intermediate state consists of more complex systems of particles ("inelastic" processes) and to express the contribution of these inelastic processes as the effect of the pion-pion interaction on low energy pion nucleon scattering.

1)

G.F. Chew, M.L. Goldberger, F.L. Low and Y. Nambu: Phys. Rev. 106, 1337 (1957). Hereafter called CGLN.

2)

W.R. Frazer and J.R. Fulco: Phys. Rev. 119, 1420 (1960)

3)

S.C. Frautschi and J.D. Walecka: Phys. Rev. 120, 1486 (1960)

4)

J. Bowcock, N. Cottingham and D. Lurié: Nuovo Cimento 16, 918 (1960) and 19, 142 (1961). Hereafter BCL. Notations are taken from those papers.

5)

W.R. Frazer: Proc. Roch. Conf. 1960, p. 282

6)

G. Höhler and K. Dietz: Zeit. Phys. 160, 453 (1960) Hereafter HD.

7)

J. Hamilton, and T.D. Spearman: Ann. of Phys. 12, 172 (1961)

8)

J. Hamilton, P. Menotti, T.D. Spearman and W.S. Woolcock: Nuovo Cimento 20, 519 (1961)

9)

M. Cini and S. Fubini: Ann. of Phys. 3, 352 (1960).

2.

The application of Cini and Fubini's method to the pion-nucleon scattering has been worked out in detail by Bowcock, Cottingham and Lurié⁴). These authors express the usual invariant amplitudes $A^{(\pm)}(s, t)$ and $B^{(\pm)}(s, t)$ in the following form:

$$(1.1a) \quad A^{(\pm)}(s, t) = \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \alpha_{\pm}^{(\pm)}(s', t) \left[\frac{1}{s'-s} \pm \frac{t}{s'-s} \right] ds' + X_A^{(\pm)}(s, t)$$

$$(1.1b) \quad B^{(\pm)}(s, t) = g_r^2 \left(\frac{1}{M^2-s} \mp \frac{1}{M^2-\bar{s}} \right) + \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \beta_{\pm}^{(\pm)}(s', t) \left[\frac{1}{s'-s} \mp \frac{t}{s'-s} \right] ds' + X_B^{(\pm)}(s, t).$$

where $\alpha_{\pm}^{(\pm)}(s, t)$ and $\beta_{\pm}^{(\pm)}(s, t)$ are the contributions to $\text{Im}A$ and $\text{Im}B$ due to the "elastic" processes defined above. The quantities here denoted by $X_A^{(\pm)}$ and $X_B^{(\pm)}$ are real in the approximation used by BCL, and express the effect of the pion-pion interaction. Their dependence on the s and t variables is given in BCL's work.

A calculation for the amplitudes of S and P waves in pion-nucleon scattering had already been worked out in 1957 by Chew, Goldberger, Low and Nambu¹). The starting point of CGLN differs from that of BCL in that the contribution of the "inelastic" processes is not taken into account, and so any effect of the pion-pion interaction is disregarded. Apart from this difference the BCL and CGLN works have the three following approximations in common:

A) The calculations are made in the "non-relativistic approximation": in other words both within and outside the integrals, the authors consider only the first terms of a series

in powers of $\frac{1}{M}$ for the expressions that occur. The justification of this approximation within the integrals is based on the following approximation B);

B) Within the dispersion integrals when $\text{Im}A$ and $\text{Im}B$ are developed in partial waves, only the resonant part $T=J=3/2$ is taken into account.

C) In order to calculate the S and small P amplitudes a behaviour like δ -function for $\text{Im} f_{33}$ is assumed.

A calculation of the amplitude for the S wave with the same outline as in CGLN has been effected by Höhler and Dietz⁶⁾ in a relativistic manner and assuming for $\text{Im} f_{33}$ a function fitting the experimental data¹⁰⁾.

In other words, HD reject the approximations A) and C) retaining only the approximation B) and do not take the pion-pion interaction into account.

In this work we make the relativistic calculation^{*)} of the contribution to the S and P amplitudes of the terms of (1.1) which do not depend on the pion-pion interaction. The purpose is to know how much this contribution agrees with the experiment, in order to confirm or modify the estimates made by BCL, in their non-relativistic approximation, of the amount of the effect which pion-pion interaction has on low energy pion-nucleon scattering. Our calculation is quite obvious and unambiguous as far as the Born terms are concerned, but a serious ambiguity arises with the dispersive integrals. In regard to these, we retain the approximation B) of CGLN and BCL, but there now arises the problem of how to choose the resonant function $\text{Im} f_{33}$ to be used. The question is no slight one, because, as we shall demonstrate in the following, the Höhler function¹⁰⁾ which fits the experimental

¹⁰⁾ G. Höhler: Nuovo Cimento 16, 585 (1960)

^{*)} The numerical calculations have been made with an IBM 1620 computer at the "Centro di Calcolo Numerico dell' Università di Genova".

data well enough is not a good solution of the integral equation for the resonant phase shift for values of c.m. momentum $q > 1.5$, while, on the other hand, it seems that to satisfy this integral equation better it would be necessary to introduce a function which remarkably differs from the experimental data in the decreasing part of the resonance. The reason for this inconsistency is evidently to be found on the theory being incomplete, and this incompleteness is particularly felt at higher energies.

It follows that the theoretical results will be affected by an uncertainty due to their sensitivity to the high energy behaviour of the function used to describe the resonance: the significant results will be those where this sensitivity is small.

