CM-P00056797

PERIPHERAL MODEL FOR INELASTIC PROCESSES

E. Ferrari and F. Selleri *)

CERN - Geneva

*) Present address : Istituto di Fisica, Università di Bologna
Italy.

TABLE OF CONTENTS

	<u>page</u>
<u>SECTION 1.</u> The importance of the peripheral mechanism of interactions in the description of physical processes ---	1
1.1) Introduction -----	1
1.2) Present situation in the peripheral description of physical phenomena -----	2
<u>SECTION 2.</u> Mathematical formulation of the peripheral model. Case of three particles in the final state -----	7
2.1) S-matrix and cross-sections -----	7
2.2) Peripheral S-matrix element -----	9
2.3) Invariants and kinematics -----	11
2.4) Calculation of cross-sections -----	21
2.5) Occurrence of different peripheral graphs for the same process -----	39
<u>SECTION 3.</u> Many particles in the final state -----	53
3.1) The most general peripheral diagram for many particle production -----	53
3.2) The particular case of one or two particles coming out from one of the vertices -----	57
Appendix 1 -----	71
Appendix 2 -----	77
Appendix 3 -----	81
Appendix 4 -----	87
References and footnotes -----	97

PERIPHERAL MODEL FOR INELASTIC PROCESSES

SECTION 1

The importance of the peripheral mechanism of interactions in the description of physical processes

1.1 INTRODUCTION

In the past few months there have been indications that in high-energy phenomena peripheral interactions play an important role ¹⁾. By "peripheral" one usually means that kind of interaction which is transmitted by the less massive system which can be exchanged between the colliding particles (for the most part of strong interactions this system consists of a single pion). We will extend the meaning of this word and call peripheral any interaction which is transmitted through the exchange of a single virtual particle (even not the lightest possible one). A more proper denomination of this kind of interaction should be "one particle exchange interaction", but the name "peripheral" will be used throughout for simplicity.

In this paper we want to discuss in detail the predictions obtained when inelastic processes are described in terms of a dominant peripheral interaction ²⁾. The existing work on the subject will be recalled and summarized : the approximations used for obtaining numerical results will be analyzed and discussed : and a set of formulae will be given, which in principle allow the calculation of any physical quantity relative to the inelastic process under consideration. These formulae will be also of practical use under certain approximations. A part of the discussion will be devoted to kinematical problems. The work will be divided in three Sections and four Appendices.

Section 1 will present a summary of the present status in the application of the peripheral approach to inelastic phenomena. Sections 2 and 3 will deal with the explicit derivation of formulae for the energy spectra, angular distributions and total cross-sections predicted by the peripheral model. Section 2 deals with the particular, but important case of two particles transforming into three. Section 3 deals with the case of more than three particles in the final state. Appendices 1, 2 and 3 contain some of the phase-space calculations. The last Appendix (Appendix 4) gives a list of the papers which contain numerical evaluations of the peripheral effects relative to some experimentally observed reactions.

1.2 PRESENT SITUATION IN THE PERIPHERAL DESCRIPTION OF PHYSICAL PHENOMENA

The importance of diagrams with only one particle exchanged was first analyzed with reference to 4-body processes (in particular to elastic scattering). In this case a peripheral diagram reduces substantially to a perturbative diagram, with possible corrections (form factors) in the vertices. Also when a one particle exchange description of such processes is possible, it is often found that such a mechanism does not reproduce the experimental behaviour. However, in some cases it is possible to separate and observe the contributions of the peripheral diagrams to the 4-body processes. An example is given by nucleon-nucleon scattering, where the peripheral diagram (with the exchange of a single pion) proved very useful in eliminating the many ambiguities which occurred in a phase-shift analysis ³⁾. In the process of Compton scattering on protons, the importance of the exchange of one intermediate π^0 has been emphasized ⁴⁾. More recently the process $\gamma + p \rightarrow K^+ + \Sigma^-$ has been analyzed by Sakurai ⁵⁾, who found that the experimental angular distributions can be fitted by the diagram containing only one intermediate K^+ exchanged, if one assumes even parity for the ΣPK coupling and a suitable value of the coupling constant.

Considerations of peripheral diagrams in inelastic processes have been first made by Chew and Low⁶⁾. They proposed a method of extrapolation from the physical into the unphysical region in order to get information on the cross-sections for processes which cannot be reproduced in the laboratory. The philosophy of Chew and Low was based on the fact that any diagram with a one-particle exchange contributes a pole to the amplitude of a physical process, in the physical variable corresponding to the squared four-momentum of the exchanged particle. Such a pole occurs for unphysical values of the variable to which it is related : in particular, the exchange of the lightest particle gives the singularity that is nearest to the physical region. In principle this pole can be reached by means of an extrapolation procedure from the physical region, and its residue can be determined. This residue is proportional to the physical amplitude for other processes, involving the exchanged particle : then, from a suitable analysis of an inelastic process it is possible to obtain information for processes (such as e.g., π - π scattering) which cannot be directly reproduced in the laboratory. In Chew and Low's approach, the experimental knowledge of the inelastic process was assumed as a starting point to obtain further information, and no predictions on it were attempted. A different point of view was assumed, however, in the most part of the following peripheral calculations⁷⁾⁻⁹⁾, which started from the following consideration. If the pole contained in the peripheral diagram is the nearest to the physical region and is not too far from it, it should be expected that in the part of the physical region near the pole the peripheral diagram is still dominant. This region corresponds to a "quasi real" intermediate particle, i.e., to a "small" momentum transfer in the vertex where this particle is emitted (e.g., the lower vertex in Fig. 1). In the preceding sentence "small" means of the order of a few times the distance of the pole to the physical region. The assumption of the dominance of the peripheral diagram is justified by the fact that, in this region, the value of the propagator containing the pole is large, and falls off rapidly with increasing momentum transfer. This effect is particularly marked if the exchanged particle is light ; it tends to disappear for heavier exchanged

particles. Therefore, the cases of interest for which the peripheral model is expected to work refer generally to the exchange of a single pion ; in some cases, one can consider also the exchange of a single K-meson.

Starting from the above point of view, Goebel ⁷⁾ showed that in the process of single pion production in pion-nucleon collisions the cross-section should have a maximum for small momentum transfers to the target nucleon, if the $\pi\text{-}\pi$ interaction is not negligibly small.

The observation of this maximum was further discussed by Bonsignori and Selleri ⁸⁾ who found the first direct evidence of $\pi\text{-}\pi$ interaction confirmed by many following investigations ¹⁰⁾. In reference ⁸⁾ it is also shown that there is a non-negligible peripheral contribution to the processes of pion production in nucleon-nucleon collisions. On the basis of further work on this subject ^{11),12)}, it has been found that the one-pion exchange term plays a very important role in such processes, in particular in the energy region of a few GeV in the lab. system. This point of view has been also extended further by tentatively calculating peripheral effects also far from the region of "quasi real" intermediate particles ¹¹⁾⁻¹³⁾. In this case the dominance of the pole term over the other contributions has no direct theoretical justification, but has to be taken as an assumption. By this approach, it is possible to get complete predictions for the experimental energy and angle distributions of all the final particles : these predictions can be calculated numerically, and compared with experiment. Of course, one cannot expect the experimental data to be reproduced by such a simple picture, especially if one limits oneself to calculating the functions entering in the vertices of a peripheral diagram as if the intermediate particle were real (pole approximation) ¹⁴⁾. This attitude, however, can be taken for a first qualitative study of certain phenomena ^{11),13),15)}. It is also possible to calculate a peripheral diagram in a more refined way by taking into account the corrections in the vertices induced by the virtuality of the intermediate particle. This point of view

coincides with the previous one at "small" momentum transfers and is supposed to give a more realistic evaluation of the peripheral effects at higher momentum transfers. It has been discussed in detail by Ferrari and Selleri ¹⁶⁾ for any process in which the vertex (virtual π) + $\mathcal{N} \rightarrow \pi + \mathcal{N}$ enters. The amplitude for such a vertex has been shown in reference ¹⁶⁾ to be given by a known function of the energy and the momentum transfer multiplied by the "pionic form factor" of the nucleon. Therefore, the calculation of a peripheral diagram containing this vertex is undetermined only by an unknown multiplicative function of the momentum transfer alone. This leaves open the possibility of an experimental check of the extension of the peripheral approach to the whole physical region. If the theory were found to disagree from experiment only for a momentum transfer (and not initial energy) dependent function, then one could reasonably infer that the peripheral effect is important also at large momentum transfers and try to determine the form factor from the experiment ¹⁷⁾. By following this approach, a first tentative evaluation of the pionic form factor has been made in reference ¹²⁾. We point out that this procedure is not necessarily limited to the pionic form factor of the nucleon and could eventually be applied to other phenomena such as the production of strange particles.

Another important consequence of the peripheral effects has been pointed out by Drell ¹⁸⁾. He showed that photons are probably more effective than nucleons for the productions of high energy beams of π -mesons. This fact is due to the expected statistical productions of pions in nucleon-nucleon collisions, as compared with the photoproduction mechanism where a photon likes to give almost all its energy to a single π -meson by expelling it off the meson cloud of the nucleon. This effect has been estimated to be so important as to compensate the factor $\alpha = \frac{1}{137}$ which usually depresses electromagnetic effects with respect to nuclear effects. In a more recent work, Drell and Hiida ¹⁹⁾ have shown that the quasi-elastic bump observed in high energy nucleon-nucleon scattering ²⁰⁾ can be explained with another peripheral effect, namely the diffractive scattering of the incoming nucleon from a virtual pion of the target nucleon cloud.

Another peripheral effect can lead to the production of very high energy photons in the forward direction in pion-nucleon collisions, as it has been pointed out by Salzman and Salzman²¹⁾.

Finally, we want to mention an interesting work by Amati, Fubini, Stanghellini and Tonin²²⁾, which may lead to important developments and applications. By using the Mandelstam representation for the high energy elastic amplitudes they show that, if one assumes the inelastic interactions to be peripheral, the diffraction scattering can be calculated. From the form of the obtained result a model for the inelastic processes is directly suggested, in which the interaction is the "most peripheral" one, in the sense that a process with any multiplicity appears to take place through a chain of vertices, joined together from a single π -meson, in each of which a pair of particles with small relative energy is emitted. From such a model it is possible in principle to calculate observable quantities like multiplicities, angular distributions, etc., and predict the asymptotic behaviour of the total cross-sections.

SECTION 2

Mathematical formulation of the peripheral model.
Case of three particles in the final state

2.1 S-MATRIX AND CROSS-SECTIONS

Consider a physical process in which two initial particles with 4-momenta p_1 and p_2 transform into n final particles with 4-momenta $q_1 \dots q_n$. From the S-matrix element for the process

$$\langle f | S | i \rangle = S_{fi}$$

we define an invariant M-matrix element through the relation

$$S_{fi} = \delta_{fi} + i (2\pi)^{4 - \frac{3}{2}(n+2)} 2^{-b/2} \left[\prod_{i=1}^f m_i \right]^{1/2} [p_{10} p_{20} q_{10} \dots q_{n0}]^{-1/2} \cdot \delta^4(P_f - P_i) M_{fi} \quad (2.1)$$

where b and f are the total numbers of bosons and fermions present in the initial and final states. Of course, $f+b = n+2$. P_f (P_i) is the total final (initial) 4-momentum. p_0 is the energy component of the 4-vector p ²³⁾.

