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RELATIVISTIC SCALAR PRODUCTS OF γ MATRICES

J.S.R. Chisholm *)

CERN - Geneva

A B S T R A C T

If S and S' are products of γ matrices containing an odd number of γ_μ ($\mu = 0, 1, 2, 3$) and any number of γ_5 , and if $\gamma_\alpha \dots \gamma^\alpha$ is the relativistic scalar product, then

$$\text{Tr}[\gamma_\alpha S] \text{Tr}[\gamma^\alpha S'] = 2 \text{Tr}[(S + S_R) S']$$

where S_R is obtained from S by reversing the order of all γ matrices.

*) On leave of absence from the School of Mathematics Trinity College, Dublin, Ireland.

In relativistic perturbation theory, a recurrent technical problem is the evaluation of a product of traces of γ matrices. The matrices which occur in traces may be :

- a) single matrices γ_μ ($\mu = 0, \dots, 3$), perhaps forming relativistic scalar products $\gamma_\mu p^\mu \equiv \gamma_0 p_0 - \underline{\gamma} \cdot \underline{p}$ with relativistic 4 momenta p^μ ;
- b) the matrix $\gamma_5 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3$; or
- c) relativistic scalar products $\gamma_\alpha \dots \gamma^\alpha$.

We follow the notation of Schweber ¹⁾ throughout.

The problem of eliminating scalar products $\gamma_\alpha \dots \gamma^\alpha$ when both terms lie within one trace was solved some years ago ²⁾. It was shown that if $S = \gamma_\rho \gamma_\sigma \dots \gamma_\tau$ is an "odd string" containing an odd number of matrices γ_μ ($\mu = 0, \dots, 3$), and

$$S_R = \gamma_\tau \dots \gamma_\sigma \gamma_\rho$$

is the "reversed string", formed by writing the matrices in the reverse order, then

$$\gamma^\alpha S \gamma_\alpha = -2 S_R \quad ; \quad (1)$$

for an "even string" $S \gamma_\omega$ ($\omega = 1, \dots, 4$), it follows that

$$\gamma^\alpha S \gamma_\omega \gamma_\alpha = 2 \left[\gamma_\omega S + S_R \gamma_\omega \right]. \quad (2)$$

These formulae are also true for strings S which contain an odd number of γ_μ and any number of γ_5 , the reversed string being defined by reversing the order of all γ matrices. This follows because the insertion of $\gamma_5 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3$ does not alter the

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"parity" of a string and because $(\gamma_5)_R \equiv -i \gamma_3 \gamma_2 \gamma_1 \gamma_0 = \gamma_5$. We define odd and even strings generally to be those containing odd and even numbers of γ_μ ($\mu = 0, \dots, 3$), irrespective of γ_5 factors.

This note gives the solution of the problem of eliminating scalar products $\gamma_\alpha \dots \gamma^\alpha$ when the matrices occur in different traces, so that we have to express

$$\text{Tr} [\gamma_\alpha S] \text{Tr} [\gamma^\alpha S'] \quad (3)$$

in terms of the strings S and S' alone. Now S and S' are (4×4) matrices and can be expressed in the form ¹⁾

$$S = \sum_{i=1}^{16} T_i X_i \quad (4a)$$

and

$$S' = \sum_{j=1}^{16} T_j Y_j, \quad (4b)$$

where the matrices $T_1, \dots, T_{16}$ are multiples of the matrices $I, \gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu \gamma_5$ and γ_5 . For convenience we take $T_1 = i \gamma_1, T_2 = i \gamma_2, T_3 = i \gamma_3$ and $T_4 = \gamma_0$, so that

$$\gamma_\alpha \dots \gamma^\alpha = \sum_{R=1}^4 T_R \dots T_R. \quad (5)$$

Using (4), (5) and the property $\text{Tr} [T_i T_j] = 4 \delta_{ij}$, (3) becomes

$$\begin{aligned}
& \sum_{k=1}^4 \text{Tr} \left[T_k \sum_{i=1}^{16} T_i X_i \right] \text{Tr} \left[T_k \sum_{j=1}^{16} T_j Y_j \right] \\
&= 16 \sum_{k=1}^4 X_k Y_k \\
&= 4 \text{Tr} \left[\left(\sum_{i=1}^4 T_i X_i \right) \left(\sum_{j=1}^{16} T_j Y_j \right) \right].
\end{aligned} \tag{6}$$

In (6), the second sum is S' ; if we can find an operator P such that

$$P(S) \equiv P\left(\sum_{i=1}^{16} T_i X_i\right) = \sum_{i=1}^4 T_i X_i, \tag{7}$$

then (3) and (6) reduce to $4\text{Tr}[P(S)S']$, which is of the desired form.

If the string S in (3) is an even string, the product is zero; so we are only concerned with odd strings S . For these, only the T matrices representing γ_μ and $\gamma_\mu \gamma_5$ occur in the sum (4a); the operator P in (7) must eliminate the terms containing $\gamma_\mu \gamma_5$. If we define an operator Q by

$$Q(T) = \gamma_\alpha T \gamma^\alpha,$$

then $Q(\gamma_\mu) = -2\gamma_\mu$, while $Q(\gamma_\mu \gamma_5) = 2\gamma_\mu \gamma_5$. So if we

take $P = \frac{1}{2} - \frac{1}{4} Q$, Eq. (7) is satisfied for odd strings S . But by (1), $Q(S) = -2S_R$ for odd strings; hence

$$P(S) = \frac{1}{2} (S + S_R). \quad (8)$$

So from (3), (6), (7) and (8), we have the final result that for non-zero traces,

$$\text{Tr} [\gamma_\alpha S] \text{Tr} [\gamma^\alpha S'] = 2 \text{Tr} [(S + S_R) S']. \quad (9)$$

When only one scalar product $\gamma_\alpha \dots \gamma^\alpha$ occurs in a calculation, the result (9) is probably of no value. But if more than one scalar product occurs, it begins to be of value. If one attempts to automate the calculation of higher order matrix elements and cross-sections, using the systematic procedure developed in an earlier paper ³⁾, one of the major problems encountered is the proliferation of such scalar products. It is now possible to teach a computer the simple general rules (1), (2) and (9), in order to deal with any number of these products.

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R E F E R E N C E S

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