CM-P00056846

AN APPROACH TO THE THEORY OF SINGLE PION PRODUCTION
IN NUCLEON-NUCLEON COLLISIONS

E. Ferrari

CERN-Geneva

and

F. Selleri ⁺⁾

Istituto di Fisica dell'Università -- Bologna
Istituto Nazionale di Fisica Nucleare - Sez. di Bologna

A B S T R A C T

The matrix element for the process of single pion production in nucleon-nucleon collisions is constructed under the hypothesis of a dominant contribution from the one-pion exchange term. The effects of the virtuality of the π -meson are fully taken into account. It is found that the finite size of the nucleon cannot be neglected. The method used allows the determination of the "pionic form factor" of the nucleon. It is shown that, by the choice of a form factor similar to the e.m. ones, a satisfactory agreement with the experiment is obtained for incident energies of 1-3 GeV. For higher energies, the connection with a possible "Regge pole behaviour" for such reactions is briefly discussed.

⁺⁾ Present address : LPCHE - Centre d'Etudes Nucléaires - Saclay
B.P. No. 2 - Gif-sur-Yvette - (S. et O.) -
France.

1. INTRODUCTION

The processes of pion production in nucleon-nucleon collisions are among the oldest examples of inelastic elementary particle interactions, especially because they are used for the production of pion beams.

A complete theoretical explanation of such phenomena has, however, not yet been found, after the failure of the first perturbative approaches ¹⁾. In the energy range from threshold up to 600 MeV a phenomenological theory has been developed by Mandelstam ²⁾; it reproduces the experimental data rather well. In this approach, the matrix elements for the pion production processes are supposed to be constant, and their variations are attributed only to final state pion-nucleon and nucleon-nucleon interaction effects. The partial wave amplitudes contributing to the reaction are grouped into three classes, the amplitudes of each class being assumed to be equal and energy independent. The three parameters left in the theory are then used for fitting the experimental data; the agreement is good and thus gives support to the assumptions made.

At higher energies ($\gtrsim 1$ GeV) the deuteron production becomes very small. This fact can be interpreted as an indication that final state interactions between nucleons are no longer important. This conclusion is indirectly confirmed by the success of the Lindenbaum-Sternheimer isobar model ³⁾, by which the experimental behaviour of the c.m. energy spectra for the final particles can be reproduced. One finds in this way that the low energy final state $\pi\pi$ interaction between pions and nucleons is important in the 1-3 GeV region ⁴⁾. The limitations of the isobar model are, however, rather strong and lie in the fact that it does not predict either absolute normalisations or angular distributions.

A further step in the interpretation of the single pion production processes was done by the remark that the one pion exchange, or "peripheral" mechanism of interaction ⁵⁾ could quantitatively reproduce some rather marked experimental features of these processes ⁶⁾. These are, for instance, the

forward-backward c.m. peaking of the nucleons and their corresponding high and low-energy laboratory peaks ⁷⁾. Furthermore, the peripheral model in a certain sense included the Lindenbaum-Sternheimer model, since it predicted the formation of the 33 isobar (see Iizuka and Klein ⁸⁾). It must, however, be remarked that the quantitative agreement was always limited to the region of very small momentum transfers, i.e., to the case of "quasi-real" intermediate pions. At higher momentum transfers the general behaviour of the energy spectra was still about the same as the experimental one, but the normalisation was too large by a factor of two with respect to experiment.

In spite of this fact, we want to present here an approach to the theory of such reactions based on the peripheral picture in the whole physical region. The consideration which may give a basis to this procedure is that, at large momentum transfers, a discrepancy of the type of the one observed between the predictions of the model and the experiment is not surprising, since the calculation of the peripheral effects cannot be made correctly in that region. Even if we suppose that the one-pion exchange describes the interaction completely, in the practical calculations we are not able to evaluate :

- i) the form factors describing pion emission and propagation ; ii) the corrections which have to be performed in the pion-nucleon scattering matrix element because of the virtuality of the initial pion.

When one considers the intermediate pion as real everywhere, one gets disagreement with the experiment : we can, however, think that, if in some way we succeed in evaluating the neglected effects, the agreement with experiment may be restored also for high momentum transfers. There are some experimental features which can give support to this feeling : e.g., even if the peripheral total cross sections for reactions (1.2) and (1.3) below are too large, their ratio is equal to the experimental one in the whole interval of energy between 1 and 3 GeV.

In order to carry out this programme, it is necessary to solve the problem of the off-shell pion-nucleon scattering ; this has been done in a paper by the authors ⁹⁾. It has been shown there that one-dimensional dispersion

relations, supplemented with the hypothesis of dominance of the $T = J = 3/2$ partial wave amplitude under the integrals, allow one to write the off-shell pion-nucleon scattering amplitude as a function of the on-shell amplitude (experimentally known), in the energy region of the ρ resonance.

As a consequence of this result, one can compute correctly a peripheral graph at arbitrarily large (space-like) momentum transfer. Actually, the calculation is still not completely determined because of the presence of an unknown multiplicative function of the squared four momentum Δ^2 of the exchanged pion. Calling $G(\Delta^2)$ this function, one has

$$G(\Delta^2) = K^2(\Delta^2) K'(\Delta^2) \quad (1.1)$$

where $K(\Delta^2)$ is the form factor of the proper pion-nucleon vertex and $K'(\Delta^2)$ is the ratio between complete and perturbative pion propagators.

The advantage with respect to the previous situation lies in the fact that the unknown part of the peripheral matrix element depends now only on one variable, namely the momentum transfer Δ^2 . Furthermore, the relation of this unknown part with the form factors is given by formula (1.1).

These results allow a more quantitative investigation of the importance of the peripheral interaction and a possible determination of $G(\Delta^2)$. Suppose that we have experimental data on single-pion production at a given energy. We can try to use the arbitrariness left in the calculation of the peripheral matrix element by looking for that function of Δ^2 (if any) which, when introduced in the theory, brings it in agreement with the experiment. We can now use the function determined at a certain energy for the calculation of the same effects at another energy. If we find that by this procedure we can again reproduce the experimental data, we can interpret this fact as an evidence for complete dominance of the one-pion exchange mode of interaction. The discrepancies generated from many pion exchanges are indeed expected to depend also

on the initial energy.

In a recent letter ¹⁰⁾ the authors have shown that the same $G(\Delta^2)$ does indeed reproduce the experimental data for the reactions

$$p + p \rightarrow p + n + \pi^+ \quad (1.2)$$

$$p + p \rightarrow p + p + \pi^0 \quad (1.3)$$

at lab. kin. energies of 970 and 2850 MeV.

This fact has been interpreted as an evidence for complete dominance of the peripheral mechanism of interaction in the above reactions. If we accept this interpretation (which will be discussed in the next section), we can obtain some information about the spatial distribution of the nuclear charge in the nucleon from the knowledge of the function $\int G(\Delta^2) \mathcal{J}^{\frac{1}{2}} = K(\Delta^2) \int K'(\Delta^2) \mathcal{J}^{\frac{1}{2}}$. We want to point out that, in the case of the pionic form factor, it seems to be very hard to determine the behaviour of K and K' separately. In fact the nuclear charge distribution (namely the function K) can be explored only by means of a virtual pion which, so to say, "carries" with itself a factor K' (due to its propagation) and another structure function (again K in our case) associated to the other vertex where the virtual pion is connected ¹¹⁾.

We want to stress that, in a certain sense, our approach has to be considered strictly phenomenological. We never touch such basic questions as, e.g., why the one-pion exchange term should be dominant in inelastic processes and not in elastic low energy processes. Our formulae have, furthermore, to be considered valid above the region of deuteron formation (laboratory energies above ~ 800 MeV), since the final state interaction between the nucleons is always neglected. A more complete calculation including this interaction is being carried out ¹²⁾.

In the present paper we shall give a systematic formulation of the above ideas, and show that for the existing experimental data a unique reasonable choice of $G(\Delta^2)$ can reproduce all the measured cross sections fairly well. We shall also write down the general formulae predicted by the model for the cross sections.

In section 2, the criterion for the determination of $G(\Delta^2)$ is discussed in detail. It is shown that for the low energy nucleon-nucleon elastic scattering processes (where many particle contributions are known to be important) one cannot reproduce the experimental data at different energies by multiplying the one-pion exchange matrix element by a Δ^2 -dependent function.

Section 3 is devoted to kinematics and to the derivation of the formulae predicted by the model for angular distributions and energy spectra of the outgoing particles. The comparison of these formulae with experiment is made in section 4. The agreement is generally found to be good : some further approximations introduced in the model are discussed. Finally, section 5 contains a recapitulation of the conclusions obtained and a brief discussion of the possible connection between our approach and the recently suggested "Regge pole" approach for strong interactions at high energies ¹³⁾.

2. MANY PION TERMS IN ELASTIC NUCLEON-NUCLEON SCATTERING

It is well known that second order perturbation theory fails to explain the data on elastic nucleon-nucleon scattering. It can easily be verified that this failure is of fundamental nature (in the sense that the many particle exchange terms necessarily play an important rôle) and is not just due to our inability of calculating correctly the one-pion exchange contributions. The simplest way to check this statement may be the following : from the hypothesis of isotopic spin one in the crossed channel, it follows that the ratio of neutron-proton to proton-proton total cross sections must be constant and equal to 5. Since experimentally this ratio is instead found to be energy dependent and always < 5 , we can conclude that the exchange of systems with zero isospin must be important.

In this section we will consider low energy nucleon-nucleon scattering in order to check the reliability of the criterion (presented in Refs. ⁹⁾ and ¹⁴⁾ and mentioned in the previous section) for the validity of the one-pion exchange approximation.