The differential cross-section over all the physical variables, $d\sigma$, can be expressed, both in terms of S_{fi} and M_{fi} , through

8.

$$d\sigma = (2\pi)^6 \frac{P_0 P_{20}}{F} \frac{1}{TV} \sum |S_{fi}|^2 d^3 q_1 \dots d^3 q_n \quad (2.2)$$

$$d\sigma = (2\pi)^{4-3n} \frac{2^{-b}}{F} \left[\prod_{i=1}^f m_i \right] \sum |M_{fi}|^2 \delta^4(p_f - p_i) \frac{d^3 q_1 \dots d^3 q_n}{q_{10} \dots q_{n0}} \quad (2.3)$$

where $\sum$ indicates a sum over final and an average over initial spin states, and

$$F = \left\{ (p_1, p_2) - M_1^2 M_2^2 \right\}^{1/2} = M_2 p_1^L = W p_1^B \quad (2.4)$$

$p_1^L(p_1^B)$ is the magnitude of the laboratory (centre of mass) 3-momentum of the incident particle. $W^2 = -(p_1 + p_2)^2$ is the square of the total c.m. energy, and M_1, M_2 are the masses of the initial particles p_1, p_2 . The lab. is defined as the system in which $p_2 = 0$.

In (2.2) TV the "volume of the 4-dimensional universe" is used in place of a more elaborate expression²⁴⁾. It is cancelled by the TV factor appearing in $\delta^4(0) = (2\pi)^{-4} TV$, which arises from the square of the δ -function contained in S_{fi} .

Starting from (2.3) we obtain in the particular case of $n=2$

$$|M_{fi}|^2 = (2\pi)^2 \frac{2^{-b}}{\prod_{i=1}^f m_i} W^2 \frac{q_i}{q_f} \frac{d\sigma}{d\Omega} \quad (2.5)$$

q_i and q_f are the initial and final c.m. 3-momenta and $\frac{d\sigma}{d\Omega}$ is the differential cross-section in the c.m. system.

2.2 PERIPHERAL S-MATRIX ELEMENT

In this sub-section and in the following ones, we will discuss in detail the case of three particles in the final state ²⁵⁾. This case is of particular interest, since for it the peripheral model enables us to obtain complete predictions for the angular and energy distributions of the final particles. The arguments treated here will be easily extended in Section 3 to the case of more than three final particles. The 4-momenta and the masses of the particles occurring in the reaction will be denoted by : p_1, p_2 and M_1, M_2 for the initial particles : q_1, q_2, q and m_1, m_2, m for the final particles. Energy-momentum conservation requires

$$p_1 + p_2 = q_1 + q_2 + q. \quad (2.6)$$

In order to fix the ideas, let us suppose p_2 and q_2 to be fermions and the intermediate exchanged particle to be a boson. (As we have pointed out, this corresponds to the cases of practical interest to which the peripheral approach can be applied.) Now we want to put into mathematical form the following interaction picture : one of the initial particles, e.g., p_2 , acts as a "spectator" in the reaction : it simply changes its 4-momentum to q_2 by emitting a virtual particle of 4-momentum $k = p_2 - q_2$ and mass μ . This particle k interacts with p_1 according to the scheme

$$k + p_1 = q + q_1. \quad (2.7)$$

In this case the structure of the S-matrix element in momentum space is ($\Delta^2 = k^2$)

$$S_{fi} = i(2\pi)^8 \delta^4(p_f - p_i) \langle q_2 | j(0) | p_2 \rangle \Delta_F'(\Delta^2) \langle q q_1 | j(0) | p_1 \rangle \quad (2.8)$$

j is the current operator for the virtual intermediate particle and Δ_F' its complete propagation function ²⁶⁾. In terms of Feynman graphs, Eq. (2.8) can be represented by the diagram of Fig. 1, where the bubbles include all the possible higher order effects. The vertex function $\langle q_2 | j(0) | p_2 \rangle$ is a function only of the variable $(p_2 - q_2)^2 = \Delta^2$, and will be called $V(\Delta^2)$. The function $\langle qq_1 | j(0) | p_1 \rangle$ is proportional to the analytic continuation for $(q + q_1 - p_1)^2 > 0$ of the M matrix element $\mathcal{M}_1$ for the physical process (2.7). More particularly, we obtain for the S -matrix element for process (2.7)

Fig. 1

$$S_1 = \langle q q_1^{\text{out}} | k p_1^{\text{in}} \rangle = i(2\pi)^4 \delta^4(p_f - p_i) \frac{(2\pi)^{-3/2}}{\sqrt{2k_0}} \langle qq_1 | j(0) | p_1 \rangle \quad (2.9)$$

By comparison with the general definition (2.1), we get

$$\langle qq_1^{\text{out}} | j(0) | p_1 \rangle = (2\pi)^{-7/2} \lambda [2q_0 q_{10} p_{10}]^{-1/2} \mathcal{M}_1 \quad (2.10)$$

Here q is supposed to be a boson; λ equals $[m_1 M_1]^{1/2}$ or $\frac{1}{2}$, according to whether particle p_1 (and therefore also particle q_1) is a fermion or a boson respectively.

When $\Delta^2 \rightarrow -\mu^2$, the square of the matrix element $\mathcal{M}_1$ can be related, through (2.5), to the physical differential cross-section for process (2.7) and thus to an observable quantity. In this limit, one is allowed to introduce experimental information in the calculation. For an arbitrary value of Δ^2 , however, the full Δ^2 -dependence of V , Δ_F' and $\mathcal{M}_1$ will generally not be known: we shall discuss later a way of performing approximate evaluations of these quantities. For the moment, however, let us deal with the kinematics of process (2.6).

2.3 INVARIANTS AND KINEMATICS

Any kinematical quantity concerning the five particles entering in reaction (2.6) can be expressed by means of 5 independent invariants. We shall choose the following ones

$$\begin{aligned} W^2 &= -(p_1 + p_2)^2 \\ \Delta^2 &= (q_2 - p_2)^2 \\ \omega^2 &= -(q + q_1)^2 \\ t^2 &= (q_1 - p_1)^2 \\ u^2 &= -(q + q_2)^2 \end{aligned} \quad (2.11)$$

W^2 is the square of the the total energy in the c.m. system (which will be in the following referred to as B-system): ω^2 is the square of the total energy in the c.m. system of q_1 and q (Q-system): u^2 is the analogous quantity in the c.m. system of q_2 and q (P-system). Δ^2 and t^2 are 4-momentum transfers.

It will be useful to consider also the following invariants

$$\begin{aligned}
\bar{\Delta}^2 &= (q_2 - p_1)^2 \\
\bar{t}^2 &= (q_1 - p_2)^2 \\
r^2 &= (q - p_1)^2 \\
\bar{r}^2 &= (q - p_2)^2 \\
z^2 &= -(q_1 + q_2)^2
\end{aligned}
\tag{2.12}$$

which are expressed in terms of those of the fundamental set (2.11) through the relations :

$$\bar{\Delta}^2 = W^2 - m_2^2 - M_1^2 - M_2^2 - \omega^2 - \Delta^2 \tag{2.13}$$

$$\bar{t}^2 = W^2 - m_1^2 - M_1^2 - M_2^2 - u^2 - t^2 \tag{2.14}$$

$$r^2 = \Delta^2 + \omega^2 - t^2 - m^2 - m_1^2 - M_1^2 \tag{2.15}$$

$$\bar{r}^2 = t^2 + u^2 - \Delta^2 - m^2 - m_2^2 - M_2^2 \tag{2.16}$$

$$z^2 = W^2 - \omega^2 - u^2 + m^2 + m_1^2 + m_2^2 \tag{2.17}$$

following from energy-momentum conservation.

The barred invariants are obtained from the unbarred ones by the exchange $p_1 \longleftrightarrow p_2$. The first four invariants (2.12) are 4-momentum transfers. The fifth (z^2) is the square of the total energy in the c.m. system of q_1 and q_2 .

Any observable quantity can be expressed as a function of the above invariants. It is possible, in principle, to use only the set (2.11). Some expressions are however simpler if expressed in terms of (2.12) : in these cases, invariants (2.12) will be used, by keeping in mind that at any stage of the calculation we can reduce to the set (2.11) through (2.13) to (2.17).

The derivation of the relations between physical quantities and invariants is straightforward, but rather tedious. We think it useful to report here some of these relations, and more particularly the ones which will be needed in the applications. The following symbols will be used

$$E(W; m^2, M^2) = \frac{1}{2W} (W^2 + m^2 - M^2) \quad (2.18)$$

$$P(W; m^2, M^2) = \frac{1}{2W} \left[W^4 - 2W^2(m^2 + M^2) + (m^2 - M^2)^2 \right]^{1/2} \quad (2.19)$$

E and P represent c.m. energy and c.m. 3-momentum respectively of a particle with mass m colliding with a particle with mass M at total c.m. energy W . One has

$$P(W; m^2, M^2) = P(W; M^2, m^2) \\ E(W; M^2, m^2) + E(W; m^2, M^2) = W \quad (2.20)$$

$$P^2(W; m^2, M^2) = E^2(W; m^2, M^2) - m^2 = E^2(W; M^2, m^2) - M^2.$$

Let us list the most interesting quantities in the following reference systems : a) the lab. (L) system ($\vec{p}_2 = 0$) ; b) the B-system ($\vec{p}_1 + \vec{p}_2 = 0$) ; c) the Q-system ($\vec{q} + \vec{q}_1 = 0$) ; d) the P-system ($\vec{q} + \vec{q}_2 = 0$).

L-SYSTEMKinetic energy of p_1

$$T = \frac{1}{2M_2} [W^2 - (M_1 + M_2)^2] \quad (2.21)$$

3-momentum of p_1

$$p_1^L = \frac{1}{2M_2} \left\{ [W^2 - (M_1 + M_2)^2] [W^2 - (M_1 - M_2)^2] \right\}^{1/2} \quad (2.22)$$

$$= \frac{W}{M_2} P(W; M_1^2, M_2^2)$$

Kinetic energies of q, q_1, q_2

$$T_L = \frac{1}{2M_2} [\bar{r}^2 + (M_2 - m)^2] \quad (2.23)$$

$$T_1^L = \frac{1}{2M_2} [\bar{t}^2 + (M_2 - m_1)^2] \quad (2.24)$$

$$T_2^L = \frac{1}{2M_2} [\Delta^2 + (M_2 - m_2)^2] \quad (2.25)$$

3-momenta of q, q_1, q_2 ²⁷⁾

$$q^L = \frac{\bar{r}}{M_2} P(\bar{r}; -M_2^2, -m^2); \quad q_1^L = \frac{\bar{t}}{M_2} P(\bar{t}; -M_2^2, -m_1^2); \quad q_2^L = \frac{\Delta}{M_2} P(\Delta; -M_2^2, -m_2^2) \quad (2.26)$$

Scattering angles of q, q_1, q_2 ²⁸⁾

$$\cos \theta^L = \frac{1}{2q_1^L p_1^L} \left[2T(T^L + m) + 2M_1 T^L - (m - M_1)^2 - r^2 \right] \quad (2.27)$$

$$\cos \theta_1^L = \frac{1}{2q_1^L p_1^L} \left[2T(T_1^L + m_1) + 2M_1 T_1^L - (m_1 - M_1)^2 - t^2 \right] \quad (2.28)$$

$$\cos \theta_2^L = \frac{1}{2q_2^L p_1^L} \left[2T(T_2^L + m_2) + 2M_1 T_2^L - (m_2 - M_1)^2 - \bar{\Delta}^2 \right] \quad (2.29)$$

B-SYSTEM

Total energies of p_1 and p_2

$$p_{10}^B = E(W; M_1^2, m_2^2) \quad p_{20}^B = E(W; M_2^2, M_1^2) \quad (2.30)$$