One must therefore see if the uncertainty in the theoretical results allows any significant conclusion to be drawn. As we shall show, the small P phase shifts indicate without any doubt that a contribution of the pion-pion interaction, even if somewhat reduced with respect to that expected by BCL, is essential to fit the experiment. The results for the S wave are less significant: in this case, one must do some subtraction in order to eliminate the theoretical uncertainty. In this way one succeeds in determining unambiguously what the terms of (1.1) independent of the pion-pion interaction contribute to the real part of the S amplitude. In principle it would be possible here to determine the parameters of the pion-pion interaction and then to use the results for the P wave to check their consistency with the experiment. On the other hand, the introduction of the new parameter due to the subtraction in the S wave, implies that the parameters of the pion-pion interaction can be determined through the experimental data at a rather high energy, i.e. in the range from 200 to 500 MeV, where the experimental errors are large.

In the present experimental situation, therefore, a determination of the parameters of the pion-pion interaction by means of the data on the pion-nucleon scattering remains, although well defined in principle, somewhat ambiguous and qualitative.

2. DEDUCTION OF THE FORMULAS

We start, then, from the relations (1.1) and calculate only the terms independent of the pion-pion interaction.

From Cini and Fubini's approximation it follows that the quantities $\alpha_{\ell}^{(\pm)}(s, t)$ and $\beta_{\ell}^{(\pm)}(s, t)$ are of the form:

$$(2.1) \quad \begin{aligned} \alpha_{\ell}^{(\pm)}(s, t) &= \alpha_1^{(\pm)}(s) + t \alpha_2^{(\pm)}(s) \\ \beta_{\ell}^{(\pm)}(s, t) &= \beta_1^{(\pm)}(s) + t \beta_2^{(\pm)}(s) \end{aligned}$$

We also have

$$(2.2) \quad \begin{aligned} \alpha_{\ell}^{(\pm)}(s, t) &= \text{Im } A(s, t) \\ \beta_{\ell}^{(\pm)}(s, t) &= \text{Im } B(s, t) \end{aligned}$$

Strictly speaking the last relations hold good only below the threshold of the inelastic processes. However we shall retain their validity over all the resonance $T = J = 3/2$. Neglecting all contributions to the partial waves developments of $A(s, t)$ and $B(s, t)$ except the resonance $T = J = 3/2$, the relations (2.2) and (2.1) give:

$$(2.3) \quad \begin{cases} \alpha_1^{(\pm)}(s) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{4\pi}{3} \left[3 \frac{W+M}{E+M} + \frac{W-M}{E-M} \right] \text{Im } f_{33}(s) \\ \beta_1^{(\pm)}(s) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{4\pi}{3} \left[\frac{3}{E+M} - \frac{1}{E-M} \right] \text{Im } f_{33}(s) \\ \alpha_2^{(\pm)}(s) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{2\pi}{q^2} \frac{W+M}{E+M} \text{Im } f_{33}(s) \\ \beta_2^{(\pm)}(s) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{2\pi}{q^2} \frac{1}{E+M} \text{Im } f_{33}(s) \end{cases}$$

6.

Using the well known formulae for projection on the partial waves we obtain from the (1.1), apart from the terms $x \frac{(+)}{A}$, $x \frac{(+)}{B}$:

$$(2.4) \quad \text{Re } P_0^{(\pm)} = P_0^{(\pm)} + D_0^{(\pm)}$$

$$(2.5) \quad P_0^{(\pm)} = 2 \frac{M^2}{W} f^2 \left\{ -\frac{E+M}{W+M} \mp \frac{1}{2} [(E+M)(W-M)G_0 + (E-M)(W+M)G_1] \right\}$$

$$(2.6) \quad D_0^{(\pm)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{1}{\pi} \int_{M+\mu}^{\infty} K_0^{(\pm)}(W, W') \frac{F_{33}(W')}{q'^2} dW'$$

where :

$$(2.7) \quad K_0^{(\pm)}(W, W') = \frac{W'}{3W} \left[3 \frac{E+M}{E'+M} \frac{q'^2 - q^2}{W' - W} + \frac{1}{W+W'} \left(\frac{E+M}{E'-M} q'^2 - \frac{E-M}{E'+M} q^2 \right) \right]$$

$$\pm \frac{W'}{3W} \left\{ \frac{E+M}{2} \left[F_0 [F - (W-M)g] + 3F_2 \frac{W-W+2M}{E'+M} q^2 \right] - \frac{E-M}{2} \left[F_1 [F + (W+M)g] + 3F_2 \frac{W'+W+2M}{E'+M} q^2 \right] \right\}$$

$$(2.8) \quad f(W, W') = 3 \frac{W'+M}{E'+M} (q'^2 - q^2) - \frac{W'-M}{E'-M} q'^2$$

$$g(W, W') = \frac{3}{E'+M} (q'^2 - q^2) - \frac{q'^2}{E'-M}$$

$$(2.9) \quad F_2(W, W') = \int_{-1}^1 \frac{x^2 dx}{W'^2 + W^2 - 2(M^2 + \mu^2 + q^2) + 2q^2 x}$$

$$(2.10) \quad G_\alpha(W) = \int_{-1}^1 \frac{x^\alpha dx}{W^2 - M^2 - 2(u^2 + q^2) + 2q^2 x}$$

$$(2.11) \quad f_{1-}^{(\pm)} = P_{1-}^{(\pm)} + D_{1-}^{(\pm)}$$

$$(2.12) \quad P_{1-}^{(\pm)} = 2 \frac{M^2}{W} f^2 \left\{ \frac{E-M}{W-M} + \frac{1}{2} \left[(E+M)(W-M) G_1 + (E-M)(W+M) G_0 \right] \right\}$$

$$(2.13) \quad D_{1-}^{(\pm)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{1}{\pi} \int_{M+\mu}^{\infty} K_{1-}^{(\pm)}(W, W') \frac{\text{Im} f_{33}(W')}{q'^2} dW'$$

where :

$$(2.14) \quad K_{1-}^{(\pm)}(W, W') = \frac{W'}{3W} \left[\frac{1}{W'-W} \left(\frac{E+M}{E'+M} q^2 - \frac{E-M}{E'-M} q'^2 \right) - 3 \frac{E-M}{E'+M} \frac{q'^2 - q^2}{W'-W} \right] \\ + \frac{W'}{3W} \left\{ \frac{E+M}{2} \left[F_L [F - (W-M)q] + 3F_2 \frac{W'-W+2M}{E'+M} q^2 \right] - \frac{E-M}{2} \left[F_0 [F + (W+M)q] + 3F_2 \frac{W'+W+2M}{E'+M} q^2 \right] \right\}$$

$$(2.15) \quad \text{Re} f_{33} = P + D + C$$

8.