This criterion rests essentially on the statement that whenever contributions different from the one-pion exchange are important in a given process, it is not possible to find a unique cut-off function $G(\Delta^2)$ such as to bring the theoretical predictions of the peripheral model in agreement with the experimental data at different energies. At a given energy, one may find a suitable expression for such a $G(\Delta^2)$; but, however, it must be expected that the various functions $G(\Delta^2)$ obtained in this way at different energies should be different to an extent which is given just by the importance of the many particle exchange contributions. This fact can be confirmed, e.g., by using perturbation theory.

Let us then check, on a phenomenological level, that indeed for nucleon-nucleon scattering the discrepancy between the one-pion exchange theory and the experiment is also dependent on the initial energy.

There are two one-pion exchange diagrams for nucleon-nucleon scattering ; they differ for the exchange of the initial (or final) nucleons. They give for the differential c.m. cross section ¹⁵⁾

$$\frac{d\sigma}{d\Omega} = \frac{4m^4}{W^2} \frac{f^4}{\mu^4} \left\{ \left[\frac{\Delta^2 G(\Delta^2)}{\Delta^2 + \mu^2} \right]^2 + \alpha^2 \left[\frac{\bar{\Delta}^2 G(\bar{\Delta}^2)}{\bar{\Delta}^2 + \mu^2} \right]^2 + \alpha \frac{\Delta^2 G(\Delta^2) \bar{\Delta}^2 G(\bar{\Delta}^2)}{(\Delta^2 + \mu^2)(\bar{\Delta}^2 + \mu^2)} \right\} \quad (2.1)$$

where $\alpha = 1$ ($\alpha = 2$) for pp (np) scattering. W is the total c.m. energy ; m (μ) the nucleon (pion) mass ; $f^2 = 0.08$ is the square of the pion-nucleon coupling constant. In terms of the c.m. angle θ and c.m. momentum p we have

$$\Delta^2 = 2p^2(1 - \cos\theta) \quad (2.2)$$

$$\bar{\Delta}^2 = 2p^2(1 + \cos\theta) \quad (2.3)$$

The function G has been defined in (1.1) in terms of the form factors. They have been introduced in order to keep into account the corrections due to the nucleon and pion finite sizes.

A possible way of obtaining G would be the one described above, i.e., by tentative at a given W , to find that function which brings (2.1) in agreement with the experimental data. This procedure is, however, tedious and we adopt a different but equivalent method. Since at 90°

$$\Delta^2 = \bar{\Delta}^2 = 2p^2 = \frac{W^2}{2} - 2m^2 \quad (2.4)$$

in (2.1) a factor $G^2(2p^2)$ multiplies the perturbative cross section. Since (2.4) establishes a one to one correspondence between W^2 and Δ^2 , the predicted behaviour of $(\frac{d\sigma}{d\Omega})_{90^\circ}$ as a function of energy is uniquely related to the behaviour of G as a function of Δ^2 . Then, under the hypothesis of dominance of the peripheral interaction, G^2 itself can be obtained empirically as the ratio between the experimental proton-proton cross section at 90° and the same quantity calculated from (2.1) with $G = 1$.

By using the experimental data collected in Table VI of Ref. ¹⁶⁾, for $G(\Delta^2)$ we obtain a function which can be well reproduced by the following empirical fit (for $\Delta^2 \geq 0.5 \mu^2$)

$$G_{el}(\Delta^2) = 1.77 \left(\frac{\Delta^2}{\mu^2} \right)^{-3} + 0.5 \quad (2.5)$$

A first serious objection against (2.5) is its increasingly large value below $\Delta^2 \simeq 2.5 \mu^2$. For Δ^2 around $0.5 \mu^2$, (2.5) gives values of $G_{el}(\Delta^2)$ of the order of $5 \div 10$. This is hardly compatible with the normalization $G(-\mu^2) = 1$.

This fact alone already indicates that many pion exchange terms must be important in nucleon-nucleon scattering. For further check, if we calculate $\frac{d\sigma}{d\Omega}$ from formula (2.1), by using for G its "empirical" determination (2.5), we obtain angular distributions strongly different from the experimental ones in the region up to ~ 300 MeV. We conclude then that the W -dependence of

the "form factor" giving the correction to the one pion exchange theory, is quite observable. Since this W -dependence can be generated only through the many pion exchange terms, we shall assume in the following that the absence of such dependence implies negligible contribution of these terms.

We hardly need stressing that this assumption cannot be proved in a rigorous way. Other checks of the model are therefore desirable. A possible criterion for this purpose has been indicated by Treiman and Yang¹⁷⁾.

3. THE PROCESS OF SINGLE-PION PRODUCTION

In this section the theory of single pion production in nucleon-nucleon collisions is developed, by leaving the function $G(\Delta^2)$ undetermined. This function will in turn be obtained in the next section by comparison with the experimental data.

a) Kinematics

In what follows we give a rather systematic discussion of the kinematical relations for the processes of single-pion production. Also relations not strictly needed in the following sections will be listed for the sake of completeness.

A process involving five particles (initial and final) can be described by five independent kinematical quantities. We choose the following set of independent relativistic invariants as fundamental set :

$$\begin{aligned} W^2 &= -(p_1 + p_2)^2 \\ \omega^2 &= -(q + q_1)^2 \\ u^2 &= -(q + q_2)^2 \\ \Delta^2 &= (q_2 - p_2)^2 \\ t^2 &= (q_1 - p_1)^2 \end{aligned} \quad (3.1)$$

where the four-vectors are defined as in Fig. 1¹⁵⁾. W is the total c.m. energy ; ω (u) is the total energy of the particles q, q_1 (resp. q, q_2) in their c.m. system, where $\vec{q} + \vec{q}_1 = 0$ (resp. $\vec{q} + \vec{q}_2 = 0$).

Δ^2 and t^2 are squared four-momentum transfers. Other invariants which can be formed are :

$$\bar{\Delta}^2 = (q_2 - p_1)^2$$

$$\bar{t}^2 = (q_1 - p_2)^2$$

$$r^2 = (q - p_1)^2$$

(3.2)

$$\bar{r}^2 = (q - p_2)^2$$

$$Z^2 = -(q_1 + q_2)^2$$

Of these, the first four are squared four-momentum transfers, the fifth is the squared total energy of q_1 and q_2 in the system where $\vec{q}_1 + \vec{q}_2 = 0$. The barred invariants are obtained from the unbarred ones by the exchange $p_1 \longleftrightarrow p_2$.

Of course, the invariants of the set (3.2) can be expressed in terms of the fundamental ones. The relations are :

$$\bar{\Delta}^2 = W^2 - 3m^2 - \omega^2 - \Delta^2$$

$$\bar{t}^2 = W^2 - 3m^2 - u^2 - t^2$$

$$r^2 = \omega^2 + \Delta^2 - t^2 - \mu^2 - 2m^2$$

(3.3)

$$\bar{r}^2 = u^2 + t^2 - \Delta^2 - \mu^2 - 2m^2$$

$$Z^2 = W^2 - \omega^2 - u^2 + \mu^2 + 2m^2$$

Any kinematical quantity (energies, scattering angles, etc.) in any frame of reference, can now be expressed in terms of the invariants of the fundamental set (3.1). We shall list below the most important of these relations. We put

$$E(a; b^2, c^2) = \frac{1}{2a} (a^2 + b^2 - c^2) \quad (3.4)$$

$$P(a; b^2, c^2) = \frac{1}{2a} [a^4 - 2a^2(b^2 + c^2) + (b^2 - c^2)^2]^{1/2} \quad (3.5)$$

E and P represent c.m. energy and c.m. 3 momentum respectively of a particle with mass b colliding with a particle with mass c at total c.m. energy a . We consider now the following reference systems :

- i) the laboratory (L) system ($\vec{p}_2 = 0$)
- ii) the c.m. (B) system ($\vec{p}_1 + \vec{p}_2 = 0$)
- iii) the Q system ($\vec{q} + \vec{q}_1 = 0$)
- iv) the P system ($\vec{q} + \vec{q}_2 = 0$)
- v) the D system ($\vec{q}_1 + \vec{q}_2 = 0$).

L system

One has

$$T = \frac{1}{2m} [W^2 - 4m^2] \quad (3.6)$$

$$T^L = \frac{1}{2m} [\bar{r}^2 + (m - \mu)^2] \quad (3.7)$$

$$T_1^L = \frac{1}{2m} \bar{t}^2 \quad (3.8)$$

$$T_2^L = \frac{1}{2m} \Delta^2 \quad (3.9)$$

for the kinetic energies of the particles p_1 , q , q_1 and q_2 respectively. Furthermore, the relations

$$p^L = \frac{W}{m} \left(\frac{W^2}{4} - m^2 \right)^{1/2} \quad (3.10)$$

$$q^L = \frac{\bar{r}}{m} P(\bar{r}; -m^2, -\mu^2) \quad (3.11)$$

$$q_1^L = \frac{\bar{t}}{m} P(\bar{t}; -m^2, -m^2) \quad (3.12)$$

$$q_2^L = \frac{\Delta}{m} P(\Delta; -m^2, -m^2) \quad (3.13)$$

give the 3 momenta of the same particles. For the scattering angles one has

$$\cos \theta^L = \frac{1}{2q^L p^L} \left[2T(T + \mu) + 2mT^L - (m - \mu)^2 - r^2 \right] \quad (3.14)$$

$$\cos \theta_1^L = \frac{1}{2q_1^L p^L} \left[2T(T_1^L + m) + 2mT_1^L - t^2 \right] \quad (3.15)$$

$$\cos \theta_2^L = \frac{1}{2q_2^L p^L} \left[2T(T_2^L + m) + 2mT_2^L - \Delta^2 \right] \quad (3.16)$$

for the particles q , q_1 and q_2 respectively.