Three-momentum of p_1 and p_2

$$p_1^B = p_2^B = k^B = P(W; M_1^2, M_2^2) \quad (2.31)$$

Total energies and 3-momenta of q, q_1, q_2

$$q_0^B = E(W; m^2, z^2) \quad q^B = P(W; m^2, z^2) \quad (2.32)$$

$$q_{10}^B = E(W; m_1^2, u^2) \quad (2.33)$$

$$q_1^B = P(W; m_1^2, u^2) \quad (2.34)$$

$$q_{20}^B = E(W; m_2^2, \omega^2) \quad (2.35)$$

$$q_2^B = P(W; m_2^2, \omega^2). \quad (2.36)$$

Scattering angles of q, q_1, q_2

$$\cos \theta^B = -\frac{1}{2k^B q_2^B} [r^2 + m^2 + M_1^2 - 2q_0^B p_0^B] \quad (2.37)$$

$$\cos \theta_1^B = -\frac{1}{2k^B q_1^B} [t^2 + m_1^2 + M_1^2 - 2p_{10}^B q_{10}^B] = \frac{1}{2k^B q_1^B} [\bar{t}^2 + m_1^2 + M_2^2 - 2p_{20}^B q_{10}^B] \quad (2.38)$$

$$\cos \theta_2^B = \frac{1}{2k^B q_2^B} [\bar{\Delta}^2 + m_2^2 + M_2^2 - 2p_{20}^B q_{20}^B] = -\frac{1}{2k^B q_2^B} [\bar{\Delta}^2 + m_2^2 + M_1^2 - 2p_{10}^B q_{20}^B] \quad (2.39)$$

Angle between $\vec{q}_1, \vec{q}_2$

$$\cos \lambda^B = \frac{1}{2q_1^B q_2^B} [-W^2 + \omega^2 + u^2 - m^2 + 2q_{10}^B q_{20}^B] \quad (2.40)$$

Q-SYSTEM

Total energies and 3-momenta of p_1 and p_2 ²⁷⁾

$$p_{10}^Q = E(\omega; M_1^2, -\Delta^2) \quad p_1^Q = P(\omega; M_1^2, -\Delta^2) \quad (2.41)$$

$$p_{20}^Q = E(\omega; M_2^2, -\bar{\Delta}^2) \quad p_2^Q = P(\omega; M_2^2, -\bar{\Delta}^2) \quad (2.42)$$

Total energies and 3-momenta of q , q_1 and q_2

$$q_0^Q = E(\omega; m^2, m^2) \quad q^Q = P(\omega; m_1^2, m^2) \quad (2.43)$$

$$q_{10}^Q = E(\omega; m_1^2, m^2) \quad q_1^Q = q^Q \quad (2.44)$$

$$q_{20}^Q = \frac{W}{\omega} E(W; -m_2^2, \omega^2) \quad q_2^Q = \frac{W}{\omega} P(W; m_2^2, \omega^2) \quad (2.45)$$

Scattering angle of q_1 with respect to p_1

$$\cos \theta_1^Q = \frac{1}{2q_1^Q p_1^Q} [2q_{10}^Q p_{10}^Q - m_1^2 - M_1^2 - t^2] \quad (2.46)$$

Scattering angle of q_1 with respect to p_2

$$\cos \epsilon_1^Q = \frac{1}{2q_1^Q p_2^Q} [2q_{10}^Q p_{20}^Q - m_1^2 - M_2^2 - \bar{t}^2] \quad (2.47)$$

Angle between $\vec{p}_1$ and $\vec{p}_2$

$$\cos \alpha^Q = \frac{1}{2 p_1^Q p_2^Q} \left[2 p_{10}^Q p_{20}^Q + M_1^2 + M_2^2 - W^2 \right] . \quad (2.48)$$

In addition to these kinematical quantities, it will be useful to define here two new quantities which do not enter directly in the kinematical description of process (2.6), but will be used in the following : namely the "on-shell limits" of p_1^Q (p_2^Q) as $\Delta^2 (\bar{\Delta}^2) \rightarrow -\mu^2$. These quantities represent the c.m. 3-momenta of a particle of mass μ colliding against p_1 and p_2 respectively at total c.m. energy ω , and will be called k_1^Q, k_2^Q . They are given by

$$k_1^Q = P(\omega; M_1^2, \mu^2) \quad (2.49)$$

$$k_2^Q = P(\omega; M_2^2, \mu^2) . \quad (2.50)$$

P-SYSTEM

Total energies and 3-momenta of p_1 and p_2

$$p_{10}^P = E(u; M_1^2, -\bar{t}^2) \quad p_1^P = P(u; M_1^2, -\bar{t}^2) \quad (2.51)$$

$$p_{20}^P = E(u; M_2^2, -t^2) \quad p_2^P = P(u; M_2^2, -t^2) \quad (2.52)$$

Total energies and 3-momenta of q , q_2 and q_1

$$q_0^P = E(u; m^2, m_2^2) \quad q^P = P(u; m^2, m_2^2) \quad (2.53)$$

$$q_{20}^P = E(u; m_2^2, m^2) \quad q_2^P = q^P \quad (2.54)$$

$$q_{10}^P = \frac{W}{u} E(W; -m_1^2, u^2) \quad q_1^P = \frac{W}{u} P(W; m_1^2, u^2). \quad (2.55)$$

Scattering angle of q_2 with respect to p_2

$$\cos \theta_2^P = \frac{1}{2q_2^P p_2^P} \left[2q_{20}^P p_{20}^P - m_2^2 - M_2^2 - \Delta^2 \right] \quad (2.56)$$

Scattering angle of q_2 with respect to p_1

$$\cos \epsilon_2^P = \frac{1}{2q_2^P p_1^P} \left[2q_{20}^P p_{10}^P - m_2^2 - M_1^2 - \bar{\Delta}^2 \right] \quad (2.57)$$

Angle between $\vec{p}_1$ and $\vec{p}_2$

$$\cos \alpha^P = \frac{1}{2p_1^P p_2^P} \left[2p_{10}^P p_{20}^P + M_1^2 + M_2^2 - W^2 \right] \quad (2.58)$$

"On-shell limits" of $p_1^P (p_2^P)$ for $\bar{t}^2 (t^2) \rightarrow -\mu^2$

$$k_1^P = P(u; M_1^2, M^2) \quad (2.59)$$

$$k_2^P = P(u; M_2^2, M^2) \quad (2.60)$$

In order to conclude this list of kinematical relations, we give the dependence of any of the quantities defined above on the invariants of the fundamental set in Table I. This can be of practical use in the calculations, when integration over certain invariants has to be carried out for fixed values of the other invariants.

TABLE I

Quantities	Dependence on	Quantities	Dependence on
$T, p_1^L, p_{10}^B, p_{20}^B, k^B$	W^2	$\cos \theta_1^L, \cos \theta_1^B, p_{10}^P, p_1^P$	
$q_{10}^B, q_1^B, q_{10}^P, q_1^P$	W^2, u^2	$\cos \alpha^P, T_1^L, q_1^L$	W^2, u^2, t^2
$q_{20}^P, q_0^P, q_1^P, k_1^P, k_2^P$	u^2	$q_0^B, q_1^B, \cos \lambda^B$	W^2, ω^2, u^2
$q_{20}^B, q_2^B, q_{20}^Q, q_2^Q$	W^2, ω^2	$\cos \theta_1^Q$	ω^2, Δ^2, t^2
$q_{10}^Q, q_0^Q, q_1^Q, k_1^Q, k_2^Q$	ω^2	$\cos \theta_2^P, T_1^L, q_1^L$	u^2, t^2, Δ^2
T_2^L, q_2^L	Δ^2	$\cos \theta_1^L, \cos \theta_1^B, \cos \epsilon_1^Q, \cos \epsilon_2^P$	$W^2, \omega^2, u^2, \Delta^2, t^2$
p_{10}^Q, p_1^Q	ω^2, Δ^2		
$\cos \theta_2^L, \cos \theta_2^B, p_{20}^Q, p_2^Q, \cos \alpha^Q$	W^2, ω^2, Δ^2		

2.4 CALCULATION OF CROSS-SECTIONS

In this sub-section we shall specify the peripheral S-matrix element for the process described in Fig. 1, and derive from it those differential cross-sections which can be of physical interest. The phase space considerations which give the limits of the integrals are contained in Appendix 1. We will always consider the invariant W^2 as a fixed quantity, which is determined uniquely through (2.21) by the lab. kinetic energy T of the incident particle²⁹⁾

$$W^2 = 2M_2 T + (M_1 + M_2)^2, \quad (2.61)$$

By combining (2.8) and (2.9), and using the notation introduced in sub-section 2.2, we get

$$S_{fi} = i (2\pi)^{7/2} \lambda \delta^4(p_f - p_i) V(\Delta^2) \Delta'_F(\Delta^2) (2q_0 q_{10} p_{10})^{-1/2} m_1(\omega, t^2; \Delta^2). \quad (2.62)$$

If we restrict ourselves to the practical case of a target particle of spin $\frac{1}{2}$ and consider the exchange of a single spinless boson of mass μ , we have

$$V(\Delta^2) = i (2\pi)^{-3} K(\Delta^2) G_r \bar{u}(q_2) O_u(p_2) \left(\frac{m_2 M_2}{p_{20} q_{20}} \right)^{1/2} \quad (2.63)$$

$$\Delta'_F(\Delta^2) = - \frac{i}{(2\pi)^4} \frac{K'(\Delta^2)}{(\Delta^2 + \mu^2)} \quad (2.64)$$

G_r is the rationalized and renormalized coupling constant of the vertex $(q_2 p_2 k)$, O is an operator which is equal to γ_5 or 1 according to the parity of the boson, $K(\Delta^2)$ is the form factor of the proper vertex part,

$K'(\Delta^2)$ is the form factor of the propagator (i.e., the ratio of the complete propagator to the perturbative propagator). These functions have been so defined as to satisfy

$$K(-\mu^2) = K'(-\mu^2) = 1, \quad (2.65)$$

We may remark that the complete form factor of the vertex is given by $F(\Delta^2) = K(\Delta^2)K'(\Delta^2)$. When (2.63) and (2.64) are put into (2.62), one gets

$$S_{fi} = i(2\pi)^{-\frac{7}{2}} \frac{\lambda}{\sqrt{2}} G_r \frac{K(\Delta^2)K'(\Delta^2)}{\Delta^2 + \mu^2} \left(\frac{m_2 M_2}{q_0 q_{10} q_{20} p_{10} p_{20}} \right)^{\frac{1}{2}} \delta^4(p_f - p_i) [\bar{u}(q_2) O u(p_2)] M_1(\omega, t^2; \Delta^2). \quad (2.66)$$

The expression for the cross-section is obtained by using (2.62), (2.1) and (2.3) (F is given by (2.4))

$$d\sigma = (2\pi)^9 \frac{\lambda^2}{2} \frac{q_{20} p_{20}}{F} |\Delta'_F(\Delta^2)|^2 |V(\Delta^2)|^2 |M_1(\omega, t^2; \Delta^2)|^2 \cdot \delta^4(p_f - p_i) \frac{d^3 q_1 d^3 q_2}{q_0 q_{10} q_{20}} \quad (2.67)$$

where summations over final spins and averages over initial spins have to be performed whenever necessary.

By carrying out the spin summation and average over p_2 and q_2 , one gets

$$d\sigma = (2\pi)^{-4} \frac{\lambda^2 G^2}{4F} \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2} K^2(\Delta^2) K'^2(\Delta^2) |M_1(\omega, t^2; \Delta^2)|^2 \cdot \delta^4(p_f - p_i) \frac{d^3 q_1 d^3 q_2}{q_0 q_{10} q_{20}} \quad (2.68)$$

where $G^2 = G_r^2 / 4\pi$ and the plus (minus) sign has to be chosen if the coupling in the lower vertex is scalar (pseudoscalar). Spin summations and averages of $|\mathcal{M}_1(\omega^2, t^2; \Delta^2)|^2$ have not been explicitly written.