$$(2.16) \quad P = -2 \frac{M^2}{W} f^2 \left[(E+M)(W-M) G_1 - (E-M)(W+M) G_{20} \right]$$

$$(2.17) \quad D = \frac{1}{\pi} \frac{(W+M)^2 - u^2}{W^2} q^2 \oint_{M+\mu}^{\infty} \frac{W'^2}{(W'+M)^2 - u^2} \frac{\text{Im} f_{33}(W')}{q'^2} \frac{dW'}{W' - W}$$

$$(2.18) \quad C = \frac{1}{\pi} \int_{M+\mu}^{\infty} K(W, W') \frac{\text{Im} f_{33}(W')}{q'^2} dW'$$

where :

$$(2.19) \quad K(W, W') = \frac{W'}{3W} \left\{ \frac{E+M}{2} \left[F_1 [F - (W-M)g] + 3F_2 \frac{W' - W + 2M}{E' + M} q^2 \right] \right. \\ \left. + \frac{E-M}{2} \left[F_{20} [F + (W+M)g] + F_{31} \frac{W' + W + 2M}{E' + M} q^2 \right] \right\}$$

$$(2.20) \quad F_{20} = -\frac{3}{2} F_2(W, W') + \frac{1}{2} F_0(W, W')$$

$$F_{31} = -\frac{9}{2} F_3(W, W') + \frac{3}{2} F_1(W, W')$$

$$G_{20} = -\frac{3}{2} G_2(W) + \frac{1}{2} G_0(W)$$

The formulae deduced in this section are the basis of our discussion of the relativistic effects. We shall divide this discussion into two parts: the first regarding the Born terms only and the second regarding the dispersion integrals. As this concerns the S and small P waves, we shall refer in the following only to the "minus" combinations of the amplitudes. These are the combinations considered by BCL, because only these are affected by a pion-pion interaction in the $T = J = 1$ state.

3. RELATIVISTIC EFFECTS IN THE BORN TERMS

The relativistic Born terms for the S and P waves are given by the relations (2.5), (2.12) and (2.16). The corresponding terms in the static limit have been obtained by CGLN and are used by BCL. Denoting these terms by $P^{(NR)}$, they are^{*)}:

$$(3.1) \quad P_0^{(-)(NR)} = 2f^2 \omega$$

$$(3.2) \quad P_{1-}^{(-)(NR)} = -\frac{2}{3} f^2 \frac{q^2}{\omega}$$

$$(3.3) \quad P_{33}^{(NR)} = \frac{4}{3} f^2 \frac{q^2}{\omega}$$

The quantity ω is defined as: $\omega = W - M$.

^{*)} In the literature the term (3.1) multiplied by the phase space factor M is sometimes given as the non-relativistic limit of the $\bar{W}$ Born term for the S-wave. This is the non-relativistic limit used by HD and already obtained by CGLN (formula 3.2 of ref. 1). With this factor one obtains a threshold value practically equal to the threshold value of the corresponding relativistic term whereas, otherwise, the threshold value of (3.1) differs by about 10%. However the term (3.1) is used in the BCL analysis, and so we shall refer to it in comparison with the relativistic terms.

In Fig. 3.1 are represented the exact and non-relativistic Born terms for $f_0^{(-)}$, $f_{1-}^{(-)}$ and f_{33} .

As one can see, e.g. at $q = 1.5$, for the $f_0^{(-)}$ and $f_{1-}^{(-)}$ waves there is a difference of about 40% between the exact and the non-relativistic terms and of course this difference increases with q . On the contrary, for f_{33} this difference is very small throughout the range of q here considered: it is completely negligible at $q = 1.5$ and reaches 10% at about $q = 3$. The reason for this different behaviour of the relativistic corrections to the Born terms in the different waves lies in the different sensitivity of the direct and crossed terms to the relativistic corrections. While the direct terms are strongly affected by the relativistic corrections for small q values too, the crossed terms begin to show the relativistic effects only for values of q about equal to 3. This becomes evident when one analyses the exact and non-relativistic Born terms in their direct and crossed contributions, as it is shown on fig. 3.2:

The quantities plotted here are:

$$P_{0dir}^{(-)} = -2 \frac{M^2}{W} f^2 \frac{E+M}{W+M} ; \quad P_{0cr}^{(-)} = \frac{M^2}{W} f^2 [(E+M)(W-M)G_0 + (E-M)(W+M)G_1]$$

$$P_{0dir}^{(-)(NR)} = -2f^2 M + 3f^2 \omega ; \quad P_{0cr}^{(-)(NR)} = 2f^2 M - f^2 \omega$$

$$P_{1dir}^{(-)} = -2 \frac{M^2}{W} f^2 \frac{E-M}{W-M} ; \quad P_{1cr}^{(-)} = \frac{M^2}{W} f^2 [(E+M)(W-M)G_1 + (E-M)(W+M)G_0]$$

$$P_{1dir}^{(-)(NR)} = -\frac{f^2 q^2}{\omega} ; \quad P_{1cr}^{(-)} = \frac{1}{3} \frac{f^2 q^2}{\omega}$$

As the Born term for the f_{33} wave is due only to the crossed part, it becomes clear how little it is affected by the relativistic corrections in the range of q from zero to 3.