B system

The energies and the 3 momenta are given by

$$p_{10}^B = p_{20}^B = \frac{W}{2} ; \quad p_1^B = p_2^B = p^B = \left(\frac{W^2}{4} - m^2 \right)^{1/2} \quad (3.17)$$

$$q_0^B = E(W; \mu^2, z^2) ; \quad q^B = P(W; \mu^2, z^2) \quad (3.18)$$

$$q_{10}^B = E(W; m^2, u^2) ; \quad q_1^B = P(W; m^2, u^2) \quad (3.19)$$

$$q_{20}^B = E(W; m^2, \omega^2) ; \quad q_2^B = P(W; m^2, \omega^2) \quad (3.20)$$

while for the scattering angles (with respect to $\vec{p}_1$) one has

$$\cos \theta^B = - \frac{1}{2q_0^B p^B} \left[r^2 + \mu^2 + m^2 - W q_0^B \right] \quad (3.21)$$

$$\cos \theta_1^B = - \frac{1}{2q_1^B p^B} \left[t^2 + 2m^2 - W q_{10}^B \right] = + \frac{1}{2q_1^B p^B} \left[\bar{t}^2 + 2m^2 - W q_{10}^B \right] \quad (3.22)$$

$$\cos \theta_2^B = - \frac{1}{2q_2^B p^B} \left[\bar{\Delta}^2 + 2m^2 - W q_{20}^B \right] = + \frac{1}{2q_2^B p^B} \left[\Delta^2 + 2m^2 - W q_{20}^B \right] \quad (3.23)$$

for the particles q , q_1 and q_2 respectively.

Finally, the angle between the momenta $\vec{q}_1$ and $\vec{q}_2$ in the B system is given by

$$\cos \lambda^B = \frac{1}{2q_1^B q_2^B} \left[-W^2 + \omega^2 + \mu^2 + 2q_{10}^B q_{20}^B \right] \quad (3.24)$$

Q system

For energies and momenta one has

$$p_{10}^Q = E(\omega; m^2, -\Delta^2) \quad ; \quad p_1^Q = P(\omega; m^2, -\Delta^2) \quad (3.25)$$

$$p_{20}^Q = E(\omega; m^2, -\bar{\Delta}^2) \quad ; \quad p_2^Q = P(\omega; m^2, -\bar{\Delta}^2) \quad (3.26)$$

$$q_0^Q = E(\omega; \mu^2, m^2) \quad ; \quad q^Q = P(\omega; \mu^2, m^2) \quad (3.27)$$

$$q_{10}^Q = E(\omega; m^2, \mu^2) \quad ; \quad q_1^Q = P(\omega; m^2, \mu^2) \quad (3.28)$$

$$q_{20}^Q = \frac{W}{\omega} E(W; -m^2, \omega^2) \quad ; \quad q_2^Q = \frac{W}{\omega} P(W; m^2, \omega^2) \quad (3.29)$$

It should be noted that $q^Q = q_1^Q$ (according to the definition of the Q system). For the most interesting angles, one has

$$\cos \theta^Q = \frac{1}{2q_1^Q p_1^Q} [2q_{10}^Q p_{10}^Q - 2m^2 - t^2] \quad \text{angle } (\vec{q}_1, \vec{p}_1) \quad (3.30)$$

$$\cos \epsilon^Q = \frac{1}{2q_1^Q p_2^Q} [2q_{10}^Q p_{20}^Q - 2m^2 - \bar{t}^2] \quad \text{angle } (\vec{q}_1, \vec{p}_2) \quad (3.31)$$

$$\cos \alpha^Q = \frac{1}{2p_1^Q p_2^Q} [2p_{10}^Q p_{20}^Q + 2m^2 - W^2] \quad \text{angle } (\vec{p}_1, \vec{p}_2) \quad (3.32)$$

P system

For energies and momenta, one has

$$p_{10}^P = E(u; m^2, -\bar{t}^2) \quad ; \quad p_1^P = P(u; m^2, -\bar{t}^2) \quad (3.33)$$

$$p_{20}^P = E(u; m^2, -t^2) \quad ; \quad p_2^P = P(u; m^2, -t^2) \quad (3.34)$$

$$q_0^P = E(u; \mu^2, m^2) \quad ; \quad q^P = P(u; \mu^2, m^2) \quad (3.35)$$

$$q_{20}^P = E(u; m^2, \mu^2) \quad ; \quad q_2^P = P(u; m^2, \mu^2) \quad (3.36)$$

$$q_{10}^P = \frac{W}{u} E(W; -m^2, u^2) \quad ; \quad q_1^P = \frac{W}{u} P(W; m^2, u^2) \quad (3.37)$$

It should be noted that $q^P = q_2^P$ (according to the definition of the P system). For the most interesting angles, one has

$$\cos \theta^P = \frac{1}{2q_1^P p_2^P} \left[2q_{20}^P p_{20}^P - 2m^2 - \Delta^2 \right] \text{angle}(\vec{q}_2, \vec{p}_2) \quad (3.38)$$

$$\cos \epsilon^P = \frac{1}{2q_1^P p_1^P} \left[2q_{20}^P p_{10}^P - 2m^2 - \Delta^2 \right] \text{angle}(\vec{q}_2, \vec{p}_1) \quad (3.39)$$

$$\cos \alpha^P = \frac{1}{2p_1^P p_2^P} \left[2p_{10}^P p_{20}^P + 2m^2 - W^2 \right] \text{angle}(\vec{p}_1, \vec{p}_2) \quad (3.40)$$

D system

For energies and momenta, one has

$$p_{10}^D = E(z; m^2, -\bar{r}^2) ; p_1^D = P(z; m^2, -\bar{r}^2) \quad (3.41)$$

$$p_{20}^D = E(z; m^2, -r^2) ; p_2^D = P(z; m^2, -r^2) \quad (3.42)$$

$$q_0^D = \frac{W}{z} E(W; -m^2, z^2) ; q^D = \frac{W}{z} P(W; m^2, z^2) \quad (3.43)$$

$$q_{10}^D = q_{20}^D = \frac{z}{2} ; q_1^D = q_2^D = \left(\frac{z^2}{4} - m^2 \right)^{1/2} \quad (3.44)$$

For the most interesting angles, one has

$$\cos \theta^D = \frac{1}{2q_1^D p_1^D} \left[2q_{10}^D p_{10}^D - 2m^2 - t^2 \right] \quad \text{angle } (\vec{q}_1, \vec{p}_1) \quad (3.45)$$

$$\cos \epsilon^D = \frac{1}{2q_1^D p_2^D} \left[2q_{10}^D p_{20}^D - 2m^2 - t^2 \right] \quad \text{angle } (\vec{q}_1, \vec{p}_2) \quad (3.46)$$

$$\cos \alpha^D = \frac{1}{2p_1^D p_2^D} \left[2p_{10}^D p_{20}^D + 2m^2 - W^2 \right] \quad \text{angle } (\vec{p}_1, \vec{p}_2) \quad (3.47)$$

b) Peripheral matrix element

The peripheral matrix elements contributing to any process of single pion production in nucleon-nucleon collisions are four. They correspond to the diagrams shown in Fig. 1. We have denoted the four momenta of the initial nucleons by p_1 and p_2 . In particular, p_2 is assumed to be that nucleon which is at rest in the laboratory system. For simplicity p_2 is always put in the lower part of a diagram. The four momenta of the outgoing pion and nucleons are denoted by q and q_1, q_2 respectively.

The $\mathcal{M}$ -matrix element for our process, defined by

$$S_{fi} = -i(2\pi)^{-\frac{7}{2}} \left[\frac{m^4}{2q_0 q_{10} q_{20} p_{10} p_{20}} \right]^{\frac{1}{2}} \delta^4(q + q_1 + q_2 - p_1 - p_2) \mathcal{M}_{fi} \quad (3.48)$$

will be given by

$$\mathcal{M}_{fi} = M_a - M_b - M_c + M_d \quad (3.49)$$

where the subscripts a, b, c, d refer to the corresponding diagrams of Fig. 1. The minus signs in the above formula are due to the identity of the initial and final nucleons (we assume full validity of charge independence).

The explicit form of the four M matrix elements is

$$M_a^\beta = \bar{u}(q_2) \tau_\alpha G_r u(p_2) \Pi(\Delta^2) \bar{u}(q_1) \left[-A^{\beta\alpha}(\omega^2, t^2; \Delta^2) + i \not{q} B^{\beta\alpha}(\omega^2, t^2; \Delta^2) \right] u(p_1) \quad (3.50)$$

$$M_b^\beta = \bar{u}(q_2) \tau_\alpha G_r u(p_1) \Pi(\bar{\Delta}^2) \bar{u}(q_1) \left[-A^{\beta\alpha}(\omega^2, \bar{t}^2; \bar{\Delta}^2) + i \not{q} B^{\beta\alpha}(\omega^2, \bar{t}^2; \bar{\Delta}^2) \right] u(p_2) \quad (3.51)$$

$$M_c^\beta = \bar{u}(q_1) \tau_\alpha G_r u(p_2) \Pi(\bar{E}^2) \bar{u}(q_2) \left[-A^{\beta\alpha}(u^2, \bar{\Delta}^2; \bar{E}^2) + i \not{q} B^{\beta\alpha}(u^2, \bar{\Delta}^2; \bar{E}^2) \right] u(p_1) \quad (3.52)$$

$$M_d^\beta = \bar{u}(q_1) \tau_\alpha G_r u(p_1) \Pi(t^2) \bar{u}(q_2) \left[-A^{\beta\alpha}(u^2, \Delta^2; t^2) + i \not{q} B^{\beta\alpha}(u^2, \Delta^2; t^2) \right] u(p_2) \quad (3.53)$$

where G_r is the renormalised and rationalised pion-nucleon coupling constant and α (β) is the charge index of the intermediate virtual (final real) pion. In the above formulae the spin variables have not been written in the spinors and the function Π is defined by

$$\Pi(A) = \frac{K(A) K'(A)}{(A + \mu^2)} \quad (3.54)$$

K and K' being the vertex and propagator form factors of the virtual pion with squared four momentum A ; they have been introduced in section 1.