Before handling the general formulae (2.67) and (2.68), it is convenient to start here a brief discussion of the structure of the matrix element $\mathcal{M}_1(\omega^2, t^2; \Delta^2)$ relevant to reaction (2.7). Consider first the case of the incident particle k being real ($\Delta^2 = -\mu^2$). One can perform a partial wave expansion of $\mathcal{M}_1$ in the following way :

$$\mathcal{M}_1(\omega^2, t^2) = 8\pi\omega \sum_{\ell=0}^{\infty} (2\ell+1) f_{\ell}(\omega) P_{\ell}(\cos\theta) \quad (2.69)$$

for the interaction of two spinless particles (briefly in the following : boson-boson interaction), and

$$\mathcal{M}_1(\omega^2, t^2) = \frac{4\pi\omega}{\sqrt{m_1 m_2}} \chi_f^+ \left[f_1 + \frac{(\vec{\sigma}\vec{q})(\vec{\sigma}\vec{k})}{qk} f_2 \right] \chi_i \quad (2.70)$$

or

$$\mathcal{M}_1(\omega^2, t^2) = \frac{4\pi\omega}{\sqrt{m_1 m_2}} \chi_f^+ \left[f_1 \frac{(\vec{\sigma}\vec{k})}{k} + f_2 \frac{(\vec{\sigma}\vec{q})}{q} \right] \chi_i \quad (2.71)$$

for the case of the interaction of a spinless boson with a spin $\frac{1}{2}$ fermion (briefly in the following : boson-fermion interactions). (2.70) applies when the intrinsic parity of the two final particles is equal to the intrinsic parity of the two initial particles. (2.71) applies to the opposite case. In both cases, f_1 and f_2 are given by

$$f_1 = \sum_{\ell=0}^{\infty} f_{\ell+}(\omega) P'_{\ell+1}(\cos\theta) - \sum_{\ell=2}^{\infty} f_{\ell-}(\omega) P'_{\ell-1}(\cos\theta) \quad (2.72)$$

$$f_2 = \sum_{\ell=1}^{\infty} [f_{\ell-}(\omega) - f_{\ell+}(\omega)] P'_{\ell}(\cos\theta) \quad (2.73)$$

ℓ is the orbital angular momentum of the final state (in (2.69) and (2.70) it is also the orbital angular momentum of the initial state)³⁰⁾. Subscripts $+$ and $-$ in (2.72) and (2.73) refer to total angular momentum $\ell \pm \frac{1}{2}$. χ_i and χ_f are two-dimensional Pauli spinors, relevant to the initial and final fermions, and σ_i ($i = 1, 2, 3$) are 2×2 Pauli spin matrices. All the kinematical quantities are evaluated in the c.m. system: $\vec{q}, \vec{k}$ are c.m. 3-momenta, θ is the c.m. angle of scattering, ω is the total c.m. energy. It can easily be seen that the expressions of q and k as functions of ω are

$$q = P(\omega; m_1^2, m^2) \quad k = P(\omega; M_1^2, M^2) \quad (2.74)$$

It is to be noted that these two quantities have the same formal expression as q^Q, k_1^Q given by (2.43) and (2.49). $f_{\ell}(\omega)$, $f_{\ell\pm}(\omega)$ are called "partial wave amplitudes"; they depend only on the c.m. energy ω , since the t^2 -dependence of M_1 is entirely contained in the angle of scattering. For the case of elastic scattering these amplitudes obey the unitarity condition

$$|m f_j| = |f_j|^2 \quad (2.75)$$

(j stands for $\ell, \ell+$ or $\ell-$). By starting from the following formula, obtained directly by (2.5)

$$\frac{d\sigma}{d\Omega}(\omega^2, t^2) = \frac{\lambda^2}{4\omega^2} (2\pi)^{-2} \frac{q}{k} |m_1(\omega^2, t^2)|^2 \quad (2.76)$$

one obtains

$$\frac{k}{q} \frac{d\sigma}{d\Omega} = \left| \sum_{\ell=0}^{\infty} (2\ell+1) f_{\ell}(\omega) P_{\ell}(\cos \theta) \right|^2 \quad (2.77)$$

for boson-boson interaction, and

$$\begin{aligned} \frac{k}{q} \frac{d\sigma}{d\Omega} = & \left| \sum_{\ell=0}^{\infty} [(\ell+1) f_{\ell}^{+}(\omega) + \ell f_{\ell}^{-}(\omega)] P_{\ell}(\cos \theta) \right|^2 + \\ & + \left| \sum_{\ell=1}^{\infty} [f_{\ell-}(\omega) - f_{\ell+}(\omega)] \sin \theta P'_{\ell}(\cos \theta) \right|^2 \end{aligned} \quad (2.78)$$

for boson-fermion interaction.

After integration over the angles, one gets the total cross-section

$$\sigma = \frac{4\pi q}{k} \sum_{\ell=0}^{\infty} (2\ell+1) |f_{\ell}(\omega)|^2 \quad (2.79)$$

(boson-boson interaction)

$$\sigma = \frac{4\pi q}{k} \sum_{\ell=0}^{\infty} \left[(\ell+1) |f_{\ell+}(\omega)|^2 + \ell |f_{\ell-}(\omega)|^2 \right] \quad (2.80)$$

(boson-fermion interaction).

Now we want to extend the above formulation to the case of k being not real, i.e., $k^2 = \Delta^2 > -\mu^2$. One can perform the partial wave expansion of $m_1(\omega^2, t^2; \Delta^2)$ in this case too. The Δ^2 -dependence of the right-hand side of (2.69), (2.70) and (2.71) will be introduced as follows :

- 1) through the kinematical quantities k and $\cos \Theta$, for which explicit expression can be written. In particular k (which will be called k_Δ to recall its Δ^2 -dependence) will be given by

$$k_\Delta = p(\omega; M_1^2, -\Delta^2). \quad (2.81)$$

As it happened for the quantities (2.74), also k_Δ has the same formal expression as p_1^Q given by (2.41). In fact, the Q -system for reaction (2.6) is completely equivalent to the c.m. system for reaction (2.7) when k is virtual. Similarly the scattering angle will be expressed by (2.46) giving $\cos \Theta_1^Q$.

- 2) through the partial wave amplitudes which are, in principle, dependent on Δ^2 .

While the Δ^2 -dependence of the kinematical quantities can be easily determined (and turns out to be very strong for $\cos \Theta$, also for Δ^2 near to $-\mu^2$), the Δ^2 -dependence of the partial wave amplitudes is a priori completely unknown. However, there are no particular reasons why such a dependence should be strong: one can reasonably think that for "small" values of Δ^2 (i.e., of the order of μ^2) the off-shell partial-wave amplitudes do not differ appreciably from the on-shell amplitudes, obtained from experiment, and can be replaced by them. This approximation is the one used in many calculations performed so far: it is not equivalent to a mere replacement $\Delta^2 = -\mu^2$ in the matrix element $M_1(\omega^2, t^2; \Delta^2)$, since the Δ^2 -dependent expression (2.46) for the angle of scattering is retained. The meaning of this approximation (which will be called "pole approximation") has been discussed more in detail in reference ¹⁶). In the same paper the analytic continuation of the $T = J = 3/2$ partial wave amplitude in pion-nucleon scattering to unphysical values of the mass of the incoming pion has also been determined.

Let us now define an "off-shell differential cross-section" $\frac{d\sigma}{d\Omega}(\omega^2, \cos\theta; \Delta^2)$, by a straightforward generalization of (2.76)

$$\frac{d\sigma}{d\Omega}(\omega^2, \cos\theta; \Delta^2) = \frac{\lambda^2}{4\omega^2} (2\pi)^{-2} \frac{q}{k_\Delta} |m_1(\omega^2, \cos\theta; \Delta^2)|^2. \quad (2.82)$$

The variable $\cos\theta$ has been introduced at the place of t^2 for practical convenience.

In terms of partial waves, this cross-section reads

$$\frac{k_\Delta}{q} \frac{d\sigma}{d\Omega}(\omega^2, \cos\theta; \Delta^2) = \left| \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(\omega, \Delta^2) P_\ell(\cos\theta) \right|^2 \quad (2.83)$$

(boson-boson interaction)

$$\begin{aligned} \frac{k_\Delta}{q} \frac{d\sigma}{d\Omega}(\omega^2, \cos\theta; \Delta^2) = & \left| \sum_{\ell=0}^{\infty} \left[(\ell+1) f_{\ell+}(\omega, \Delta^2) + \ell f_{\ell-}(\omega, \Delta^2) \right] P_\ell(\cos\theta) \right|^2 + \\ & + \left| \sum_{\ell=1}^{\infty} \left[f_{\ell-}(\omega, \Delta^2) - f_{\ell+}(\omega, \Delta^2) \right] \sin\theta P'_\ell(\cos\theta) \right|^2 \end{aligned} \quad (2.84)$$

(boson-fermion interaction).

In the same way one can define the off-shell total cross-section

$$\sigma(\omega, \Delta^2) = \frac{\lambda^2}{8\pi\omega^2} \frac{q}{k_\Delta} \int_{-1}^1 |m_1(\omega^2, \cos\theta; \Delta^2)|^2 d\cos\theta \quad (2.85)$$

which gives for the boson-boson and boson-fermion interaction respectively

$$\frac{k_\Delta}{q} \sigma(\omega, \Delta^2) = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1) |f_\ell(\omega, \Delta^2)|^2 \quad (2.86)$$

$$\frac{k_{\Delta}}{q} \sigma(\omega, \Delta^2) = 4\pi \sum_{p=0}^{\infty} \left[(p+1) |f_{p+}(\omega, \Delta^2)|^2 + p |f_{p-}(\omega, \Delta^2)|^2 \right]. \quad (2.87)$$

It is interesting to note that in the pole approximation the r.h.s. of (2.86) and (2.87) reduce to Δ^2 -independent quantities, which can be deduced from the analogous on-shell expressions (2.79) and (2.80) and thus from experimental data.

At this point we can handle expression (2.68) which covers the most part of the cases of practical interest for physical applications³¹⁾. We introduce the off-shell cross-section through (2.82) and recall that all kinematical quantities $(k, k_{\Delta}, q, \cos \theta)$ in the present case are to be evaluated in the Q -system, according to the correspondence rule already stated above ($q \rightarrow q^Q, k \rightarrow k_1^Q, k_{\Delta} \rightarrow p_1^Q, \cos \theta \rightarrow \cos \theta_1^Q$). We then get³²⁾

$$d\sigma = (2\pi)^{-2} \frac{G^2}{F} \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2} K^2(\Delta^2) K'^2(\Delta^2) \omega^2 \left[\frac{p_1^Q}{q^Q} \frac{d\sigma_1}{d\Omega}(\omega^2, \cos \theta_1^Q; \Delta^2) \right] \cdot \delta^4(p_f - p_i) \frac{d^3 q_1 d^3 q_2}{q_0 q_{10} q_{20}}. \quad (2.88)$$

This formula is rigorous, but cannot be applied unless the Δ^2 -behaviour of $K \cdot K'$, and of $\frac{d\sigma_1}{d\Omega}$ (i.e. of the transition amplitudes) is known. The approximation valid for small Δ^2 (pole approximation) consists in putting $\Delta^2 = -\mu^2$ in K, K' ³³⁾ and the quantity in square brackets, which for $\Delta^2 \sim \mu^2$ (see (2.83) to (2.87)) is Δ^2 -dependent practically only through the angle. Therefore (2.88) in the pole approximation reduces to

$$d\sigma = (2\pi)^{-2} \frac{G^2}{F} \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2} \omega^2 \frac{k_1^Q}{q^Q} \frac{d\sigma_1}{d\Omega}(\omega^2, \cos \theta_1^Q) \cdot \delta^4(p_f - p_i) \frac{d^3 q_1 d^3 q_2}{q_0 q_{10} q_{20}} \quad (2.89)$$

The advantage of (2.89) is that everything in it is known and can be calculated. When the information required for the use of (2.88) is not available, one can try to use (2.89) also in the region of higher Δ^2 , even though the result cannot be correct a priori ; in fact the comparison of the predictions of (2.89) with experiment can be used for getting some insight into the importance of the peripheral mechanism at high momentum transfers, and on the behaviour of the unknown functions entering in (2.88) ¹⁷⁾. However, we point out that, at least for some processes involving the scattering of a virtual pion against a nucleon in the region of the $\bar{3}\bar{3}$ -resonance, the use of the rigorous formula (2.88) is possible ¹⁶⁾.