4. DISCUSSION OF THE SHAPE OF THE $T=J=3/2$ RESONANCE

In order to evaluate the relativistic effects on the dispersion integrals we must first of all establish what shape for the resonant function $\text{Im } f_{33}(W)$ is to be used. BCL merely describe the resonance with a δ -function suitably normalized, by assuming:

$$(4.1) \quad \sin^2 \alpha_{33} = \frac{4}{3} f^2 q_r^2 \pi \delta(\omega - \omega_r)$$

where $\omega = W - M$, and W_r , q_r are the values of ω and q at the resonance.

This particularly simple assumption is reasonable in the non-relativistic limit where all the results have rather limited reliability for q values < 1.5 . Of course the results for higher energies will be more affected by the shape of the resonance.

For this reason HD, adopting a more realistic point of view, assume the effective range formula on the left of the resonance, and a negative exponential on the right:

$$(4.2) \quad \sin^2 \alpha_{33} = \begin{cases} \frac{\lambda^2 q^6}{\left(\frac{W-M}{W_r-M}\right)^2 (W_r - W) + \lambda^2 q^6} & \text{for } q \leq 1.65 \\ A \exp\left(-\frac{W}{\sigma}\right) & \text{for } q \geq 1.65 \end{cases}$$

$$(4.3) \quad A = 47.5, \quad \sigma = 0.397$$

This function fits quite well the values if $\sin^2 \alpha_{33}$ at intermediate energies obtained by Willis¹¹⁾ and Walker¹²⁾ et al. by a phase shift analysis where α_{33} is assumed as good as real.

By using the function (4.2) HD have made the relativistic calculation of $\text{Re } f_0^{(-)*}$.

We note at this point that the question now arises of how consistent the function (4.2) is with the theory here used, which is defined by the equations (2.4), (2.11), (2.15) and by the unitarity condition for f_{33} . This unitarity condition transforms the relation (2.15) into an integral equation, and care is to be taken to see if the function (4.2) verifies this equation in a satisfactory way: this is necessary if the results for the small waves, obtained in the framework of the approximations made and by describing the resonance by the function (4.2), are to be reliable.

A very simple criterion to see how the function (4.2) verifies the integral equation above, consists in the comparison

11) W. Willis: Phys. Rev. 116, 753 (1959)

12) W. D. Walker, J. Davis and W.D. Shepard: Phys. Rev. 118, 1612 (1960).

*) A similar calculation has been made by K. Dietz also for P and D waves. We thank Professor G. Höhler for this communication.

of the quantity Ref_{33} given by the expression (2.15):

$$(4.4') \quad (\text{Ref}_{33})' = P + D + C$$

with the quantity Ref_{33} given by the unitarity condition:

$$(4.4'') \quad (\text{Ref}_{33})'' = \pm \left[\text{Im} f_{33} \left(\frac{1}{q} - \text{Im} f_{33} \right) \right]^{1/2}$$

This criterion rests upon the hypothesis that a possible additional contribution to the right-hand side of (4.4') coming from the pion-pion interaction is negligible.

Table 4.1 shows the values obtained for these quantities:

TABLE 4.1

q	$(\text{Ref}_{33})'$	$(\text{Ref}_{33})''$	$\Delta\%$
0.25	0.0118	0.0125	5.6
0.50	0.0479	0.0503	4.8
0.75	0.113	0.117	3.4
1.00	0.218	0.225	3.1
1.30	0.359	0.372	3.5
1.50	0.213	0.236	9.7
1.75	-0.160	-0.168	4.8
2.00	-0.251	-0.238	5.5
2.50	-0.203	-0.175	16
3.00	-0.112	-0.0949	18

and the same results are plotted in Fig. 4.1.

14.

The principal part of the singular integral has been numerically evaluated after subtracting the singularity ¹³⁾ i.e.,:

$$G \int_a^b \frac{f(x')}{x' - x} dx' = \int_a^b \frac{f(x') - f(x)}{x' - x} dx' + f(x) \ln \frac{b-x}{x-a} \quad (a < x < b)$$

The integral on the right-hand side is no longer singular, since in our case we have:

$$\lim_{x' \rightarrow x} \frac{f(x') - f(x)}{x' - x} \neq \infty$$

As is evident the percentage differences increase with q (at $q > 2$ they are decisively greater than 10%). The nature of this discrepancy seems not to be accidental, however the Höhler formula is not the only one that can fit the experimental data. It may be that the function (4.2) would decrease too fast with the energy, and, on the other hand, a function decreasing more slowly would improve the situation described by the Table 4.2. For the moment do not wish to take up this question in further detail, and so we shall limit ourselves to consider the function (4.2) as a good fit to the experiment among possible other fits.

We may exclude that the discrepancy found could be attributed to a contribution, so far neglected, of the small waves to the equation (2.15). Indeed a contribution of the

¹³⁾

G. Puppi and A. Stanghellini: Nuovo Cimento, 5, 1305 (1957).

S wave to the right-hand side of the equation (2.15) is of the order 10^{-5} and so cannot remove the above discrepancy.