The most general differential cross section can easily be written down in terms of M_{fi} . The relation is

$$d\sigma = (2\pi)^{-5} \frac{1}{F} \frac{m^4}{2} \sum |M_{fi}|^2 \delta^4(P_f - P_i) \frac{d^3q d^3q_1 d^3q_2}{q_0 q_{10} q_{20}} \quad (3.55)$$

where $P_f = q + q_1 + q_2$ and $P_i = p_1 + p_2$; $\sum$ indicates a sum (average) over final (initial) spin states. Therefore $d\sigma$, as well as all the cross sections derived from it in the following, refers to an unpolarised initial beam.

The invariant flux factor F is given by

$$F = [(P_1 P_2)^2 - m^4]^{1/2} = W p^B = m p^L \quad (3.56)$$

where $p^B(p^L)$ is the c.m. (lab.) initial momentum. Any cross section can be calculated from (3.55), once $\sum |M_{fi}|^2$ is known. For the calculation of this quantity, let us remark that in principle one has to deal with the sum of ten terms, namely, of four square moduli and six interferences.

Expressions (3.43) to (3.46) are however such as to make two of the interferences vanish. Precisely

$$2 \operatorname{Re} \sum M_a M_d^* = 2 \operatorname{Re} \sum M_b M_c^* = 0 \quad (3.57)$$

These terms vanish because of the pseudoscalar nature of the pion.

Furthermore, two of the other interference terms are very small and always negligible in practical cases. Precisely we can put

$$2 \operatorname{Re} \sum M_a M_c^* \simeq 0 ; \quad 2 \operatorname{Re} \sum M_b M_d^* \simeq 0 \quad (3.58)$$

This fact has been verified quantitatively by Da Prato ⁶⁾ at 2.85 GeV and can be understood quite generally by considering a simple example. Let us suppose that the pion and the nucleon emerging from the same vertex in any of the diagrams in Fig. 1 are in a relative state with $T = J = 3/2$ (this is often a good approximation). Then the relative state of the pion and the other nucleon in the same diagram is a superposition of many angular momenta and in general of both isospin $T = 1/2$ and $T = 3/2$, since no correlation at all between these two particles is implied by the structure of the diagram. Therefore, e.g., if in M_a q and q_1 are in a pure $T = J = 3/2$ state, only that part of M_c having q and q_1 in the state $T = J = 3/2$ gives non-vanishing contribution to the interference $\sum M_a M_c^*$. Similarly, only that part of M_a having q and q_2 with $T = J = 3/2$ can contribute to this interference. Since only a small part of both M_a and M_c contributes to their interference term, it follows that the interference itself must be small, especially at high energy where many states of J are likely to be present.

Actual numerical calculations of these terms give a value which is at most a few percent of the square moduli. Only at low energies ($\lesssim 600$ MeV) these terms are not completely negligible but in this region our theory is not expected to be valid because of the neglected effects of final state nucleon-nucleon interaction. We will then use Eq. (3.58) systematically in the following without further comments.

On the other hand we cannot neglect the interferences between M_a and M_b and between M_c and M_d , since these have no a priori reasons for being small: they are indeed found to contribute some 10 - 20% to the cross sections at energies around 1 GeV.

Let us now consider the case of a given observable reaction [e.g., (1.2)]. The isospin structure of the four matrix elements $M_a \dots M_d$ can be determined in the easiest way by direct inspection of the four diagrams of Fig. 1. It turns out to be a function of the total isospin t ($t = 1/2$ or $3/2$) of the pion-nucleon scattering process appearing in one of the two vertices. Also the coupling constants of the three-line vertices may differ by multiplicative factors according to the various states of charge of nucleons and pions. We define eight "basic" matrix elements $M_a^t \dots M_d^t$ ($t = 1/2, 3/2$); they describe the peripheral diagrams a) ... d) for pion-nucleon scattering taking place in the pure isospin state t , and for the value of the coupling constant $f^2 = 0.08$. The coefficients of such basic matrix elements are called $c_1^t \dots c_4^t$. Then, by taking into account (3.57) and (3.58), we can write the following expression of $|m_{fi}|^2$

$$|m_{fi}|^2 = \left| \sum_t (c_1^t M_a^t - c_2^t M_b^t) \right|^2 + \left| \sum_t (c_3^t M_c^t - c_4^t M_d^t) \right|^2 \quad (3.59)$$

The c 's are listed in Table I for all possible single-pion production reactions. It is always intended that q indicates the final pion in the diagrams of Fig. 1; when the two final nucleons are different, the neutron will be denoted by q_2 , the proton by q_1 .

TABLE I

Reaction	$c_1^{1/2}$	$c_1^{3/2}$	$c_2^{1/2}$	$c_2^{3/2}$	$c_3^{1/2}$	$c_3^{3/2}$	$c_4^{1/2}$	$c_4^{3/2}$
$p+p \rightarrow p+n+\pi^+$	0	$\sqrt{2}$	0	$\sqrt{2}$	$\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{2}}{3}$
$p+p \rightarrow p+p+\pi^0$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
$n+p \rightarrow n+p+\pi^0$	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{2}{3}$
$n+p \rightarrow n+n+\pi^+$	$\frac{2\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{2}}{3}$	$\frac{2\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{2}}{3}$
$n+p \rightarrow p+p+\pi^-$	$\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{2}}{3}$	$\frac{2\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{2}}{3}$	$\frac{2\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$
$n+n \rightarrow n+p+\pi^-$	$\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{2}}{3}$	$\frac{\sqrt{2}}{3}$	$-\frac{\sqrt{2}}{3}$	0	$\sqrt{2}$	0	$\sqrt{2}$
$n+n \rightarrow n+n+\pi^0$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$

It is easily checked that the following relations hold

$$d\sigma(nn \rightarrow nn^0) = d\sigma(pp \rightarrow pp^0)$$

$$d\sigma(np \rightarrow nn^+) = d\sigma(np \rightarrow pp^-)$$

$$d\sigma(pp \rightarrow pn^+) = d\sigma(nn \rightarrow np^-)$$

as it has to be expected for charge symmetry.

It is also to be noted that, for pn collisions, an extra relative minus sign between c_1 and c_2 , and between c_3 and c_4 has been introduced ; it compensates the minus signs occurring in (3.59). In fact, the initial particles are not identical in this case. Such sign would arise automatically in a complete charge independent treatment.

The r.h.s. of (3.59) appears now as the sum of two symmetrical parts : the one can indeed be obtained from the other by the exchange of q_1 and q_2 . This exchange is equivalent to the following set of substitutions :

$$q_1 \leftrightarrow q_2 \Rightarrow \begin{cases} c_1^t \leftrightarrow c_3^t \\ c_2^t \leftrightarrow c_4^t \\ \omega^2 \leftrightarrow u^2 \\ \Delta^2 \leftrightarrow \bar{E}^2 \\ t^2 \leftrightarrow \bar{\Delta}^2 \end{cases} \quad (3.60)$$

Therefore, we will calculate only the first term in (3.59) ; the second one is then simply obtained from it by performing the substitutions (3.60). The easiest way to obtain $\left| \sum_t (c_{1a}^{tt} - c_{2b}^{tt}) \right|^2$ is to perform the calculation in the c.m. system of the two particles coming out together, namely in the system $\vec{q} + \vec{q}_1 = 0$ (Q system). In this way the relativistic invariance is lost during the calculation. It can be restored, however, by expressing the result in terms of invariants, e.g., those of the fundamental set (3.1).

It is useful to write down the result of such a calculation in terms of the partial wave amplitudes for off-shell pion-nucleon scattering defined in Ref. 9). By writing again $M_a = \sum_t c_{1a}^{tt}$, $M_b = \sum_t c_{2b}^{tt}$ we get

$$\sum_{\text{Spin}} |M_a - M_b|^2 = \sum_{\text{Spin}} |M_a|^2 + \sum_{\text{Spin}} |M_b|^2 - 2 \operatorname{Re} \sum_{\text{Spin}} M_a M_b^* \quad (3.61)$$

and by using the expressions (3.50) and (3.51) for M_a and M_b , specified for the physical charge states under consideration, we get

$$\sum |M_a|^2 = (2\pi)^2 \frac{G_r^2}{m^4} \Delta^2 \pi^2 (\Delta^2) \omega^2 \left[|f_1 + \cos \theta^Q f_2|^2 + |\sin \theta^Q f_2|^2 \right] \quad (3.62)$$

$$\sum |M_b|^2 = (2\pi)^2 \frac{G_r^2}{m^4} \bar{\Delta}^2 \pi^2 (\bar{\Delta}^2) \omega^2 \left[|f_1' + \cos \epsilon^Q f_2'|^2 + |\sin \epsilon^Q f_2'|^2 \right] \quad (3.63)$$

$$-2 \operatorname{Re} \sum M_a M_b^* = (2\pi)^2 \frac{G_r^2}{m^4} \pi (\Delta^2) \pi (\bar{\Delta}^2) \omega^2. \quad (3.64)$$

$$\begin{aligned} & \cdot \operatorname{Re} \left[(\omega + m) R_{(-)} (f_1' f_1^* \cos \alpha^Q + f_2' f_2^* + f_2' f_1^* \cos \theta^Q + f_1' f_2^* \cos \epsilon^Q) + \right. \\ & \left. + (\omega - m) R_{(+)} (f_1' f_1^* + f_2' f_2^* \cos \alpha^Q + f_2' f_1^* \cos \epsilon^Q + f_1' f_2^* \cos \theta^Q) \right] \end{aligned}$$

where

$$\begin{aligned} f_1 &= \sum_t c_1^t f_1^t \\ f_2 &= \sum_t c_1^t f_2^t \\ f_1' &= \sum_t c_2^t f_1'^t \\ f_2' &= \sum_t c_2^t f_2'^t \end{aligned} \quad (3.65)$$

and

$$R_{(+)} = [(p_{10}^Q + m)(p_{20}^Q + m)]^{1/2} ; R_{(-)} = [(p_{10}^Q - m)(p_{20}^Q - m)]^{1/2} \quad (3.66)$$

$$G_r^2 = \frac{16\pi m^2}{\mu^2} f^2 \quad (3.67)$$

The off-shell amplitudes f_1^t, f_2^t referring to a definite state of the isotopic spin of the πN system, are simply related to the invariant off-shell pion-nucleon scattering matrix element, or to the invariant amplitudes A and B describing it, through formulae (15) to (17) of Ref. 9), where the superscript $(\pm)$ has to be replaced by t. [It is easily seen that $f_{1,2}^{(\pm)}$ and $f_{1,2}^t$ satisfy a linear relation of the type of Eq. (14) of Ref. 9).]