In the following for the various distributions predicted, we will always give both the rigorous formula (derived from (2.88)) and the formula obtained in the pole approximation (derived from (2.89)).

In order to go further, one has to perform a certain number of integrations and reduce to the cases of interest, which involve one or at most two variables. Since quantities like $\frac{d^3q}{q_0}$ are invariant, one can evaluate each of them in the reference system that is the most convenient for practical use. If we want to write explicitly these quantities for q_1 and q_2 , e.g., in the B-system, we get

$$\frac{d^3q_1}{q_{10}} = q_1^B dq_{10}^B d\cos\theta_1^B d\varphi_1^B; \quad \frac{d^3q_2}{q_{20}} = q_2^B dq_{20}^B d\cos\theta_2^B d\varphi_2^B \quad (2.90)$$

(φ_1^B, φ_2^B are azimuthal angles), and similarly in the other systems. Also the following formulae will be useful :

$$q_2^B dq_{20}^B d\cos\theta_2^B = \frac{d\omega^2 d\Delta^2}{4F} \quad (2.91)$$

$$q_1^B dq_{10}^B d\cos\theta_1^B = \frac{du^2 dt^2}{4F} \quad (2.92)$$

where F has been defined in (2.4).

We are mainly interested in the total cross-section, the energy distribution of the particles q_1 and q_2 ³⁴⁾ in the lab. and in the c.m. system, the angular distribution of the same particles in the c.m. system and the energy distribution in the lab. at a fixed angle. We shall consider the cases of particle q_2 and q_1 separately.

PARTICLE q_2

For the distributions relevant to particle q_2 a straightforward procedure can be found, which further reduces the number of integrations needed.

From the inspection of the dependence of all the physical quantities concerning q_2 on the invariants (either in the L or in the B-System : see Table I), we see that all the information we need is contained in the partial cross-section $\frac{d^2\sigma}{d\omega^2 d\Delta^2}$. We have then to integrate over u^2 and t^2 (or, in other words, over particle q_1). This can be achieved in a simple way, directly from (2.88) (or (2.89)). One first writes the invariant $\frac{d^3q_2}{q_{20}}$ by using (2.90) and (2.91); the integration on the azimuthal angle $d\varphi_2$ simply gives 2π for unpolarized initial particles. The other two differential invariants are written in the Q-system. After integration over the δ -function, one obtains

$$\int \frac{d^3q d^3q_1}{q_0 q_{10}} \delta^4(p_f - p_i) \rightarrow \int \frac{q^Q}{\omega} d\omega d\theta_1^Q d\varphi_1^Q \quad (2.93)$$

where φ_1^Q is the azimuthal angle of q_1 in the Q-system. Since in the Q-system to fix Δ^2 and ω^2 is equivalent to fixing the 3-vectors $\vec{p}_1, \vec{p}_2$ and $\vec{q}_2$ in their plane (see equations from (2.41) to (2.48)) but does not give restriction at all on the orientation of $\vec{q}^Q$ in the space, the variation of u^2 and t^2 within their limits (at fixed ω^2 and Δ^2) will be equivalent to the

variation of the angles θ_2^Q, φ_1^Q covering the whole solid angle. The integration over particle q_1 is then reduced to a very simple angular integration. In fact, $\cos \theta_1^Q$ appears only in the differential cross-section $\frac{d^2\sigma}{d\omega^2 d\Delta^2}$ and φ_1^Q does not appear at all in the integral. Thus we get in the general case

$$\frac{d^2\sigma}{d\omega^2 d\Delta^2} = \frac{1}{8\pi} \frac{G^2}{F^2} K^2(\Delta^2) K^{1/2}(\Delta^2) p_1^Q \omega \sigma_1(\omega, \Delta^2) \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2}. \quad (2.94)$$

The total off-shell cross-section for process (2.7), $\sigma_1(\omega, \Delta^2)$, is given, as a function of the off-shell transition amplitudes, by (2.86) or (2.87) written in the Q-system.

In the pole approximation, we get

$$\frac{d^2\sigma}{d\Delta^2 d\omega^2} = \frac{1}{8\pi} \frac{G^2}{F^2} k_1^Q \omega \sigma_1(\omega) \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2}. \quad (2.95)$$

Of course, $\sigma_1(\omega)$ is the physical total cross-section for reaction (2.7), which can be taken from experiment.

Now let us specify the formulae of interest.

TOTAL CROSS-SECTION

It is given by ³⁵⁾

$$\sigma_{\text{tot}} = \int_{(m_1+m)^2}^{(W-m_2)^2} d\omega^2 \int_{\Delta_{\min}^2(\omega)}^{\Delta_{\max}^2(\omega)} \frac{d^2\sigma}{d\omega^2 d\Delta^2} \quad (2.96)$$

where

$$\Delta_{\max}^2(\omega) = 2q_{20}^B p_{20}^B - m_2^2 - M_2^2 + 2q_2^B k^B \quad (2.97)$$

$$\Delta_{\min}^2(\omega) = 2q_{20}^B p_{20}^B - m_2^2 - M_2^2 - 2q_2^B k^B \quad (2.98)$$

(for the derivation, see Appendix I). In the pole approximation (2.95), the integration over Δ^2 can be carried out analytically and gives

$$G_{\text{rot}} = \frac{1}{8\pi} \frac{G^2}{F^2} \int_{(m_1+m)^2}^{(\omega-m_2)^2} d\omega^2 k_1^Q \omega \delta_1(\omega) \left\{ \log \frac{\Delta_{\max}^2(\omega) + \mu^2}{\Delta_{\min}^2(\omega) + \mu^2} - \right. \\ \left. - \frac{[\mu^2 - (M_2 \pm m_2)^2] [\Delta_{\max}^2(\omega) - \Delta_{\min}^2(\omega)]}{(\Delta_{\max}^2(\omega) + \mu^2)(\Delta_{\min}^2(\omega) + \mu^2)} \right\} \quad (2.99)$$

ENERGY DISTRIBUTION IN THE LAB.

Because of relation (2.25), it is given by

$$\frac{d\delta}{dT_2^L} = 2M_2 \frac{d\delta}{d\Delta^2} = \frac{M_2}{4\pi} \frac{G^2}{F^2} K^2(\Delta^2) K'^2(\Delta^2) \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2} \cdot \\ \cdot \int_{(m_1+m)^2}^{\omega_{\max}^2(\Delta^2)} d\omega^2 p_1^Q \omega \delta_1(\omega, \Delta^2) \quad (2.100)$$

in the general case, and by

$$\frac{d\delta}{dT_2^L} = \frac{M_2}{4\pi} \frac{G^2}{F^2} \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2} \int_{(m_1+m)^2}^{\omega_{\max}^2(\Delta^2)} d\omega^2 k_1^Q \omega \delta_1(\omega) \quad (2.101)$$

in the pole approximation. Here Δ^2 has to be expressed as a function of T_2^L according to (2.25), and

$$\omega_{\max}^2(\Delta^2) = W^2 + m_2^2 - \frac{W p_{20}^B}{M_2^2} (\Delta^2 + M_2^2 + m_2^2) + \frac{p^B W}{M_2^2} \left\{ [\Delta^2 + (m_2 - M_2)^2] [\Delta^2 + (m_2 + M_2)^2] \right\}^{1/2} \quad (2.102)$$

(see Appendix I for the derivation).

ENERGY LAB. DISTRIBUTION AT FIXED ANGLE

$$\begin{aligned} \frac{d^2 \mathcal{G}}{dT_2^L d \cos \theta_2^L} &= 4 M_2 q_2^L p_1^L \frac{d^2 \mathcal{G}}{d\omega^2 d\Delta^2} = \\ &= \frac{q_2^L}{2\pi} \frac{G^2}{F} K^2(\Delta^2) K'^2(\Delta^2) \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2} p_1^Q \omega \mathcal{G}_1(\omega, \Delta^2) \end{aligned} \quad (2.103)$$

in the general case, and

$$\frac{d^2 \mathcal{G}}{dT_2^L d \cos \theta_2^L} = \frac{q_2^L}{2\pi} \frac{G^2}{F} \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2} k_1^Q \omega \mathcal{G}_1(\omega) \quad (2.104)$$

in the pole approximation.

In the above formulae relation (2.4) has been used. Δ^2 has to be expressed as a function of T_2^L through (2.23), and ω^2 is given by

$$\omega^2 = 2 q_2^L p_1^L \cos \theta_2^L + (M_1 + M_2 - m_2)^2 + 2T(M_2 - m_2) - 2T_2^L(T + M_1 + M_2) \quad (2.105)$$

which follows from (2.25) and (2.29).

ANGULAR DISTRIBUTION IN THE C.M. SYSTEM

$$\frac{d\sigma}{d\omega\theta_2^B} = \int_{(m_1+m)^2}^{(W-m_2)^2} d\omega^2 \, 2k_2^B \frac{d^2\sigma}{d\omega^2 d\Delta^2} = \frac{1}{4\pi} \frac{G^2}{k^B W^2} \int_{(m_1+m)^2}^{(W-m_2)^2} d\omega^2 \, p_1^Q \omega q_2^B \cdot$$

$$\cdot K^2(\Delta^2) K'^2(\Delta^2) \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2} \sigma_1(\omega, \Delta^2) \quad (2.106)$$

in the general case, and

$$\frac{d\sigma}{d\omega\theta_2^B} = \frac{1}{4\pi} \frac{G^2}{k^B W^2} \int_{(m_1+m)^2}^{(W-m_2)^2} d\omega^2 \, k_1^Q \omega q_2^B \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2} \sigma_1(\omega) \quad (2.107)$$

in the pole approximation.

In the above formulae Δ^2 has to be expressed as a function of ω^2 and $\cos \theta_2^B$ according to (2.39).

ENERGY DISTRIBUTION IN THE C.M. SYSTEM

$$\frac{d\sigma}{dq_{20}^B} = \frac{d\sigma}{dT_2^B} = 2W \frac{d\sigma}{d\omega^2} = \frac{1}{4\pi} \frac{G^2 \omega}{k^{B^2} W} \int_{\Delta_{\min}^2(\omega)}^{\Delta_{\max}^2(\omega)} d\Delta^2 \, K^2(\Delta^2) K'^2(\Delta^2) \cdot$$

$$\cdot \frac{\Delta^2 + (m_2 \pm M_2)^2}{(\Delta^2 + \mu^2)^2} p_1^Q \sigma_1(\omega, \Delta^2) \quad (2.108)$$

in the general case, and

$$\frac{d\sigma}{dT_2^B} = \frac{1}{4\pi} \frac{G^2}{k^{B^2} W} k_1^Q \omega \sigma_1(\omega) \left\{ \log \frac{\Delta_{\max}^2(\omega) + \mu^2}{\Delta_{\min}^2(\omega) + \mu^2} - \right.$$

$$\left. - \frac{[\mu^2 - (M_2 \pm m_2)^2] [\Delta_{\max}^2(\omega) - \Delta_{\min}^2(\omega)]}{[\Delta_{\max}^2(\omega) + \mu^2] [\Delta_{\min}^2(\omega) + \mu^2]} \right\} \quad (2.109)$$

in the pole approximation.

The limits $\Delta_{\min}^2(\omega)$, $\Delta_{\max}^2(\omega)$ are given by (2.97) and (2.98) :
 ω^2 has to be expressed as a function of q_{20}^B (or $T_2^B = q_{20}^B - m_2$) according to (2.35).