We have, then, tried to see if it would be possible to remove or at least to reduce the discrepancy between the expressions (4.4') and (4.4'') for Ref_{33} by obtaining a direct solution, even if approximate, of the equation (2.15)

It is clear that for an ideally exact solution, (4.4') and (4.4'') coincide. Because in the range of q from zero to 3 the relativistic Born term differs very little from the non-relativistic one, as furthermore the D term is also very similar to non-relativistic term and the crossed term C is negligible with respect to the direct one (if one uses the Hhler formula the C term amounts only to a few percent of the D term), there are good reasons to believe that following the procedures already used in the literature to solve integral equations like (2.5) one would obtain a solution like the Breit-Wigner function of the non-relativistic limit. Indeed, after putting:

$$(4.5) \quad h(W) = \frac{W^2}{(W+M)^2 - \mu^2} \frac{F_{33}(W)}{q^2}$$

the equation (2.14) becomes:

$$(4.6) \quad \text{Re } h(W) = \frac{1}{q^2} \frac{W^2}{(W+M)^2 - \mu^2} P(W) + \frac{1}{\pi} \oint_{M+\mu}^{\infty} \frac{\text{Im } h(W')}{W' - W} dW' + \frac{1}{q^2} \frac{W^2}{(W+M)^2 - \mu^2} C(W)$$

We now introduce the following approximations:

i) We approximate the inhomogeneous term with a pole:

$$(4.7) \quad \frac{1}{q^2} \frac{W^2}{(W+M)^2 - \mu^2} P(W) = \frac{\alpha}{W - \beta}$$

The parameters α and β are determined by the least squares method.

By choosing:

$$(4.8a) \quad \alpha = 0.0366$$

$$(4.8b) \quad \beta = 6.54$$

the approximation (4.7) holds in the range of q from zero to 3 within few percents, while for higher values of q the two sides of the (4.7) assume completely different behaviours.

ii) We neglect the crossed dispersive term $C(W)$ postponing our verification of the consequences of this assumption until later.

With these approximations the equation (4.6) becomes:

$$(4.9) \quad \text{Re } h(W) = \frac{\alpha}{W - \beta} + \frac{1}{\pi} \beta \int_{M+\mu}^{\infty} \frac{\text{Im } h(W')}{W' - W} dW'$$

or

$$(4.10) \quad \text{Re } \frac{F_{33}(W)}{q^2} = \frac{(W+M)^2 - \mu^2}{W^2} \frac{\alpha}{W - \beta} + \frac{1}{\pi} \frac{(W+M)^2 - \mu^2}{W^2} \beta \int_{M+\mu}^{\infty} \frac{W'}{(W'+M)^2 - \mu^2} \frac{\text{Im } F_{33}(W')}{W' - W} \frac{dW'}{q'^2}$$

It is useful to compare the equation (4.10) with the corresponding one of the non-relativistic limit (equation 4.1 of CGLN neglecting the dispersive crossed term). It is apparent that the difference lies in the factor $\frac{(W+M)^2 - \mu^2}{W^2}$ and in changing the pole and the residue. The pole is shifted from $M = 6.72$ to $\beta = 6.54$ and the residue from

$$\frac{4}{3}f^2 \simeq 0.11 \text{ to } \frac{(\beta+M)^2 - 1}{\beta^2} \alpha \simeq 0.15$$

Our equation (4.9) is very similar to that obtained by Frazer and Fulco ²⁾ who write down a dispersion relation for the amplitude $h_1(W)$ on the basis of its analytic properties in the complex W plane and then obtain for $h_{33}(W)$ a simplified equation by neglecting the singularities far from the physical region and using a pole to describe the short cut on the real W axis coming from the pole in the $\bar{S}$ variable. The difference due to the different methods and approximations used affects only the parameters describing the pole, for Frazer and Fulco our β coincides with M and the value of our α is $\simeq \frac{4}{3}f^2 \simeq 0.027$.

The equation (4.9) can be solved using the N/D method ¹⁴⁾ by attributing the pole to N and the cut to D . As in ref. ²⁾, we must perform three subtractions in order to make the integral on $\text{Im}D$ convergent, and we obtain the result:

$$(4.11) \quad \frac{\alpha}{W-\beta} \frac{(W+M)^2 - \mu^2}{W^2} q^2$$

$$f_{33}(W) = \frac{1 - (W-\beta)^2 + (W-\beta)^2 \left[\nu - \frac{1}{\pi} \alpha (W-\beta)^3 \oint_{M+\mu}^{\infty} \frac{q'^3 [(W'+M)^2 - \mu^2] dW'}{(W'-\beta)^4 (W'-W) W'^2} - i\alpha \frac{q^3}{W-\beta} \frac{(W+M)^2 - \mu^2}{W^2} \right]}{1 - (W-\beta)^2 + (W-\beta)^2 \left[\nu - \frac{1}{\pi} \alpha (W-\beta)^3 \oint_{M+\mu}^{\infty} \frac{q'^3 [(W'+M)^2 - \mu^2] dW'}{(W'-\beta)^4 (W'-W) W'^2} - i\alpha \frac{q^3}{W-\beta} \frac{(W+M)^2 - \mu^2}{W^2} \right]}$$

¹⁴⁾

G.F. Chew and S. Mandelstam, Phys.Rev. 119, 467 (1960).

The parameters r and p are due to the subtractions. In the right-hand side of eq. (4.11) we neglect the third and the fourth term in the denominator: this is suggested by the analogy of Eq. (4.11) with the corresponding equation of the static theory¹⁵⁾ from which it follows $p \ll \tau$ and that neglecting the term with the integral is consistent with the similar neglect of the dispersive crossed term in Eq. (4.6). We shall not go into these considerations any further, limiting ourselves to justify a posteriori the approximations introduced here.