Similarly, the relation between $f_{1,2}^t$ and the partial wave amplitudes $f_{\ell\pm}^t$ will be given by the following formulae, similar to Eqs. (19) and (20) of Ref. 9)

$$f_1^t = \sum_{\ell=0}^{\infty} f_{\ell+}^t(\omega, \Delta^2) P'_{\ell+1}(\omega \theta^Q) - \sum_{\ell=2}^{\infty} f_{\ell-}^t(\omega, \Delta^2) P'_{\ell-1}(\omega \theta^Q) \quad (3.68)$$

$$f_2^t = \sum_{\ell=1}^{\infty} [f_{\ell-}^t(\omega, \Delta^2) - f_{\ell+}^t(\omega, \Delta^2)] P'_\ell(\omega \theta^Q) \quad (3.69)$$

P'_ℓ are derivatives of Legendre polynomials. The primed f 's defined in (3.63) and (3.65) are obtained from the unprimed f 's by the exchange $p_1 \leftrightarrow p_2$, i.e., by the substitutions ¹⁸⁾

$$\Delta^2 \rightarrow \bar{\Delta}^2 ; \omega \theta^Q \rightarrow \omega \epsilon^Q \quad [\text{i.e., } t^2 \rightarrow \bar{t}^2] .$$

Since low-energy off-shell πN scattering is dominated by the $T = J = 3/2$ state ⁹⁾, for $\omega \lesssim 1.5 m$ it is a good approximation to keep in the f's only the partial-wave amplitude of this state. By generalizing this result, one can divide the range of variation of ω in different intervals in each of which a single partial-wave is supposed to be dominant, corresponding to the physical situation which shows marked effects on the total cross section from different resonances ¹⁹⁾. Therefore, we will write down approximate formulae for $\sum |M_a|^2$, $\sum |M_b|^2$, $-2 \operatorname{Re} \sum M_a^* M_b$ obtained by keeping only one partial wave amplitude in the f's. In this case, since the dominant wave is in a definite state of isotopic spin, we will drop the sum $\sum_t$ and the superscript t in c_1 , c_2 and in the f's; it is intended that c_1 and c_2 may vary from one interval of energy to the other.

In this approximation, we get

$$\begin{aligned} \sum |M_a|^2 = & c_1^2 (2\pi)^2 \frac{G_r^2}{m^4} \Delta^2 \pi^2 (\Delta^2) \omega^2 |f_{\ell\pm}(\omega, \Delta^2)|^2. \\ & \cdot \left\{ P_{\ell\pm 1}'^2(\cos\theta^Q) + P_{\ell}'^2(\cos\theta^Q) + 2\cos\theta^Q P_{\ell+1}'(\cos\theta^Q) P_{\ell}'(\cos\theta^Q) \right\} \end{aligned} \quad (3.70)$$

$$\begin{aligned} \sum |M_b|^2 = & c_2^2 (2\pi)^2 \frac{G_r^2}{m^4} \bar{\Delta}^2 \pi^2 (\bar{\Delta}^2) \omega^2 |f_{\ell\pm}(\omega, \bar{\Delta}^2)|^2. \\ & \cdot \left\{ P_{\ell\pm 1}'^2(\cos\epsilon^Q) + P_{\ell}'^2(\cos\epsilon^Q) + 2\cos\epsilon^Q P_{\ell\pm 1}'(\cos\epsilon^Q) P_{\ell}'(\cos\epsilon^Q) \right\} \end{aligned} \quad (3.71)$$

$$\begin{aligned} -2 \operatorname{Re} \sum M_a M_b^* = & c_1 c_2 (2\pi)^2 \frac{G_r^2}{m^4} \pi(\Delta^2) \pi(\bar{\Delta}^2) \omega^2 \cdot \\ & \cdot \operatorname{Re} \left[f_{\ell\pm}^*(\omega, \Delta^2) f_{\ell\pm}(\omega, \bar{\Delta}^2) \right] \phi_{\ell\pm} \end{aligned} \quad (3.72)$$

where

$$\begin{aligned} \phi_{\ell\pm} = & (\omega+m) R_{(-)} \left[p'_{\ell\pm 1}(\omega\epsilon^Q) p'_{\ell\pm 1}(\omega\theta^Q) \omega\alpha^Q + p'_\ell(\omega\epsilon^Q) p'_\ell(\omega\theta^Q) - \right. \\ & \left. - p'_\ell(\omega\epsilon^Q) p'_{\ell\pm 1}(\omega\theta^Q) \omega\theta^Q - p'_{\ell\pm 1}(\omega\epsilon^Q) p'_\ell(\omega\theta^Q) \omega\epsilon^Q \right] + \\ & + (\omega-m) R_{(+)} \left[p'_{\ell\pm 1}(\omega\epsilon^Q) p'_{\ell\pm 1}(\omega\theta^Q) + p'_\ell(\omega\epsilon^Q) p'_\ell(\omega\theta^Q) \omega\alpha^Q - \right. \\ & \left. - p'_\ell(\omega\epsilon^Q) p'_{\ell\pm 1}(\omega\theta^Q) \omega\epsilon^Q - p'_{\ell\pm 1}(\omega\epsilon^Q) p'_\ell(\omega\theta^Q) \omega\theta^Q \right] \end{aligned} \quad (3.73)$$

The quantities p_{10}^Q , p_{20}^Q , θ^Q , $\cos\epsilon^Q$, $\cos\alpha^Q$ have been defined in formulae (3.25), (3.26), (3.30), (3.31) and (3.32) respectively.

c) Calculation of cross sections

In the previous sub-section we have given the expression of the squared M matrix element in terms of the partial-wave amplitudes for off-shell πN scattering. This expression has been obtained both in general form (formulae (3.62) to (3.64)) and in the single partial-wave approximation (formulae (3.70) to (3.73)). It is now clear that either the total or any differential cross section can be calculated from (3.55). Since in the r.h.s. of (3.55) nine differentials appear, the derivation of a distribution implies the calculation of eight integrals, four of which are trivially performed through the δ^4 -function. Furthermore, if one refers to an initial unpolarized beam, the integration on the over-all azimuthal angle simply gives 2π . Thus, in this case, the calculation of a point of the spectrum in question implies the calculation of a triple integral. This problem can, in any case, be solved with the use of an electronic computer. In some cases,

however, two integrals can be performed analytically, and the resulting formulae look rather simple.

We give in the following a list of the cross sections for the physical variables of practical interest. For this purpose, it is convenient to split $d\sigma$ (see (3.55)) into the sum of two terms, the one containing M_a and M_b , the other containing M_c and M_d . We shall then write

$$d\sigma = d\sigma_1 + d\sigma_2 \quad (3.74)$$

Energy spectra and angular distributions for a nucleon

Let us consider nucleon q_2 for definiteness. We will show below that for nucleon q_1 the formulae are essentially the same, with trivial changes. We get

$$\frac{d\sigma}{dq_{20}d\Omega_2} = \frac{(2\pi)^{-5}}{F} \frac{m^4}{2} |\vec{q}_2| \int \sum |m_{fi}|^2 \delta^4(p_f - p_i) \frac{d^3q d^3q_1}{q_0 q_{10}} \quad (3.75)$$

where, of course, $|\vec{q}_2|$ has to be calculated in the same reference system as q_{20} and Ω_2 .

The splitting (3.74) turns out immediately to be useful. All the integrations indicated in (3.75) can indeed be carried out rather easily for $d\sigma_1$. One gets in the single partial-wave approximation :

$$\begin{aligned} \frac{d\sigma_1}{dq_{20}d\Omega_2} = & \frac{1}{\pi} \frac{G^2}{F} |\vec{q}_2| \omega q^2 \left\{ c_1^2 \Delta^2 \Pi^2(\Delta^2) |f_{\ell\pm}(\omega, \Delta^2)|^2 + \right. \\ & + c_2^2 \bar{\Delta}^2 \Pi^2(\bar{\Delta}^2) |f_{\ell\pm}(\omega, \bar{\Delta}^2)|^2 + c_1 c_2 \Pi(\Delta^2) \Pi(\bar{\Delta}^2) \cdot \\ & \cdot f_{\ell\pm}^*(\omega, \Delta^2) f_{\ell\pm}(\omega, \bar{\Delta}^2) \phi_0^{\ell\pm}(\omega, \Delta^2) \left. \right\} a_{\ell\pm} \end{aligned} \quad (3.76)$$

where we have put $G^2 = G_r^2/4\pi = \frac{4M^2}{\mu^2} f^2$ and

$$a_{\ell+} = \ell+1 \quad a_{\ell-} = \ell \quad (3.77)$$

$$\phi_0^{\ell\pm}(\omega, \Delta^2) = (\omega+m) R_{(-)} P_{\ell\pm 1}(\omega \alpha^Q) + (\omega-m) R_{(+)} P_{\ell}(\omega \alpha^Q) \quad (3.78)$$

The expressions of Δ^2 , $\bar{\Delta}^2$ and ω as functions of q_{20} and $\cos\theta_2$ can be obtained by inverting some of the formulae of section 3 a), as it will be explained later. In particular, in the L system one uses (3.9) and (3.16), and in the B system one uses (3.20) and (3.23).