PARTICLE q_1

For particle q_1 the situation is more complicated than for particle q_2 . It can again be stated that all information needed in the lab. and c.m. system is contained in the partial cross-section $\frac{d^2\sigma}{du^2 dt^2}$: but this cross-section cannot be expressed by a simple formula free from integrations, like (2.94). We will first give the formulae linking this partial cross-section to the quantities of physical interest, and then outline a procedure for calculating it. Since the dependence of the integrand on the variables of integration is very complicated, all the integrations appearing in the following formulae will have to be performed numerically, by means of an electronic computer³⁶⁾.

ENERGY DISTRIBUTION IN THE LAB.

$$\frac{d\sigma}{dT_1^L} = 2M_2 \frac{d\sigma}{d\bar{t}^2} = 2M_2 \int_{(m_2+m)^2}^{u_{\max}^2(\bar{t}^2)} \frac{d^2\sigma}{du^2 dt^2} du^2 \quad (2.110)$$

Here $\bar{t}^2$ is related to the kinetic energy T_1^L through (2.24), and

$$u_{\max}^2(\bar{t}^2) = W^2 + m_1^2 - \frac{W p_{20}^B}{M_2^2} (\bar{t}^2 + m_1^2 + M_2^2) + \frac{k^B W}{M_2^2} \left\{ [\bar{t}^2 + (m_1 - M_2)^2] [\bar{t}^2 + (m_1 + M_2)^2] \right\}^{1/2} \quad (2.111)$$

In the cross-section $\frac{d^2\sigma}{du^2 dt^2}$, t^2 has to be considered everywhere as a function of u^2 and T_1^L given by

$$t^2 = W^2 - M_1^2 - 2M_2 m_1 - u^2 - 2M_2 T_1^L \quad (2.112)$$

which follows from (2.14) and (2.24).

ENERGY LAB. DISTRIBUTION AT FIXED ANGLE

$$\frac{d^2\sigma}{dT_1^L d\cos\theta_1^L} = 4M_2 q_1^L p_1^L \frac{d^2\sigma}{du^2 d\bar{t}^2} = 4M_2 q_1^L p_1^L \frac{d^2\sigma}{du^2 dt^2} \quad (2.113)$$

with the substitutions $\bar{t}^2 \rightarrow T_1^L$ given by (2.24), and

$$t^2 = 2T(T_1^L + m_1) + 2M_1 T_1^L - (m_1 - M_1)^2 - 2q_1^L p_1^L \cos\theta_1^L \quad (2.114)$$

$$u^2 = (M_1 + M_2 - m_1)^2 - 2T_1^L (M_1 + M_2 + T) + 2T(M_2 - m_1) + 2q_1^L p_1^L \cos\theta_1^L$$

which follow from (2.28) and (2.112).

ANGULAR C.M. DISTRIBUTION

$$\frac{d\sigma}{d\cos\theta_1^B} = 2k^B \int_{(m_2+m_1)^2}^{(W-m_1)^2} q_1^B \frac{d^2\sigma}{du^2 dt^2} du^2. \quad (2.115)$$

Under the integral, t^2 has to be expressed as a function of $\cos\theta_1^B$ and u^2 through (2.38).

ENERGY C.M. DISTRIBUTION

$$\frac{d\phi}{dq_{10}^B} = \frac{d\phi}{dT_1^B} = 2W \frac{d\phi}{du^2} = 2W \int_{t_{\min}^2(u)}^{t_{\max}^2(u)} \frac{d^2\phi}{du^2 dt^2} dt^2 \quad (2.116)$$

with

$$t_{\max}^2(u) = 2q_{10}^B p_{10}^B - m_1^2 - M_1^2 + 2q_1^B k^B \quad (2.117)$$

$$t_{\min}^2(u) = 2q_{10}^B p_{10}^B - m_1^2 - M_1^2 - 2q_1^B k^B \quad (2.118)$$

u^2 has to be expressed as a function of q_{10}^B (or $T_1^B = q_{10}^B - m_1$) through (2.33).

Now we will show a method for integrating the general formulae (2.88) or (2.89) over Δ^2 and ω^2 (i.e., over particles q and q_2) in order to get the quantity $\frac{d^2\phi}{du^2 dt^2}$, used in the preceding computations. One can follow the procedure outlined for particle q_2 . First, the differential invariant $\frac{d^3 q_1}{q_{10}}$ is calculated by means of (2.90) and (2.92): the integration over the azimuthal angle $d\varphi_1^B$ can be performed and gives 2π . Then one writes the other two differential invariants in the P-system (even though there are in the formula no cross-sections evaluated in this system, but the cross-section entering in the integrand contains variables characteristic of the Q-system). After integration over the δ -function, we get

$$\int \frac{d^3 q d^3 q_2}{q_0 q_{20}} \delta^4(p_f - p_i) \rightarrow \frac{q^P}{u} \int d\varphi_2^P d\varphi_2^P \quad (2.119)$$

where θ_2^P and φ_2^P are the polar and azimuthal angles of q_2 in the P-system.

For the argument explained in the previous discussion, at fixed u^2 and t^2 , θ_2^P and φ_2^P can vary without restriction between $(-1, 1)$ and $(0, 2\pi)$ respectively. Now we have to find the link between these integration variables and the quantities ω^2 and Δ^2 on which essentially the integrand depends.

Δ^2 can be expressed as a function of u^2, t^2 and $\cos \theta_2^P$ through (2.56). In order to obtain ω^2 , the procedure is more complicated. First one connects the angle ϵ_2^P [see (2.57)] to θ_2^P, φ_2^P and α^P (the latter is a function of u^2 and t^2 only: see (2.58)) through the trigonometric formula

$$\cos \epsilon_2^P = \cos \alpha^P \cos \theta_2^P + \sin \alpha^P \sin \theta_2^P \cos \varphi_2^P. \quad (2.120)$$

Next, one obtains $\bar{\Delta}^2$ as a function of $\cos \epsilon_2^P, u^2$ and t^2 through (2.57). Finally, one gets ω^2 as a function of $\Delta^2, \bar{\Delta}^2$ through (2.13). Therefore, ω^2 turns out to be a function of all the four variables of integration ($u^2, t^2, \cos \theta_2^P, \varphi_2^P$).

From here, the other quantities (like the various energies and angles in the Q-system) can be deduced by means of the set of formulae (2.41) to (2.48).

We can then summarize the expression obtained for $\frac{d^2\sigma}{du^2 dt^2}$ (37)

$$\frac{d^2\sigma}{du^2 dt^2} = \frac{1}{8\pi} \frac{G^2}{F^2} \frac{q^P}{u} \int_{-1}^1 d\cos \theta_2^P K(\Delta^2) K'(\Delta^2) \frac{\Delta^2 + (M_2 \pm m_2)^2}{(\Delta^2 + \mu^2)^2} \cdot \int_0^{2\pi} d\varphi_2^P \omega^2 \frac{P_1^Q}{q^Q} \frac{d\delta_1}{d\Omega} (\omega^2, \cos \theta_1^Q; \Delta^2) \quad (2.121)$$

in the general case, and

$$\frac{d^2\sigma}{du^2 dt^2} = \frac{1}{8\pi} \frac{G^2}{F^2} \frac{q^2}{u} \int_{-1}^1 d\cos\theta_2^P \frac{\Delta^2 + (M_2 + m_2)^2}{(\Delta^2 + \mu^2)^2} \cdot \int_0^{2\pi} d\varphi_2^P \omega^2 \frac{k_1^Q}{qQ} \frac{d\zeta_1}{d\Omega} (\omega^2, \cos\theta_1^Q) \quad (2.122)$$

in the pole approximation.

In the above formulae all the quantities under the integral sign have to be expressed as functions of u^2 , t^2 and of the variables of integration according to the prescriptions given. It is apparent that the complexity of the substitutions requires the calculation to be performed numerically. In any case, the presence of fixed limits of integrations seems to be convenient for practical applications ³⁸⁾.

2.5 OCCURRENCE OF DIFFERENT PERIPHERAL GRAPHS FOR THE SAME PROCESS

All the formulae derived in the preceding sub-section refer strictly to the graph described in Fig. 1 [hereafter referred to as graph '1'], where p_2 is considered as the target particle and the two particles q, q_1 emerge together from the upper vertex, where the projectile-particle p_1 enters. It is easy to realize, however, that in many processes it is possible for two particles to emerge from the vertex where the target particle enters. Such a situation is shown in Fig. 2.

Fig. 2

Here $k' = p_1 - q_2$. This graph will simply be referred to as graph '2'. It is apparent that all formulae valid for this case can be deduced from the ones valid for graph '1' by interchanging p_1 and p_2 . The unbarred invariants have therefore to be substituted with the barred ones and vice versa. This implies also the exchanges $M_1 \leftrightarrow M_2$, $\theta_{1,2}^{P,Q} \leftrightarrow \epsilon_{1,2}^{P,Q}$ everywhere. On the other hand, the connection between physical quantities and invariants stays unchanged : e.g., we still have $\frac{d\mathcal{G}}{dT_2^L} = 2M_2 \frac{d\mathcal{G}}{d\Delta^2}$ and so on. We think it useful to list all the formulae referring to this case. Of course, relations (2.13), (2.14) will be always needed when one wants to express the barred invariants in terms of the unbarred ones.

GENERAL FORMULAE FOR THE CROSS-SECTION

The formulae analogous to (2.67), (2.68), (2.88) and (2.89) for the graph of Fig. 2 are :

$$d\mathcal{G} = (2\pi)^9 \frac{\lambda^2}{2F} p_{10} q_{20} \left| V(\bar{\Delta}^2) \right|^2 \left| \Delta'_F(\bar{\Delta}^2) \right|^2 \left| M_1(\omega^2, t^2; \bar{\Delta}^2) \right|^2 \cdot \delta^4(P_f - P_i) \frac{d^3 q d^3 q_1 d^3 q_2}{q_0 q_{10} q_{20}} \quad (2.123)$$

$$d\mathcal{G} = (2\pi)^{-4} \frac{\lambda^2 \bar{G}^2}{4F} \frac{\bar{\Delta}^2 + (m_2 \pm M_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} K^2(\bar{\Delta}^2) K'^2(\bar{\Delta}^2) \left| M_1(\omega^2, t^2; \bar{\Delta}^2) \right|^2 \cdot \delta^4(P_f - P_i) \frac{d^3 q d^3 q_1 d^3 q_2}{q_0 q_{10} q_{20}} \quad (2.124)$$

$$d\mathcal{G} = (2\pi)^{-2} \frac{\bar{G}^2}{F} \frac{\bar{\Delta}^2 + (m_2 \pm M_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} K^2(\bar{\Delta}^2) K'^2(\bar{\Delta}^2) \omega^2 \cdot \left[\frac{p_2^Q}{q^Q} \frac{d\mathcal{G}_1(\omega^2, \omega \epsilon_1^Q; \bar{\Delta}^2)}{d\Omega} \right] \delta^4(P_f - P_i) \frac{d^3 q d^3 q_1 d^3 q_2}{q_0 q_{10} q_{20}} \quad (2.125)$$

$$d\sigma = (2\pi)^{-2} \frac{\bar{G}^2}{F} \frac{\bar{\Delta}^2 + (m_2 + M_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} \omega^2 \frac{k_2^Q}{q^Q} \frac{d\sigma_1}{d\Omega}(\omega^2, \omega \epsilon_1^2) \delta^4(p_f - p_i) \frac{d^3 q_1 d^3 q_2}{q_0 q_{10} q_{20}} \quad (2.126)$$

In these formulae $V(\bar{\Delta}^2)$ is the function describing the vertex $(p_1 q_2 k')$, $\bar{G}^2 = \frac{\bar{G}r^2}{4\pi}$, where $\bar{G}r$ is the coupling constant and K the form factor of the same vertex (in the particular case (2.63)), σ_1 is the cross-section for the reaction

$$k' + p_2 \rightarrow q + q_1 \quad (2.127)$$

p_2^Q is just the off-shell initial 3-momentum for this reaction, and k_2^Q is its on-shell limit (given by (2.50)). From the above formulae the derivation of the distributions of interest is straightforward.

