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RENORMALIZATION GROUP AND REGGE BEHAVIOUR

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In recent papers ^{1),2),3)}, the equations of the renormalization group have been used to obtain Regge pole behaviour. In this note, the validity of these results is discussed. We show that the renormalization group equations contain a certain amount of arbitrariness, when a perturbation theory amplitude is used as input. The resulting solutions (which include those of Regge type) coincide only in a domain of s which does not contain the asymptotic region $s \rightarrow \infty$.

In order to see how the renormalization group works, let us take the case of a scalar field $\phi(x)$ ¹⁾, with mass m and a self-interaction Lagrangian $mg\phi^3(x)$, where g is a dimensionless coupling constant. The amplitude for the elastic two-particle process is written as $T(\frac{t}{\lambda}, \frac{s}{\lambda}, \frac{m^2}{\lambda}, g^2)$, where s and t are the Mandelstam variables, and λ the parameter of the renormalization group. Following Arbusov et al., we shall define $M(x, y, \mu, g^2)$ by :

$$T(x, y, \mu, g^2) = \frac{m^2 g^2}{m^2 - s} M(x, y, \mu, g^2) \quad (1)$$

$$\text{where } x = \frac{t}{\lambda}, y = \frac{s}{\lambda}, \mu = \frac{m^2}{\lambda}.$$

The renormalization group equation for M , with respect to the x variable, can be written as :

$$\frac{\partial}{\partial x} \log M(x, y, \mu, g^2) = \frac{1}{x} \frac{\partial}{\partial \xi} \log M \left[\xi, \frac{y}{x}, \frac{\mu}{x}, g^2 \psi(x, \mu, g^2) \right]_{\xi=1} \quad (2)$$

Here $g^2 \psi(x, \mu, g^2)$ defines the "invariant charge", and the function $\psi(x, \mu, g^2)$ satisfies the equations ⁴⁾ :

2.

$$\frac{\partial}{\partial x} \psi(x, \mu, g^2) = \frac{\psi(x, \mu, g^2)}{x} - \frac{\partial}{\partial \xi} \psi \left[\xi, \frac{\mu}{x}, g^2 \psi(x, \mu, g^2) \right]_{\xi=1} \quad (3)$$

$$\psi(1, \mu, g^2) = 1 \quad (4)$$

In the following $\psi(x, \mu, g^2)$ is assumed to be known. Now, Eq. (2) is solved as follows : suppose that we know an approximate expression $M_0(x, y, \mu, g^2)$ for M , as given for example by perturbation theory and consider the equation :

$$\frac{\partial}{\partial x} \text{Log } M(x, y, \mu, g^2) = \frac{1}{x} \frac{\partial}{\partial \xi} \text{Log } M_0 \left[\xi, \frac{y}{x}, \frac{\mu}{x}, g^2 \psi(x, \mu, g^2) \right]_{\xi=1} \quad (5)$$

Putting $x = 1$ and using (4) one gets :

$$\frac{\partial}{\partial \xi} \text{Log } M(\xi, y, \mu, g^2)_{\xi=1} = \frac{\partial}{\partial \xi} \text{Log } M_0(\xi, y, \mu, g^2)_{\xi=1} \quad (6)$$

From this, it follows that every solution of (5) is also a solution of (2). Hence, starting with a function $M_0(x, y, \mu, g^2)$ which does not satisfy the renormalization group equation (2), Eq. (5) provides us, by simple quadrature, with a function $M(x, y, \mu, g^2)$ which does.

Arbusov et al.¹⁾ start with the amplitude M_0 given by the two following graphs^{*)}:

When s is much bigger than m^2 , M_0 takes the form:

$$M_0(x, y, \mu, g^2) = 1 + \frac{g^2}{8\pi^2} \phi\left(\frac{x}{\mu}\right) \log\left(-\frac{y}{\mu}\right) \quad (7)$$

where

$$\phi(x) = \int_4^{\infty} \frac{dx'}{\sqrt{x'(x'-4)} (x'-x)}$$

Using this expression for M_0 , Eq. (5) becomes

*) Evidently these terms do not represent the complete amplitude to order g^4 ; moreover they are not the dominant ones in the asymptotic region $s/t \rightarrow \infty$. The effect of the above choice of M_0 is to reproduce the ladder series (in the t channel). It may be noted that, in contrast with a Bethe-Salpeter treatment, the g^2 term alone would not yield this series.