It should be remarked that in (3.76) the real part has not been indicated, since $f_{\ell\pm}(\omega, \Delta^2)$ and $f_{\ell\pm}(\omega, \bar{\Delta}^2)$ have the same phase as the on-shell amplitude $f_{\ell\pm}(\omega, -\mu^2)$ [see Ref. 9), formula (22)].

What makes the calculation of $d\mathcal{G}_1/dq_{20} d\Omega_2$ easy is the fact that $d\mathcal{G}_1$ contains only M_a and M_b . For both of these matrix elements q_2 is a "recoil particle", while q_1 comes from the pion-nucleon scattering vertex. For $d\mathcal{G}_2$ the situation is different, due to the fact that q_2 is now the scattered nucleon. It is therefore no longer possible for $d\mathcal{G}_2/dq_{20} d\Omega_2$ to assume a form free from integrations like (3.76).

What one obtains is now ²⁰⁾

$$\begin{aligned} \frac{d^2\mathcal{G}_2}{dq_{20} d\Omega_2} &= \frac{1}{(2\pi)^2} \frac{G^2}{F} |\vec{q}_2| \frac{q^Q}{\omega} \int_{-1}^1 d\cos\theta^Q \int_0^{2\pi} d\varphi^Q u^2 \cdot \left\{ C_3^2 t^2 \pi^2(t^2) \cdot \right. \\ &\cdot |f_{\ell\pm}(u, t^2)|^2 \left([a_{\ell\pm} P_{\ell}(\cos\theta^P)]^2 + [\sin\theta^P P'_{\ell}(\cos\theta^P)]^2 \right) + C_4^2 \bar{t}^2 \pi^2(\bar{t}^2) \cdot \\ &\cdot |f_{\ell\pm}(u, \bar{t}^2)|^2 \left([a_{\ell\pm} P_{\ell}(\cos\epsilon^P)]^2 + [\sin\epsilon^P P'_{\ell}(\cos\epsilon^P)]^2 \right) + C_3 C_4 \pi(t^2/\pi(\bar{t}^2)) \cdot \\ &\cdot f_{\ell\pm}^*(u, t^2) f_{\ell\pm}(u, \bar{t}^2) \phi'_{\ell\pm} \left. \right\} \quad (3.79) \end{aligned}$$

where $\Phi'_{\ell\pm}$ is obtained from $\Phi_{\ell\pm}$ (see (3.73)) by the substitutions

$$\begin{aligned}\omega &\rightarrow u \\ p_{10}^Q, p_{20}^Q &\rightarrow p_{10}^P, p_{20}^P \\ \theta^Q &\rightarrow \theta^P \\ \epsilon^Q &\rightarrow \epsilon^P \\ \alpha^Q &\rightarrow \alpha^P\end{aligned}\tag{3.80}$$

A point which has to be explained is how to express the quantity within curly brackets in (3.79) as a function of the variables of integration $\cos\theta^Q$ and φ^Q and of the invariants Δ^2, ω^2 which are simply related to q_{20} and Ω_2 both in the L and B systems. This is easily accomplished by noting that this quantity depends only on $\Delta^2, \omega^2, t^2, u^2$ since the barred invariants $\bar{\Delta}^2$ and $\bar{t}^2$ do not add any further dependence, because of the first two formulae (3.3). The invariants t^2 and u^2 must now be related to $\Delta^2, \omega^2, \theta^Q$ and φ^Q . As far as t^2 is concerned we only need inverting (3.30); this is equivalent to calculating t^2 in the Q system. One gets

$$t^2 = 2q_{10}^Q p_{10}^Q - 2m^2 - 2q_1^Q p_1^Q \omega \theta^Q\tag{3.81}$$

where $q_{10}^Q, q_1^Q, p_{10}^Q, p_1^Q$ depend only on ω^2 and Δ^2 (see Eqs. (3.28) and (3.25)).

For u^2 the situation is a little more involved. First one obtains u^2 as a function of t^2 and $\bar{t}^2$ through the second of Eqs. (3.3). Next one expresses t^2 by means of (3.81) and $\bar{t}^2$ by means of

$$\bar{t}^2 = 2q_{10}^Q p_{20}^Q - 2m^2 - 2q_2^Q p_2^Q \omega \epsilon^Q\tag{3.82}$$

as functions of ω^2 , Δ^2 , θ^Q and ϵ^Q . Finally, one expresses ϵ^Q in terms of θ^Q and φ^Q , using the trigonometric formula

$$\omega \epsilon^Q = \omega \alpha^Q \cos \theta^Q + \sin \alpha^Q \sin \theta^Q \cos \varphi^Q \quad (3.83)$$

where α^Q (see (3.32)) depends only on W , ω and Δ^2 . One thus obtains $u^2 = u^2(\omega^2, \Delta^2, \theta^Q, \varphi^Q)$.

This expression, if substituted together with (3.81) into (3.70), would lead to a very complicated formula. There is, however, no point in actually developing it since only a computer calculation of (3.79) is conceivable, and there the various substitutions are made automatically.

The final remarks about how formulae (3.76) and (3.79) should be used are the following.

C.M. angular distribution. The invariants $\bar{\Delta}^2$, Δ^2 can be expressed through (3.23) and the first of (3.3) as a function of $\cos \theta_2^B$ and ω^2 . ω^2 can in turn be expressed as a function of q_{20}^B by inverting (3.20). The integration over q_{20}^B extends from m to $(W^2 - 2m\mu - \mu^2)/2W$.

C.M. energy spectrum. Again one uses (3.23) and (3.20) in order to express Δ^2 and ω^2 as functions of $\cos \theta_2^B$ and q_{20}^B . Then one integrates over q_{20}^B (which simply gives 2π) and over $\cos \theta_2^B$ between the limits -1 and $+1$.

Lab. kin. energy spectrum. One expresses Δ^2 as a function of $T_2^L = q_{20}^L - m$ by means of (3.9). Then one uses the relation

$$q_2^B dq_{20}^B d\cos \theta_2^B = \frac{d\omega^2 d\Delta^2}{4F} = \frac{m}{2F} d\omega^2 dT_2^L \quad (3.84)$$

and integrates over $d\varphi_2^B$ (obtaining 2π) and $d\omega^2$ in the range

$$(m+\mu)^2 \leq \omega^2 \leq \omega_{\max}^2(T_2^L)$$

with

$$\omega_{\max}^2(T_2^L) = m^2 - \frac{W^2 T_2^L}{m} + \frac{2p^B W}{m} [T_2^L(T_2^L + 2m)]^{1/2} \quad (3.85)$$

Let us now consider the case of the nucleon q_1 . As indicated in (3.53) the formulae for this case are the same as for q_2 when the exchanges (3.60) are performed. Then the contribution "of type $\mathcal{G}_1$ " for q_2 will transform into the contribution "of type $\mathcal{G}_2$ " for q_1 , and vice versa. In the particular simple cases in which the dominant waves are the same for both the $\mathcal{G}_1$ and the $\mathcal{G}_2$ parts, the spectra for q_1 will be obtained from the spectra from q_2 simply by exchanging c_1 with c_3 , c_2 with c_4 .

Let us now consider the calculation of the spectra of the π -meson q . Since the pion is in any diagram a scattered particle it is a priori clear that in no case we shall obtain expressions free from integrations like (3.76). We will then get expressions similar to (3.79). Referring to (3.74) we stress once more that $d\mathcal{G}_1$ differs from $d\mathcal{G}_2$ by the fact that the former (latter) refers to the situations where the nucleon q_2 is the recoil (scattered) nucleon, while q_1 is the scattered (recoil) nucleon. Since, however, for the sake of obtaining the pion distributions one has to integrate over both q_1 and q_2 , a difference between the expressions for $d\mathcal{G}_1/dq_0 d\Omega$ and $d\mathcal{G}_2/dq_0 d\Omega$ can only arise from the fact that the waves dominating the πN^2 scattering may have in $\mathcal{G}_1$ and $\mathcal{G}_2$ different isotopic spins; but for a given isotopic spin component, the two contributions are exactly the same.

We have then

$$\frac{d\sigma}{dq_0 d\Omega} = A_1 + A_2 \quad (3.86)$$

where

$$\begin{aligned} A_1 = & \frac{1}{(2\pi)^2} \frac{q^2}{F} |\vec{q}| \frac{q^D}{z} \int_{-1}^1 d\cos\theta^D \int_0^{2\pi} d\varphi^D u^2. \\ & \left\{ c_1^2 t^2 \pi^2(t^2) |f_{\ell_{\pm}}(u, t^2)|^2 \left([a_{\ell_{\pm}} P_{\ell}(\cos\theta^P)]^2 + \right. \right. \\ & \left. \left. + [\sin\theta^P P'_{\ell}(\cos\theta^P)]^2 \right) + c_2^2 \bar{E}^2 \pi^2(\bar{E}^2) |f_{\ell_{\pm}}(u, \bar{E}^2)|^2 \right. \\ & \cdot \left([a_{\ell_{\pm}} P_{\ell}(\cos\epsilon^P)]^2 + [\sin\epsilon^P P'_{\ell}(\cos\epsilon^P)]^2 \right) + \\ & \left. + c_1 c_2 \pi(t^2) \pi(\bar{E}^2) f_{\ell}^*(u, t^2) f_{\ell}(u, \bar{E}^2) \phi'_{\ell_{\pm}} \right\} \end{aligned} \quad (3.87)$$

and A_2 is given again by a formula of the type (3.87) with c_3 at the place of c_1 , c_4 at the place of c_2 , and, possibly, values of $\ell_{\pm}$ different from the ones used for A_1 . In (3.87) $a_{\ell_{\pm}}$ is defined by (3.77) and $\phi'_{\ell_{\pm}}$ is related, through Eq. (3.80), to Eq. (3.73). Of course, $|\vec{q}|$ has to be calculated in the same frame as q_0 and Ω . The quantity within curly brackets in the r.h.s. of (3.87) has to be expressed as a function of the variables of integration $\cos\theta^D$, φ^D and of the invariants z^2 and r^2 which are simply related to q_0 and Ω : see (3.7) and (3.14) for the lab. and (3.18) and (3.21) for the c.m. system.