TOTAL CROSS-SECTION

It is still given by (2.96) or (2.99) with the usual exchange of barred and unbarred invariants. $\Delta_{\max}^2(\omega)$ and $\Delta_{\min}^2(\omega)$ have now to be replaced by the following two expressions :

$$\bar{\Delta}_{\max}^2 = 2q_{20}^B p_{10}^B - m_2^2 - M_1^2 + 2q_2^B k^B \quad (2.128)$$

$$\bar{\Delta}_{\min}^2 = 2q_{20}^B p_{10}^B - m_2^2 - M_1^2 - 2q_2^B k^B, \quad (2.129)$$

In fact, these expressions are the kinematical limits of $\bar{\Delta}^2$ as functions of ω . Since for the total cross-section complete integration is needed, this can be performed by assuming $\bar{\Delta}^2$ as integration variable.

PARTICLE q_2

ENERGY DISTRIBUTION IN THE LAB.

$$\frac{dG}{dT_2^L} = \frac{M_2}{4\pi} \frac{\bar{G}^2}{F^2} \int_{(m_1+m)^2}^{\omega_{\max}^2(\Delta^2)} d\omega^2 K^2(\bar{\Delta}^2) K'^2(\bar{\Delta}^2) \frac{\bar{\Delta}^2 + (m_2 \pm M_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} \cdot p_2^Q \omega G_1(\omega, \bar{\Delta}^2) \quad (2.130)$$

(general case)

$$\frac{dG}{dT_2^L} = \frac{M_2}{4\pi} \frac{\bar{G}^2}{F^2} \int_{(m_1+m)^2}^{\omega_{\max}^2(\Delta^2)} d\omega^2 \frac{\bar{\Delta}^2 + (m_2 \pm M_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} k_2^Q \omega G_1(\omega) \quad (2.131)$$

(pole approximation).

Note that the kinematical upper limit $\omega_{\max}^2(\Delta^2)$ is the same as for graph '1'. $\bar{\Delta}^2$ has to be considered as the function of Δ^2 and ω^2 given by (2.13). This substitution has to be performed also in the formulae which follow.

ENERGY LAB. DISTRIBUTION AT FIXED ANGLE

$$\frac{d^2G}{dT_2^L d\omega \theta_2^L} = \frac{q_2^L}{2\pi} \frac{\bar{G}^2}{F} K^2(\bar{\Delta}^2) K'^2(\bar{\Delta}^2) \frac{\bar{\Delta}^2 + (m_2 \pm M_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} \cdot p_2^Q \omega G_1(\omega, \bar{\Delta}^2) \quad (2.132)$$

(general case)

$$\frac{d^2G}{dT_2^L d\omega \theta_2^L} = \frac{q_2^L}{2\pi} \frac{\bar{G}^2}{F} \frac{\bar{\Delta}^2 + (m_2 \pm M_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} k_2^Q \omega G_1(\omega) \quad (2.133)$$

(pole approximation).

ANGULAR C.M. DISTRIBUTION

$$\frac{d\delta}{d\cos\theta_2^B} = \frac{1}{4\pi} \frac{\bar{G}^2}{k^B W^2} \int_{(m_1+m)^2}^{(W-m_2)^2} d\omega^2 p_2^Q \omega q_2^B k^2(\bar{\Delta}^2) k'^2(\bar{\Delta}^2) \cdot \frac{\bar{\Delta}^2 + (m_2 \pm M_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} \delta_1(\omega, \bar{\Delta}^2) \quad (2.134)$$

(general case)

$$\frac{d\delta}{d\cos\theta_2^B} = \frac{1}{4\pi} \frac{\bar{G}^2}{k^B W^2} \int_{(m_1+m)^2}^{(W-m_2)^2} d\omega^2 k_2^Q \omega q_2^B \frac{\bar{\Delta}^2 + (m_2 \pm M_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} \delta_1(\omega) \quad (2.135)$$

(pole approximation).

C.M. ENERGY DISTRIBUTION

$$\frac{d\delta}{dT_2^B} = \frac{1}{4\pi} \frac{\bar{G}^2}{k^B W} \omega \int_{\bar{\Delta}_{\min}^2(\omega)}^{\bar{\Delta}_{\max}^2(\omega)} d\bar{\Delta}^2 k^2(\bar{\Delta}^2) k'^2(\bar{\Delta}^2) \frac{\bar{\Delta}^2 + (m_2 \pm M_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} \cdot p_2^Q \delta_1(\omega, \bar{\Delta}^2) \quad (2.136)$$

(general case)

$$\frac{d\delta}{dT_2^B} = \frac{1}{4\pi} \frac{\bar{G}^2}{k^B W} k_2^Q \omega \delta_1(\omega) \left\{ \log \frac{\bar{\Delta}_{\max}^2(\omega) + \mu^2}{\bar{\Delta}_{\min}^2(\omega) + \mu^2} - \frac{[\mu^2 - (m_2 \pm M_1)^2][\bar{\Delta}_{\max}^2(\omega) - \bar{\Delta}_{\min}^2(\omega)]}{[\bar{\Delta}_{\max}^2(\omega) + \mu^2][\bar{\Delta}_{\min}^2(\omega) + \mu^2]} \right\} \quad (2.137)$$

(pole approximation).

In (2.136) and (2.137), $\bar{\Delta}_{\max}^2(\omega)$, $\bar{\Delta}_{\min}^2(\omega)$ are given by (2.128) and (2.129).

PARTICLE q_1

All formulae from (2.110) to (2.120) stay unchanged. An alternative formula for (2.116) can be obtained by making the variable substitution $t^2 \rightarrow \bar{t}^2$, and changing the integration limits correspondingly. We need only to rewrite equations (2.121) and (2.122) ³⁷⁾.

$$\frac{d^2\sigma}{du^2 dt^2} = \frac{1}{8\pi} \frac{\bar{G}^2}{F^2} \frac{q^P}{u} \int_{-1}^1 d\cos\theta_2^P \int_0^{2\pi} d\varphi_2^P K^2(\bar{\Delta}^2) K'^2(\bar{\Delta}^2) \cdot$$

$$\cdot \frac{\bar{\Delta}^2 + (m_2 \pm m_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} \omega^2 \frac{p_2^Q}{q_Q} \frac{d\sigma_1(\omega^2, \cos\epsilon_1^Q; \bar{\Delta}^2)}{d\Omega} \quad (2.138)$$

(general case)

$$\frac{d^2\sigma}{du^2 dt^2} = \frac{1}{8\pi} \frac{\bar{G}^2}{F^2} \frac{q^P}{u} \int_{-1}^1 d\cos\theta_2^P \int_0^{2\pi} d\varphi_2^P \frac{\bar{\Delta}^2 + (m_2 \pm m_1)^2}{(\bar{\Delta}^2 + \mu^2)^2} \cdot$$

$$\cdot \omega^2 \frac{k_2^Q}{q_Q} \frac{d\sigma_1(\omega^2, \cos\epsilon_1^Q)}{d\Omega} \quad (2.139)$$

(pole approximation).

Up to now, we have always supposed that the inelastic process under consideration can be represented by a single peripheral graph of the type either '1' or '2' (as it happens for the process $\pi + N \rightarrow \pi + \pi + N$, with the exchange of a single pion). For other processes (e.g., all processes initiated by nucleon-nucleon collisions) it is easy to see that both graphs of the types '1' and '2' can occur. In general, whenever in the initial or in the final state there are two particles which have the same quantum numbers (apart from the electric charge) and emerge from different vertices, it is always possible to obtain two separate graphs, the one going into the other by the

exchange of these two particles. For example, in single pion production in $N-N$ collisions (see Da Prato, reference ¹⁾) four different peripheral graphs can describe the process. The intermediate particle is always a pion, and only its charge may change. The relative importance of the various possible graphs will be different in different regions of phase-space. For example, at small Δ^2 graph '1' will be dominant over all others, due to the factor $(\Delta^2 + m^2)^{-2}$ which is present only in it. This is the reason why in the most part of the existing literature (where processes at low Δ^2 have been considered) the possible occurrence of other graphs has been neglected ⁷⁾⁻⁹⁾. For arbitrary Δ^2 , however, all graphs must be retained on the same footing. Complete predictions for the peripheral cross-sections will be obtained by summing up the contributions of all the different peripheral matrix elements relative to the various possible graphs. The matrix element to be inserted in formula (2.3) is then

$$|M_{fi}|^2 = \left| \sum_n \eta_n M^{(n)} \right|^2 \quad (2.140)$$

η_n can be equal to +1 or -1. Its exact value has to be decided case by case by looking at possible effects of the Pauli principle. In general

$\eta_n = +1$ for any n if all the particles are different. Thus the calculation of a process according to the peripheral model will be reduced to the calculation of a sum of terms which can be grouped in two classes.

- a) Square moduli of the matrix elements associated to the single graphs.
- b) Interference terms between different graphs.

The terms belonging to class a) are the most important ones. They have been written explicitly for the graphs of the type '1' or '2'. When both types of diagrams are present, one has simply to add the two contributions, (calculated from (2.96) to (2.122), and from (2.123) to (2.139) respectively). For a three-particle final state, one can have two additional types of graphs, obtained by the exchange $q_1 \longleftrightarrow q_2$ in graph '1' and '2' respectively.

These diagrams are shown in Fig. 3 and will be called graphs '3' and '4'.

Fig. 3

The distributions obtained from the square moduli of the latter graphs do not need any additional formula. In the case of graph '3', one for particle q_2 uses the formulae derived for particle q_1 in the case of graph '1', and vice versa. In the case of graph '4', the same criterion applies by starting from the formulae derived in the case of graph '2'. Of course, in the formulae given, one has to insert the correct values of the masses, of the constant G^2 and of $\frac{d\delta_1}{d\Omega}$, according to what particles interact in the vertices. Sometimes the alteration induced by the exchange of the particles reduces to a multiplicative isotopic spin factor. When this factor turns out to be small, one can neglect the contribution of the diagram in a first approximate evaluation.

The contributions of terms of class b) cannot be discussed in detail in this report. Apart from the mathematical complication of the formulae, the main difficulty lies in the fact that a correct calculation of the interference terms requires the knowledge of the transition amplitudes, i.e., a phase-shift analysis which is not available for a part of the 4-body reactions known today. However, in the few cases in which the calculation of such terms has proved

possible under reasonable approximations (Selleri, reference ¹¹⁾ ; Da Prato, reference ¹⁾) it has been found that:

- a) the interferences between graphs which differ by the exchange of the initial particles [i.e., '1' and '2', or '3' and '4'] are much more important than the interferences between graphs which differ by the exchange of the final particles. In particular in Da Prato's calculations the latter turn out to be one order of magnitude smaller than the former, and two orders of magnitude smaller than the quadratic terms.
- b) the relative magnitude of the interferences to the corresponding quadratic terms decreases rapidly with increasing energy.
- c) an increase of the number of the partial waves present in the 4-particle vertex (also when occurring in different intervals of energy) reduces the contribution of the interferences.
- d) there is no region of the phase-space in which the effect of the form factors K and K' is negligible for the interference terms, and then the pole approximation for them is less reliable than for the quadratic terms.