4.

$$\frac{\partial}{\partial x} \log M(x, y, \mu, g^2) = \frac{g^2 \psi(x, \mu, g^2)}{8\pi^2 (1+A)} \log\left(-\frac{y}{\mu}\right) \frac{\partial}{\partial x} \phi\left(\frac{x}{\mu}\right) \quad (8)$$

where
$$A = \frac{g^2 \psi(x, \mu, g^2)}{8\pi^2} \phi\left(\frac{x}{\mu}\right) \log\left(-\frac{y}{\mu}\right)$$

Equation (8) can be easily integrated in the special case $\psi(x, \mu, g^2) \equiv 1$ ($\psi \equiv 1$ is a solution of Eqs. (3) and (4), and is actually the first term in the perturbation expansion of ψ). The integration constant can be chosen in such a way that the solution of (8) is simply

$$M(x, y, \mu, g^2) = M_0(x, y, \mu, g^2) \quad (9)$$

This means that the input function $M_0(x, y, \mu, g^2)$ already satisfies the renormalization group equation (2).

However, this is not the solution given by Arbusov et al.¹⁾ According to these authors, one must expand the right-hand side of (8) in powers of g^2 ^{*)} and keep only the first term of this expansion⁶⁾:

*) Actually, Arbusov et al. make the expansion in powers of $g^2 \psi$ which is the last argument of the function M_0 in the equation (5). They get the following expression of $\alpha(t)$ (Eq. (6) in Ref. 1)):

$$\alpha(t) = -1 + \frac{g^2}{8\pi^2 m^2} \int_{-\infty}^{-t} \phi'\left(\frac{t'}{m^2}\right) \psi\left(\frac{t'}{m^2}, 1, g^2\right) dt'$$

$$\frac{\partial}{\partial x} \text{Log } M(x, y, \mu, g^2) = \frac{g^2}{8\pi^2} \text{Log} \left(-\frac{y}{\mu}\right) \frac{\partial}{\partial x} \phi\left(\frac{x}{\mu}\right) \quad (10)$$

With a particular choice of the integration constant, the solution can be written as follows

$$M(x, y, \mu, g^2) = \left(-\frac{y}{\mu}\right)^{\frac{g^2}{8\pi^2} \phi\left(\frac{x}{\mu}\right)} \quad (11)$$

The T amplitude then possesses a Regge-type behaviour in $s^{\alpha(t)}$ with

$$\alpha(t) = -1 + \frac{g^2 m^2}{8\pi^2} \int_{4m^2}^{\infty} \frac{dt'}{\sqrt{t'(t'-4m^2)} (t'-t)} \quad (12)$$

Let us compare the solutions (9) and (11). First, the differential Eq. (10) can be obtained by putting $\psi \equiv 1$ and by neglecting A with respect to 1, in the differential Eq. (8). Now, when $\psi = 1$ and $A \ll 1$, the solution (11) reads as follows

$$M(x, y, \mu, g^2) = \exp A$$

$$\simeq 1 + A = M_0(x, y, \mu, g^2)$$

Thus, the solutions (9) and (11) are not different when the following condition is fulfilled

$$\left| \frac{g^2}{8\pi^2} \phi\left(\frac{x}{\mu}\right) \text{Log}\left(-\frac{x}{\mu}\right) \right| \ll 1 \quad (13)$$

We see at once that this condition excludes the asymptotic region $s \rightarrow \infty$ ($y \rightarrow \infty$), which is the one of interest. On the other hand, in the asymptotic region, solutions (9) and (11) differ completely from each other in behaviour.

Secondly, we agree with the statement of Arbusov et al., that, in the perturbation expansion of the right-hand side of (8), the only reliable terms are those of the same order as that of the input. In particular, with the input (7), only the first order terms are to be trusted ^{*}). But we claim further that the solution obtained by limiting this expansion to its first term is no better than the solution obtained by using the complete expression. To the extent that the solution depends on higher terms of the perturbation expansion, it is not reliable. This is the case for the Regge-type solution (11), in the asymptotic region $s \rightarrow \infty$.

In other words, when using the renormalization group equations for the scattering amplitude at asymptotic energies, within the framework of perturbation theory, one cannot reasonably expect to get solutions valid beyond this framework ^{**}).

^{*}) See, however, footnote page 3, concerning the validity of (7).

^{**}) The basis of improvement on perturbation theory by renormalization group methods for the Green's functions ⁴⁾ or at large momentum transfer ⁵⁾ is discussed in a recent paper by Huang and Low (preprint).

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REFERENCES

- 1) B.A. Arbusov et al., "The asymptotic behaviour of the scattering amplitudes and the renormalization group method", Physics Letters 2, 150 (1962).
- 2) A.A. Logunov et al., "Regge poles in quantum electrodynamics", preprint JINR E-1122.
- 3) A.A. Logunov et al., "Regge poles and perturbation theory", preprint JINR E-1194.
- 4) N.N. Bogoliubov, D.V. Shirkov, "Introduction to the theory of quantized fields", New York, Interscience, 1959, Chapter VIII, Eqs. (42.24) and (42.26).
- 5) K.E. Eriksson, Il Nuovo Cimento, X, 27, 178 (1963).
- 6) B.M. Barbashov, private communication.