The parameter τ is given by the position of the resonance ($W_\tau = 8.85$) and so the Eq. (4.11) may be written as follows:

$$(4.12) \quad f_{33}(W) = \frac{\frac{\alpha}{W-\beta} \frac{(W+M)^2 - \mu^2}{W^2} q^2}{1 - \frac{W-\beta}{W_\tau - \beta} - i\alpha \frac{q^3}{W-\beta} \frac{(W+M)^2 - \mu^2}{W^2}}$$

Of course the functions (4.11) and (4.12) satisfy the unitarity condition. As is apparent, the function (4.12) is very similar to the Breit-Wigner function of the non relativistic limit except for the slow variation of the factor

$$\frac{(W+M)^2 - \mu^2}{W^2}$$

Now we must verify how accurate a solution is (4.12) for the integral equation. In this way we verify a posteriori the validity of the approximation introduced on the r.h.s. of Eqs. (2.15) and (4.11).

15) G.F. Chew: Theory of pion scattering and photoproduction, unpublished.

In order to do this, let us again consider the quantities $(\text{Ref}_{33})'$ and $(\text{Ref}_{33})''$ defined in (4.4'), (4.4'') and evaluated now by using the function (4.12). The results are shown in Table (4.2).

TABLE 4.2

q	$(\text{Ref}_{33})'$	$(\text{Ref}_{33})''$	$\Delta \%$
0.25	0.0127	0.0138	8.0
0.50	0.0519	0.0553	6.1
0.75	0.123	0.129	4.7
1.00	0.237	0.244	2.9
1.30	0.373	0.379	1.6
1.50	0.221	0.225	1.8
1.75	-0.107	-0.106	0.9
2.00	-0.205	-0.208	1.4
2.50	-0.187	-0.199	6.0
3.00	-0.146	-0.166	12

and are plotted in Fig. 4.2.

It is interesting to note that in the range from the resonance to $q = 3$ the relative situation of $(\text{Ref}_{33})'$ and $(\text{Ref}_{33})''$ is inverted with respect to the case of the Höhler formula.

As one can see from Table 4.2, the function (4.12) gives, on the average, smaller deviations than the phenomenological Höhler function. Of course the function (4.12) is still a very rough solution because of all the approximations made. An improvement could be obtained by finding a closer approximation to the inhomogeneous term (representing it with

two poles would be particularly suitable) and by accounting approximately for the crossed integral term. For the moment we shall limit ourselves to the approximate solution (4.12).

We have also considered the stability of the solution (4.12) with respect to variation of the parameter α assuming as a measure of the accuracy of the solution the quantity $\chi^2 = \sum_1^N [(Re f_{33})' - (Re F_{33})'']^2$ where the sum is extended to the values of q of Table 4.2. One finds that χ^2 reaches a minimum in the neighbourhood of the value of α given by the relation (4.8a).

In Fig. 4.3¹⁶⁻²³⁾ we compare the behaviours of $\frac{\sin^2 \alpha_{33}}{q}$ given by the function (4.2), by the Breit-Wigner formula of the non-relativistic limit, and by the function (4.12). The two last functions are practically coincident in the resonance region and become distinguished only at about $q = 3$. The experimental points are left completely below these curves: in the case of the Breit-Wigner function this is of course the well-known discrepancy of the experimental values from the Chew-Low plot beyond the resonance.

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- 16) J. Orear, J.J. Lord and A.B. Weaver: Phys. Rev. 93, 575 (1954).
 - 17) E. Fermi, N. Metropolis and E.F. Alei: Phys. Rev. 91, 155 (1953); 95, 1581 (1954).
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 - 22) J.H. Foote, O. Chamberlain, E.H. Rogers, H.M. Steiner, C. Wiegand and T. Ypsilantis: Phys. Rev. Lett. 4, 30 (1960).
 - 23) M.E. Blevins, M.M. Black and J. Leitner: Phys. Rev. 112, 1287 (1958).

The impossibility of reproducing through the right-hand side of (2.15) the real part of (4.2) given by the unitarity condition, and the really notable divergence between the imaginary parts of f_{33} given by (4.2) and (4.12) give rise to some doubts about the reliability of the real parts of the S and small P waves obtained from the relations (2.4) and (2.11). These relations, indeed, hold within the same approximations as the equation (2.15), these approximations being evidently inadequate. We shall assume as a measure of the uncertainty of the results the difference between those obtained by means of the functions (4.2) and (4.12): the first fits the experimental data well but does not verify the integral equation in a satisfactory way in the range of q from 1.5 to 3, the second is a better solution of the integral equation but disagrees remarkably with the experiment to the right of the resonance.

5. SENSITIVITY OF THE DISPERSION INTEGRALS FOR THE SMALL WAVES TO THE SHAPE OF THE RESONANCE

The integrals (2.6) and (2.13) for the S and small P waves have been numerically evaluated by using for $\text{Im}f_{33}$ first the function (4.2) and then the function (4.12). The possibility of drawing significant conclusions depends essentially on the sensitivities of these integrals to the shapes used to describe the resonance.

The results for the S and small P waves are shown in Fig. 5.1 and in Fig. 5.2, where the non-relativistic limits and the experimental points are also plotted.

Although in both cases the differences between the two types of curves are remarkable, we can nevertheless observe that the experimental points lie always above the two curves.

This fact shows that, similarly to what happens in the static limit, the relativistic calculation which disregards the pion-pion contribution also maintains an appreciable discrepancy between the theoretical previsions and the experimental data for both the S and the small P waves.

The results for the P wave are more significant than those for the S wave: the experimental points are far from both the relativistic curves, and the static limit lies always below them; therefore, one sees that the relativistic calculation implies a contribution from the pion-pion interaction even if smaller than that predicted by BCL. For the S wave the situation is less clear, and the curve of the static limit intersects both the relativistic curves.

Even if the aim of the present paper is not the evaluation of the parameters of the pion-pion interaction, we nevertheless wish to show that the theoretical indetermination for the S wave can be eliminated by making a subtraction in the S wave dispersion integral. In principle the parameters of the pion-pion interaction could then be derived from S wave, and their consistency could be checked by analysing the P wave results.