4. COMPARISON WITH EXPERIMENT

The formulae derived in this paper can be used for comparing our theory with the experiment only if the off-shell behaviour of the partial-wave amplitudes $f_{\ell\pm}(\omega, \Delta^2)$ can be determined. In such a case, one is left only with an unknown multiplicative function of Δ^2 . In our case, the unknown function left is $K(\Delta^2)$, the pionic form factor of the nucleon. In Ref. ⁹⁾ it has been shown that $K(\Delta^2)$ appears in the same way for every $f_{\ell\pm}(\omega, \Delta^2)$. One can then try the application of the criterion explained in sections 1 and 2 for the determination of the product of the unknown form factors. We have already mentioned that the theory developed in Ref. ⁹⁾ allows one to write the Δ^2 and ω^2 -dependence of the πN scattering matrix element for values of ω^2 in the 33 resonance energy region (i.e., for $\omega \lesssim 1.5$ m, corresponding to a πN Q value $\lesssim 350$ MeV). Beyond this region the application of the theory becomes very doubtful and only possible with the arbitrary assumption that the off-shell correction for the higher waves is the same as the one for the $T = J = 3/2$ wave. Even for the latter wave, at energies $\omega \gtrsim 1.5$ m, one cannot strictly use the expression obtained, because in the derivation of this expression the static approximation, among others, has been performed. For these reasons we shall proceed to the comparison with experiment in two steps :

first we will compare only the "rigorous" part of the theory, by limiting ourselves to consider low values of ω . This will allow the determination of $K^2 K'$. Then we will look (more or less empirically) for that off-shell behaviour (in the off- 33 resonance region) which gives the best agreement with the experiment, mainly for the sake of showing that the one-pion exchange theory may work also in that energy region and of having some indications about the direction where further theoretical developments might go.

a) The 33 resonance energy region

As we have already said, we limit ourselves to values of ω lower than 1.5 m. When the kinetic energy of the incoming nucleon exceeds ~ 1300 MeV in the lab. the upper limit of ω exceeds 1.5 m. Since final state nucleon-nucleon interactions put a lower limit on the incident energy at about 800 MeV, it is clear that our theory has a restricted range of application, if one wants to consider distributions of all the events experimentally measured. If, however, one makes a selection by considering only those events for which $\omega < 1.5$ m, then our theory is still applicable, with some care. The difficulty for performing such a selection lies in the fact that one should actually put an upper limit on both the Q values of the systems formed by the pion together with each of the final nucleons, since in two diagrams q and q_1 and in the other two q and q_2 come out from the same vertex. At energies $\gtrsim 2$ GeV the two limitations become incompatible. It is, however, possible to perform the limitation only on one Q value for the case of reaction (1.2), namely on $Q(\pi^+p)$ (or, in other words, on ω^2). For this reaction q_2 is the final neutron and the contribution of $d\sigma_2$ (where the mass of the πN system emerging from the same vertex is not ω , but u) is ~ 9 times smaller than the one of $d\sigma_1$. Furthermore, the "off-33" part of $d\sigma_2$ is only a fraction of $d\sigma_2$ itself. Therefore, this latter part has practically no weight in the final result and any reasonable approximation in its calculation can be acceptable.

We can summarize the reactions for which the model is expected to be the most powerful : all single π production reactions in the range 800-1300 MeV and reaction (1.2) at all energies when ω^2 is kept in the 33 resonance region. For the latter case the formulae given in the preceding sections remain unchanged ; only when an explicit integration over ω^2 occurs the change in the limits of integration must be performed. Where it does not, the additional restriction over ω^2 must be introduced in some way in the computer calculation. In practice, it can be reduced

to a limitation in the q^Q -integration : we cannot give further details here. Reaction (1.2) is the most reliable for the calculation of the form factor function $G(\Delta^2)$; and in fact we will present later a plot of $G(\Delta^2)$ determined in this way.

Let us see how to transform Eqs. (3.67) and (3.70) when the expression for the 33 off-shell matrix element is explicitly written. From Eq. (66) of Ref. 9) we get for $T = 3/2$ (with some obvious changes of notation) 21) :

$$f_{1+}(\omega, \Delta^2) = K(\Delta^2) \frac{p_1^Q}{q^Q} \Gamma(\Delta^2) [1 + C(\omega, \Delta^2)] f_{1+}(\omega) \quad (4.1)$$

where

$$\Gamma(\Delta^2) = \left(1 + \frac{\Delta^2 + \mu^2}{4m^2}\right)^{1/2} \gamma(\omega_r, \Delta^2) \quad (4.2)$$

$$C(\omega, \Delta^2) = \frac{4f^2}{3\mu^2(\omega - m)} \left[\frac{\gamma(\omega, \Delta^2)}{\gamma(\omega_r, \Delta^2)} - 1 \right] q^{Q3} \cotg \delta_{1+}(\omega) \quad (4.3)$$

$$\gamma(\omega, \Delta^2) = \left[1 + \frac{3(\Delta^2 + \mu^2)}{2m(\omega - m)} \right] \left[1 + \frac{\Delta^2 + \mu^2}{2m(\omega - m)} \right]^{-3} \quad (4.4)$$

and the subscript r means "calculated at resonance", namely at $\omega = \omega_r = 1.314 m$.

In (4.1) $f_{1+}(\omega) \equiv f_{1+}(\omega, -\mu^2) = (q^Q)^{-1} e^{i\delta_{1+}(\omega)} \sin \delta_{1+}(\omega)$ is the physical amplitude which can be extracted from the experiment, and $\delta_{1+}(\omega)$ is the physical phase-shift. The C term is in general small (it vanishes at the resonance), and in a simplified calculation can be neglected.

The insertion of (4.1) into (3.76) and (3.79) must be performed with the care of using the right variables at their place. For instance, in the first term of (3.79) $f_{1+}(u^2, t^2)$ is needed. This means that in Eqs. (4.1) to (4.4) everywhere the substitutions $\omega \rightarrow u$ and $\Delta^2 \rightarrow t^2$ must be performed. It can be useful to recall the formula

$$|f_{1+}(\omega)|^2 = \frac{\sigma_{33}(\omega)}{8\pi} \quad (4.5)$$

which connects $|f_{1+}(\omega)|^2$ to the more familiar resonant cross-section. For the latter, any fit like e.g., a Breit-Wigner formula, can be used. Furthermore, when we refer to the wave $1+$ we have

$$\phi_0^{1+} = (\omega+m) R_{(-)} \frac{3\cos^2\alpha^Q - 1}{2} + (\omega-m) R_{(+)} \cos\alpha^Q \quad (4.6)$$

and

$$\begin{aligned} \phi_{1+} = (\omega+m) R_{(-)} [9\cos\alpha^Q \cos\theta^Q \cos\epsilon^Q + 1 - 3\cos^2\theta^Q - \\ - 3\cos^2\epsilon^Q] + (\omega-m) R_{(+)} [\cos\alpha^Q + 3\cos\theta^Q \cos\epsilon^Q] \end{aligned} \quad (4.7)$$

For the case of Eq. (3.79) the substitutions $\omega \rightarrow u$ and $\alpha^Q \rightarrow \alpha^P$, $\theta^Q \rightarrow \theta^P$, $\epsilon^Q \rightarrow \epsilon^P$, performed in (4.7), give the term ϕ_{1+}^P .

Formulae (3.76) and (3.79) have been used for calculating lab. energy distributions of the nucleons for reactions (1.2) and (1.3) as well as the total cross section for the same reactions up to 1300 MeV. The total cross section for reaction (1.2) when the $\pi^+ p$ Q value is less than 350 MeV has been also evaluated up to 3 GeV. The lab. energy distributions are compared with experiment at 970 MeV (Fig. 2) and at 2000 and 2850 MeV with restriction on $Q(\pi^+ p)$ to the resonance region (Figs. 3 and 4 a))²². As one can see, a satisfactory agreement is obtained for all the spectra with a unique choice of the unknown function $G(\Delta^2)$. The assumed behaviour of $G(\Delta^2)$ is given in Fig. 6. This behaviour has been reproduced by the following empirical fit :

$$G(\Delta^2) = \frac{1 - A}{1 + \frac{\Delta^2 + \mu^2}{\alpha}} + A \quad (4.8)$$

The values $A = 0.28$ and $\alpha = 4.73 \mu^2$ have been used in the calculation of the spectra presented in Figs. 2 to 4 a). The actual analytical expression (4.8) was chosen because of its resemblance to the formulae of the Clementel-Villi type currently used in the discussion of the e.m. form factors. Other types of expressions have been tried (e.g., quadratic), and found to give bad agreement for any choice of the parameters.

In Fig. 6 the possible fluctuations of A and α are represented as an over-all uncertainty in $G(\Delta^2)$ (roughly corresponding to a standard deviation). We have not attempted a "best-fit" procedure for obtaining A and α ; such a precision goes beyond the aim of the present investigation. We stress also that the behaviour shown for G becomes less reliable as Δ^2 increases, since the contribution at large Δ^2 is always very small and the changes of G affect less and less the agreement with experiment. Equation (4.8) can be trusted with a certain confidence up to $\Delta^2 \sim 30 \mu^2$, though in the spectra presented it has been used also for larger values.

b) The off-33 energy region

Let us now discuss the question whether it is possible to use the present theory also in the region where a theory for the description of off-shell scattering is lacking. If we put no limitations on ω , we can idealize the situation for physical πN^0 scattering as follows: the $T = 3/2$ cross section can be represented with only the $J = 3/2$ wave in its whole range; the $T = 1/2$ cross section is represented only by the two resonant states $D_{3/2}$ and $F_{5/2}$ at $\omega \geq 1.51$ m and $\omega \geq 1.69$ m respectively. We do not worry here about the region $\omega \gtrsim 2$ m since such values would become important only for incident nucleon energies above 3 GeV.