Let us therefore write the kernel (2.7) in the following form:

$$(5.1) \quad K_0^{(-)}(W, W') = K^{(0)}(W, W') + K^{(-1)}(W, W')$$

where $K^{(0)}$ is of order zero in W' , and $K^{(-1)}$ of order W^{-1} . We now introduce the kernel:

$$(5.2) \quad \tilde{K}(W, W') = K^{(0)}(W, W') - \frac{F(W)}{F(W_0)} K^{(0)}(W_0, W') + K^{(-1)}(W, W')$$

where :

$$W_0 = M + \mu$$

$$F(W) = \frac{(E + M)(W - M)}{W}$$

It is easy to verify that $\tilde{K}(W, W')$ is of order W^{-1} . Hence, if we put:

$$(5.3) \quad \tilde{D}(W) = -\frac{1}{\pi} \int_{M+\mu}^{\infty} \tilde{K}(W, W') \frac{\text{Im} F_{33}(W')}{q'^2} dW'$$

the dispersive term of the right-hand side of (2.4) becomes

$$(5.4) \quad D_0^{(-)}(W) = \tilde{D}(W) - \frac{F(W)}{F(W_0)} \int_{M+\mu}^{\infty} K^{(0)}(W_0, W') \frac{\text{Im} F_{33}}{q'^2} dW'$$

As we wish to introduce the threshold value of $\text{Re} f_0^{(-)}$, because here the pion-pion contribution is different from zero ⁴⁾, we rewrite the Eq. (2.4) thus:

$$(5.5) \quad \text{Re} f_0^{(-)} = P_0^{(-)} + D_0^{(-)} + X_0$$

where $X_0(W)$ represents the contribution to the S wave deriving from the pion-pion interaction.

Therefore the equation (5.5), by using (5.3) and (5.4), becomes:

$$(5.6) \quad \text{Re} f_0^{(-)}(W) = P_0^{(-)}(W) + \tilde{D}(W) + N \frac{F(W)}{f(W_0)} + X_0(W)$$

where the constant N is defined as:

$$(5.7) \quad N = - \int_{M+\mu}^{\infty} K^{(0)}(W_0, W') \frac{\text{Im} \tilde{f}_{33}(W')}{q'^2} dW'$$

but its numerical value is given by the equation:

$$(5.8) \quad N = \text{Re} f_0^{(-)}(W_0) - P_0^{(-)}(W_0) - \tilde{D}(W_0) - X_0(W_0)$$

Here $P_0^{(-)}(W_0)$ and $\tilde{D}(W_0)$ can be obtained from (2.5) and (5.3). $X_0(W_0)$ is, in principle, expressed through the parameters of the pion-pion interaction and $\text{Re} f_0^{(-)}(W_0)$ is taken from the experiment:

$$(5.9) \quad \text{Re} f_0^{(-)}(W_0) = \frac{1}{3} (a_1 - a_3)$$

where a_1 and a_3 are the S scattering lengths.

We have evaluated the quantity $\tilde{D}(W)$ by using for $\text{Im} f_{33}$ first the function (4.2) and then the (4.12). It is

easy to see that, independently of the parametrization of the pion-pion interaction, the discrepancy between the two relativistic curves becomes completely negligible.

In order to do this, let us write the Eq. (5.6) as follows:

$$(5.10) \quad \text{Re} f_0^{(-)}(W) = g_1(W) + g_2(W)$$

where :

$$(5.11) \quad g_1(W) = P_0^{(-)}(W) + \tilde{D}(W) + N' \frac{F(W)}{F(W_0)}$$

$$(5.12) \quad N' = N + X_0(W_0) = \text{Re} f_0^{(-)}(W_0) - P(W_0) - \tilde{D}(W_0)$$

$$(5.13) \quad g_2(W) = X_0(W) - X_0(W_0)$$

In Eq. (5.10) $g_1(W)$ does not depend on the parametrization of the pion-pion interaction but depends on the shape of $\text{Im} f_{33}$, while, on the contrary, $g_2(W)$ depends only on the pion-pion interaction and is not affected by any ambiguity due to the shape of $\text{Im} f_{33}$.

In Table 5.1 we show the values obtained for $g_1(W)$ by using first the function (4.2) and then the (4.12) for the evaluation of $\tilde{D}(W)$ (second and third columns respectively). We have assumed $a_1 - a_3 = 0.255$. As one can see, the differences are at the most only a few percent.

TABLE 5.1
Values of the function $3g_1(W)$

<u>q</u>	$3g_1(W)$	
0	0.255	0.255
0.25	0.261	0.261
0.5	0.273	0.273
0.75	0.288	0.288
1	0.309	0.309
1.5	0.351	0.351
2	0.387	0.384
2.5	0.417	0.405
3	0.435	0.417

In Fig. 5.3 is plotted the function $3g_1(W)$ together with the experimental data. These are left always below, and this fact implies a contribution of the pion-pion interaction in order to fit the experiment.

That the subtraction parameter is determined by the threshold value implies that the contribution of the function $g_2(W)$ can be determined mainly by the data at higher energy, i.e., in the range from 200 to 500 MeV. In this region the experimental errors are still very large, and Fig. 5.3 shows that the difference between $g_1(W)$ and the mean experimental values, although always positive, is in each case of the magnitude of the experimental error. In this connection, it is very much to be hoped that there may be a systematic analysis effected of the phase-shifts of the pion-nucleon scattering in the energy range 200-500 MeV, in order to obtain further data and achieve greater precision in what is already known.

6. CONCLUDING REMARKS

Let us now summarize the conclusions of our relativistic calculation of the amplitudes $f_0^{(-)}$ and $f_{1-}^{(-)}$ for pion-nucleon scattering.