If we make now the following hypotheses: i) that the above situation remains true for the off-shell scattering; ii) that in the "off-33" energy region an expression of the type

$$f(\omega, \Delta^2) = K(\Delta^2) \Lambda(\Delta^2) f(\omega, -\mu^2) \quad (4.9)$$

can be used, we can try to extend our theory to the whole physical region up to incident energies ~ 3 GeV. We remark that assumption ii) (the one used in Ref. ¹⁰⁾) should look rather reasonable, since for the higher ($T = 1/2$) resonances the ω -dependence of the off-shell matrix element cannot be derived by analogy with the 33 case, and the most sensible procedure is to calculate everything at the resonance and to lump all the kinematical quantities in a single unknown Δ^2 -dependent function. The use of the same function for both the $T = 1/2$ resonances, and its extension also to the "tail" of the $T = 3/2$ wave may have less justification; however, at the moment, any further indication on this subject is lacking (because also for the off-resonance $T = 3/2$ interaction Eq. (4.1) cannot

be used) and the simplest procedure is to assume that the average off-shell correction for all waves is the same. This attitude can be applied also to i), by analogy with the on-shell case.

In the off $\bar{3}3$ region, of course, we do not worry about the behaviour of K^2K' since we cannot separate it from the kinematical part. By analogy with the calculations of Ref. ¹⁰⁾ we put

$$G(\Delta^2) \wedge (\Delta^2) = \left[1 + \frac{\Delta^2 + \mu^2}{\gamma} \right]^{-1} \quad (4.10)$$

We use the same γ in the whole off- $\bar{3}3$ region and insert (4.9) and (4.10) into (3.76) and (3.79).

Needless to say, in (3.76) and (3.79) the quantities which depend on the partial waves (i.e., on $\ell \pm$) must be specified according to which waves are supposed to be important; i.e., we use $\ell \pm = 1+; 2- \text{ and } 3-$ respectively for $T = 3/2$ and $1/2$ in the above-mentioned intervals of energy ²³⁾. In the integrated spectra, such as total cross sections, lab. energy spectra and c.m. angular distributions, this "single wave approximation" does not influence the results appreciably if the total $T = 3/2$ and $T = 1/2$ cross sections are normalized to the experimental ones.

We have applied (4.9) and (4.10) (with $\gamma = 105 \mu^2$) for the calculation of the "off $\bar{3}3$ part" of the lab. spectrum of the neutrons from reaction (1.2) at 2 GeV. The curve obtained is compared with the experiment in Fig. 4 b). The total spectrum at 2 GeV $\left[\text{obtained by superposition of the calculated spectra in Figs. 4 a) and 4 b)} \right]$ is presented in Fig. 5. We see that the agreement is good in spite of the many approximations performed. By means of the same cut-off function for the higher values of ω we have calculated the total cross sections for reactions (1.2) and (1.3) up to 3 GeV. The result is shown in Fig. 7, together with the experimental points known today ²⁴⁾. In the region above 800 MeV, where the model is more

reliable, the agreement is more than satisfactory. Some experimental points between 500 and 800 MeV have been included in the graphs for the sake of completeness : they are still compatible with the theory for reaction (1.2), but they show marked deviations for reaction (1.3). We note that in this energy region the ratio between π^+ and π^0 production seems to be different from the value 5 predicted by any model containing the hypothesis of dominance of the $T = J = 3/2$ final πN state. This deviation might be caused by final state nucleon-nucleon interaction (cf. Sect. 1).

In Fig. 8 is shown the behaviour of the total cross section for reaction (1.2) for the events where $Q(\pi^+p) \leq 350$ MeV.

Before closing this argument, we want to discuss here the meaning of the calculations performed in Ref. ¹⁰⁾. In this work, a good agreement with the experiment was obtained by using (4.9) and (4.10) everywhere, with γ of the order of $60 - 90 \mu^2$. By comparison with the more rigorous results obtained here, such a γ is a bit too high for the 33 region alone, a bit too low for the off-33 region alone ; the general agreement was due to a sort of compensation (in the high momentum transfer region) between the two effects. However, the form factor derived by this procedure turns out to be only slightly different from the one obtained here with a more reliable method. We think, therefore, that from an empirical point of view and for rough simple evaluation of integrated spectra extended to the whole physical region, the simple approximation (4.9) and (4.10), with $\gamma \sim 80 \mu^2$, can be used without a too large error. Such a type of Δ^2 -dependent cut-off function can also be tentatively applied to other peripheral processes ²⁵⁾.

5. CONCLUSIONS

The theory presented in this paper for the processes of single pion production works, we must admit it, unexpectedly well. As far as the creation of the $\Sigma(1385)$ isobar is concerned, the agreement with the experiment supports the criterion explained in the introduction and in section 2, and seems to give a positive answer for the one-pion exchange character of these processes. As a by-product of this analysis, the pionic form factor of the nucleon has been obtained through (4.8). For the creation of the higher isobars, again a fairly good agreement has been obtained (which is expressed by the graphs of Fig. 4b) and 5, as well as by the behaviour of the total cross sections), in spite of the many assumptions and approximations performed.

Apart from the practical interest of these conclusions, which has been outlined in the preceding section, also some theoretical consequences can be inferred. The dominance of the one-pion exchange also in the case where the exchange of systems having $T = 0$ is allowed, may mean that at the energies considered here the Regge pole character¹³⁾ of the systems exchanged is not yet apparent. In fact, it has been conjectured that at high energies the dominant system exchanged should be the $T = 0$ Pomeron pole; this hypothesis seems to be supported by the experiment at incident energies of 10 - 20 GeV²⁶⁾. The fact that at our energies we have obtained a good agreement everywhere in terms of a "non-reggeized" one-pion exchange should, however, not seem to be in contradiction with the Regge pole approach, since at energies higher than 3 GeV the transition from the one to the other mechanism may well be switched on²⁷⁾.

From what we have discussed, we conclude that an experimental study of reactions of the type (1.2) or (1.3), especially in the region of 5 - 10 GeV, may be interesting for an investigation of the Regge pole behaviour of quasi-elastic collisions. One has indeed a point to start, namely the present theory: a criterion for finding the breakdown of the one-pion exchange hypothesis, as explained in section 2; a way for detecting the characteristic behaviour of the Regge poles, as discussed in Ref.²⁷⁾.

In conclusion, we think that the peripheral model has been much more useful than expected for the processes of single-pion production. The study of reactions (1.2) and (1.3) at higher energies ($\gtrsim 5$ GeV) should lead to some disagreement with this model. The fact, however, that an understanding of these phenomena has been achieved at lower energies may help the understanding of the new mechanisms of interaction which switch on with increasing energy and momentum transfer.

ACKNOWLEDGEMENTS

We thank Dr. D.V. Bugg and Dr. G.A. Smith for having sent us the spectra used in this paper. We would like to thank also Professor L. Van Hove, Professor S.D. Drell and many other CERN people for interesting discussions. We are indebted to the CERN Ferranti computer staff for the numerical evaluation of our formulae.

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FIGURE CAPTIONS

Fig. 1 The four peripheral diagrams describing single-pion production in nucleon-nucleon collisions. Solid line represents nucleons, dashed lines represent pions.

Fig. 2 Experimental (histograms) and theoretical (full lines) lab. kinetic energy distributions at 970 MeV : a) neutron spectrum from reaction (1.2) ; b) proton spectrum from reaction (1.2) ; c) proton spectrum from reaction (1.3).

Fig. 3 Experimental (histograms) and theoretical (full lines) lab. kinetic energy distributions at 2.85 GeV, for reaction (1.2) and for only those events where $Q(\pi^+p) \leq 350$ MeV (33 region) : a) neutron spectrum ; b) proton spectrum.

Fig. 4 Experimental (histograms) and theoretical (full lines) lab. kinetic energy distribution of the neutrons at 2.00 GeV, for reaction (1.2) : a) for only those events where $75 \text{ MeV} \leq Q(\pi^+p) \leq 225 \text{ MeV}$; b) for the events outside this interval. The gap observed around 1600 MeV in Fig. 4 b), both in the experimental and theoretical distributions, is due only to the criterion used for the selection ; in fact, 1600 MeV is the upper limit of the energy observable in the events having $Q(\pi^+p) \geq 225 \text{ MeV}$.

Fig. 5 The same distribution as in Fig. 4, referred to all the events.

Fig. 6 Plot of the "form factor" $G(\Delta^2)$ as a function of Δ^2/μ^2 . Formula (4.8) is used with $A = 0.28$ and $\alpha = 4.73 \mu^2$. A rough estimate of the standard error is indicated at the beginning and at the end of the curve.

Fig. 7 Theoretical and experimental total cross section as a function of the incident lab. energy T up to 3 GeV : a) for reaction (1.2) ; b) for reaction (1.3). Not all the experimental points known below 800 MeV have been plotted in the figures : in particular, very low energy points (below 500 MeV) have been neglected.

Fig. 8. Theoretical total cross section for reaction (1.2) and for the events where $Q(\pi^+p) \leq 350$ MeV. T is the lab. incident energy of the protons. The experimental point at 2.85 GeV (Smith et al.) is also reported.

Fig.1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig.6

Fig. 7

Fig. 8

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Of the last three papers the first one contains a calculation for the total cross section for reaction (1.2) ; the other two are mainly concerned with lab. energies $\gtrsim 10$ GeV.