We have evaluated only the contribution independent of the pion-pion interaction: information on the latter can be obtained by comparing our results with the experimental data.

a) Reliability of the results obtained - This has been established by testing if the integral equation obtained for the resonant phase shift succeeds in reproducing the experimental resonance. We find that this reproduction takes place within few percents in the range from zero to 170 MeV, while in the range from 170 to 500 MeV the discrepancy increases progressively and comes to exceed 10%. On the other hand, a resonance which will reproduce itself throughout the range

from zero to 500 MeV seems to be always beyond the experimental data to the right of the resonance. This fact shows that the theory used is incomplete, and this is particularly felt with the increasing of the energy.

b) Estimate of the theoretical uncertainty - We have assumed as a measure of the theoretical uncertainty the difference between the results obtained by describing the resonance first by means of a fit with experimental data, and then by means of a function which will reproduce itself, through the integral equation, within few percents in the range from zero to 500 MeV.

c) Prevision regarding the pion-pion contribution -
S wave: the theoretical uncertainty prevents our drawing significant conclusions and, if this uncertainty is reduced by means of suitable subtraction procedures, the results show that experimental results more accurate than the present ones in the range from 200 to 400 MeV are necessary in order to make unambiguous previsions.

P wave: in this case, on the contrary, although there is still a not insignificant theoretical uncertainty, this does not surpass, on average in the energy range from 200 to 400 MeV, 20% of the discrepancy between the mean experimental and theoretical values. This fact makes it possible to predict, to within 20%, the contribution of the pion-pion interaction to the pion-nucleon scattering.

d) Comparison with the non-relativistic results -

For the S wave the non-relativistic amplitude differs remarkably from the relativistic one not only at high but also at low energies, and so the non-relativistic approximation

seems here to be inadequate to calculate the contribution of the pion-pion interaction. For the P wave the relativistic correction is very small at low energies and reaches about 30% around 250 MeV. The relativistic curves lie between the non-relativistic one and the experimental points: therefore the contribution of the pion-pion interaction turns out to be less than that predicted by the non-relativistic approach. As far as the Born terms are concerned, we note that the direct terms are more affected by the relativistic correction than the crossed ones. In particular, the Born term for the resonant amplitude, which is given only by the crossed part, coincides within few percents with the non-relativistic term, the difference reaching 10% only at energies around 550 MeV. This fact explains in part how a solution (good to within few percents) of the integral equation for the resonant phase shift can be given, in all the range from zero to 500 MeV, by a function very similar to the Breit-Wigner function which describes the resonance in the non-relativistic theory. It seems therefore that, in the framework of the approximations made, the mere introduction of the relativistic kinematics cannot eliminate that discrepancy that in the static theory is represented by the well-known displacement of the experimental points from the Chew-Low plot above the resonance.

We wish now to conclude with a general remark about the whole problem of pion-nucleon scattering at low and intermediate energies. In this process there is a dominant feature, i.e., the resonance in the $T = J = 3/2$ state, and then many secondary aspects, such as, for example, the behaviour of the small phase-shifts with the energy. A fundamental

step in pion-nucleon physics, as is well known, was made in 1956 with the static theory of Chew and Low, which succeeded in obtaining the coupling constant and the existence of the resonance, without giving, on the other hand, accurate information about the details. Every time that these matters of detail have been considered, thus seeking to make a substantial advance with respect to static theory, a whole complex of problems has arisen: the effects to be calculated are small and have many different causes often very difficult to account for adequately: relativistic kinematics, rescattering, high energy contribution, contribution from the pion-pion interaction and, more generally, from the inelastic processes. In addition, a small variation of the dominant feature, i.e., the shape of the resonance in the $T = J = 3/2$ state, remarkably affects these "minor" details. It is therefore clear what sort of difficulty it is that faces one when tackling the detailed characteristics of this process and why it is that the further progress has necessarily been much slower than the early results in static theory.

When this work was already finished we have seen a preprint by K. Dietz in which one finds the results of the relativistic calculation of S, P and D phase shifts obtained by describing the resonance with the Höhler formula and accounting for the pion-pion resonance by the approach proposed by Frazer⁵⁾. The results of Dietz independent of the pion-pion interaction coincide with those obtained by us under the same conditions.

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FIGURE CAPTIONS

- Fig. 3.1 Exact and non-relativistic Born terms for $f_0^{(-)}$, $f_{1-}^{(-)}$ and f_{33} .
- Fig. 3.2 Exact and non-relativistic contributions of the direct and crossed parts to the Born terms.
- Fig. 4.1 The quantities $(\text{Ref}_{33})'$ and $(\text{Ref}_{33})''$ evaluated by using the Höhler function (4.2). The C line shows the contribution of the crossed term.
- Fig. 4.2 The quantities $(\text{Ref}_{33})'$ and $(\text{Ref}_{33})''$ evaluated by using the function (4.12).
- Fig. 4.3 The behaviour of $\frac{\sin^2 \alpha_{33}}{q}$ given by the Höhler formula (4.2) and by the function (4.12). The dotted line is the Breit-Wigner function of the non-relativistic limit. Experimental points from the Refs. 11), 12), 16-23).
- Fig. 5.1 $3\text{Ref}_0^{(-)}$ obtained by the Höhler function (4.2) and by the function (4.12). The dotted curve is the non-relativistic approximation of BCL. The low energy experimental points are those obtained from a fit by Hamilton and Woolcock (see Ref. 21)), the other experimental points from Refs. 12), 17), 21).
- Fig. 5.2 The same as Fig. 5.1 for $3\text{Ref}_{1-}^{(-)}$.
- Fig. 5.3 The function $3g_1(W)$.

Fig. 3.1

Fig. 3.2

Fig. 4.1

Fig. 4.2

Fig. 4.3

Fig. 5.1

Fig. 5.2

Fig. 5.3