Point a) can be easily understood in the case of a resonance present in the cross-section σ_1 . In fact, the product of two diagrams which differ by the exchange of two final particles contains the product of two amplitudes calculated at different energies (namely ω and u). Then in the whole phase-space the region where the two resonances overlap is very small, and therefore the contribution is strongly reduced with respect to the case where the resonant amplitude is calculated at the same energy in both the interfering diagrams. Another similar way of explaining the smallness of such terms is the following. If the interaction of process (2.7) in graph '1' has a resonant character, the matrix element $M_1(\omega^2, t^2; \Delta^2)$ can be well approximated by a single amplitude with fixed isospin ($T = \bar{T}$) and total angular momentum ($J = \bar{J}$).

The particles q and q_1 emerging from, say, graph '3' have instead all the possible T and J values. In the interference between graphs '1' and '3' only that "part" of graph '3' with $T = \bar{T}$ and $J = \bar{J}$ will give non-negligible contribution, and the result will generally be small when compared with the quadratic terms ³⁹⁾.

Points b) and d) can also be easily understood, since the two momentum transfers (e.g. Δ^2 and $\bar{\Delta}^2$) entering symmetrically in an interference term cannot in general be simultaneously small, and their average separation increases with energy.

From what has been discussed above, it turns out that the approximation of neglecting the interference terms should be generally a good one, in particular when many partial waves are simultaneously present in the 4-particle vertex. (In this case the complication of the mathematical formalism and the lack of experimental detailed analysis make their evaluation practically impossible). One of the cases in which the effect of the interference terms may be of some importance is the one of interfering graphs which differ by the exchange of the initial particles and contain a partial cross-section dominated by only one (possibly resonant) partial wave amplitude. (Such is the case of single pion production in nucleon-nucleon collisions). At relatively low energies, considerations a), b) and c) do not apply to this case. However, it is possible to obtain a formula which allows us to calculate such a term : under the approximation of only one partial wave amplitude present, the energy behaviour of the latter is unnecessary, and the formula obtained involves only the total cross-section. We consider p_1 and q_1 as fermions, because the only practical cases to which this formula can be applied are those of nucleon-nucleon collisions. The general contribution to the cross-section of the interference terms between graphs '1' and '2' (to be added to (2.63) + (2.124)) is ⁴⁰⁾

$$\begin{aligned}
d\sigma_{\text{int}} = & \eta \frac{G_r \bar{G}_r}{2F} (2\pi)^{-3} \frac{\omega^2}{(\Delta^2 + \mu^2)(\bar{\Delta}^2 + \mu^2)} K(\Delta^2) K'(\Delta^2) K(\bar{\Delta}^2) K'(\bar{\Delta}^2) \cdot \\
& \cdot \text{Re} \left[g^{(+)} \sqrt{(p_0^Q - M_1)(p_{20}^Q - M_2)} \cdot (f_1' f_1^* \cos \alpha^Q + f_2' f_2^* \right. \\
& + f_2' f_1^* \cos \theta_1^Q + f_1' f_2^* \cos \epsilon_1^Q) + g^{(-)} \sqrt{(p_0^Q + M_1)(p_{20}^Q + M_2)} \cdot \\
& \cdot (f_1' f_1^* + f_2' f_2^* \cos \alpha^Q + f_2' f_1^* \cos \epsilon_1^Q + f_1' f_2^* \cos \theta_1^Q) \left. \right] \delta^4(p_f - p_i) \frac{d^3 q_1 d^3 q_2}{q_0 q_{10} q_{20}}
\end{aligned} \quad (2.141)$$

where G_r , $\bar{G}_r$ are the renormalized coupling constants of the 3-particle vertices of graphs '1' and '2' and η is a factor coming from the Pauli principle : it is equal to +1 if the initial particles are identical, to -1 if they are different. The amplitudes f_1, f_2 are defined by (2.72) and (2.73). The difference between primed and unprimed f 's is the following. In the unprimed ones, the (off-shell) partial wave amplitudes $f_{\pm}$ are to be calculated as functions of ω (c.m. energy) and Δ^2 ("mass" of the incident pion), and the Legendre polynomials are functions of $\cos \theta_1^Q$. In the primed ones, the $f_{\pm}$ are functions of ω and $\bar{\Delta}^2$, the Legendre polynomials are functions of $\cos \epsilon_1^Q$. The pole approximation is obtained by putting $K = K' = 1$ and by considering the $f_{\pm}$'s as functions of ω alone.

The coefficients $g^{(+)}$ and $g^{(-)}$ are defined as follows

$$\begin{aligned}
g^{(+)} &= \omega \pm (M_1 + M_2 - m_2) && \text{for pseudoscalar coupling in the 3-particle} \\
&&& \text{vertex } (0 = \gamma_5 \text{ in (2.63)}) \\
g^{(-)} &= \omega \pm (M_1 + M_2 + m_2) && \text{for scalar coupling in the 3-particle vertex} \\
&&& (0 = 1 \text{ in (2.63)}).
\end{aligned}$$

Formula (2.141) cannot be used for practical calculations if the phase-shift analysis for reaction (2.7) is not known. However, a simplified formula can be obtained in the single phase-shift approximation. If we consider all the

partial-wave amplitudes equal to zero, except one $(f_{\ell\pm})$ we get

$$f_1 = \pm f_{\ell\pm} p'_{\ell\pm 1} \quad (2.142)$$

$$f_2 = \mp f_{\ell\pm} p'_{\ell}$$

and (2.80) reduces to

$$\frac{k}{q} \sigma_1(\omega) = 4\pi a_{\ell}^{\pm} |f_{\ell\pm}|^2 \quad (2.143)$$

with $a_{\ell}^+ = \ell + 1$, $a_{\ell}^- = \ell$.

In this way it is easy to calculate the two cross-sections $\frac{d^2 \sigma_{\text{int}}}{d\Delta^2 d\omega^2}$ and $\frac{d^2 \sigma_{\text{int}}}{du^2 dt^2}$. From these, according to the rules of integration given in the previous sub-section, all the distributions of interest for particles q_2 and q_1 can be calculated.

The interference term between diagrams '3' and '4' is given by the same formulae, by applying to particle q_1 the corresponding distribution valid for particle q_2 in the interference term between graphs '1' and '2', and vice versa, and taking into account the correct values of the G_r 's and of σ_1 . We get ($G' = G_r \bar{G}_r / 4\pi$)

$$\frac{d^2 \sigma_{\text{int}}}{d\Delta^2 d\omega^2} = \frac{\eta}{2} \frac{G'}{(k^B \omega)^2} \frac{q^Q \omega}{(\Delta^2 + \mu^2)(\bar{\Delta}^2 + \mu^2)} K(\Delta^2) K(\bar{\Delta}^2) K'(\Delta^2) K'(\bar{\Delta}^2) \cdot \quad (2.144)$$

$$\cdot \text{Re} \left[f_{\ell\pm}^*(\omega, \Delta^2) f_{\ell\pm}(\omega, \bar{\Delta}^2) \right] a_{\ell}^{\pm} \phi_0^{\ell\pm}(\omega, \Delta^2)$$

in the general case, and

$$\frac{d^2 \delta_{int}}{d\Delta^2 d\omega^2} = \frac{\eta}{8\pi} \frac{G'}{(k^B W)^2} \frac{1}{(\Delta^2 + \mu^2)(\bar{\Delta}^2 + \mu^2)} k_1^Q \omega \delta_1(\omega) \phi_0^{\ell\pm}(\omega, \Delta^2) \quad (2.145)$$

in the pole approximation.

We have put

$$\begin{aligned} \phi_0^{\ell\pm} = & g^{(+)} \sqrt{(p_{10}^Q - M_1)(p_{20}^Q - M_2)} p_{\ell\pm 1}(\omega \alpha^Q) + \\ & + g^{(-)} \sqrt{(p_{10}^Q + M_1)(p_{20}^Q + M_2)} p_{\ell}(\omega \alpha^Q) \end{aligned} \quad (2.146)$$

and used (2.143) in order to obtain (2.145).

We also get

$$\begin{aligned} \frac{d^2 \delta_{int}}{du^2 d\Delta^2} = & \frac{\eta}{8\pi} \frac{G'}{(k^B W)^2} \frac{q^P}{u} \int_{-1}^1 d\cos\theta_2^P \frac{K(\Delta^2) K'(\Delta^2)}{\Delta^2 + \mu^2} \cdot \\ & \cdot \int_0^{2\pi} d\varphi_2^P \operatorname{Re} \left[f_{\ell\pm}^*(\omega, \Delta^2) f_{\ell\pm}(\omega, \bar{\Delta}^2) \right] \frac{\omega^2}{\bar{\Delta}^2 + \mu^2} K(\Delta^2) K'(\Delta^2) \phi_1^{\ell\pm} \end{aligned} \quad (2.147)$$

$$\frac{d^2 \delta_{int}}{du^2 d\Delta^2} = \frac{\eta}{8\pi} \frac{G'}{(k^B W)^2} \frac{q^P}{u} \int_{-1}^1 d\cos\theta_2^P \frac{1}{\Delta^2 + \mu^2} \int_0^{2\pi} d\varphi_2^P \frac{k_1^Q \omega^2}{q^Q} \frac{\delta_1(\omega)}{4\pi a_{\ell}^{\pm}} \frac{\phi_1^{\ell\pm}}{\bar{\Delta}^2 + \mu^2} \quad (2.148)$$

with

$$\begin{aligned} \phi_1^{\ell\pm} = & g^{(+)} \sqrt{(p_{10}^Q - M_1)(p_{20}^Q - M_2)} \left[p'_{\ell\pm 1}(\omega \epsilon_1^Q) p'_{\ell\pm 1}(\omega \theta_1^Q) \omega \alpha^Q + p'_{\ell}(\omega \epsilon_1^Q) p'_{\ell}(\omega \theta_1^Q) - \right. \\ & \left. - p'_{\ell}(\omega \epsilon_1^Q) p'_{\ell\pm 1}(\omega \theta_1^Q) \omega \theta_1^Q - p'_{\ell\pm 1}(\omega \epsilon_1^Q) p'_{\ell}(\omega \theta_1^Q) \omega \epsilon_1^Q \right] + \\ & + g^{(-)} \sqrt{(p_{10}^Q + M_1)(p_{20}^Q + M_2)} \left[p'_{\ell\pm 1}(\omega \epsilon_1^Q) p'_{\ell\pm 1}(\omega \theta_1^Q) + p'_{\ell}(\omega \epsilon_1^Q) p'_{\ell}(\omega \theta_1^Q) \omega \alpha^Q - \right. \\ & \left. - p'_{\ell}(\omega \epsilon_1^Q) p'_{\ell\pm 1}(\omega \theta_1^Q) \omega \epsilon_1^Q - p'_{\ell\pm 1}(\omega \epsilon_1^Q) p'_{\ell}(\omega \theta_1^Q) \omega \theta_1^Q \right] \end{aligned} \quad (2.149)$$

It can be shown that $\int \phi_i^{\ell\pm} d\cos\theta_1^Q d\varphi_1^Q = 4\pi a_{\ell}^{\pm} \phi_0^{\ell\pm}$.

SECTION 3

Many particles in the final state

3.1 THE MOST GENERAL PERIPHERAL DIAGRAM FOR MANY PARTICLE PRODUCTION

In this section we extend the considerations made in Section 2 to the case of an arbitrary number of particles present in the final state. The most general peripheral diagram describing such a process is the one drawn in Fig. 4, where n particles come out from the upper vertex and n' particles come out from the lower vertex. We will denote the total 4-momentum of the two groups of final particles by Q_n and Q'_n , and define

$$\Delta^2 = k^2 = (Q_n - p_1)^2 = (Q'_n - p_2)^2 \quad (3.1)$$

Fig. 4

The calculation of angular and energy distributions of the individual particles from such a graph is difficult and of no practical interest. The only information which can easily be obtained concerns the Δ^2 -behaviour and, as a consequence, the total cross-section. In effect, the presence of the factor $(\Delta^2 + \mu^2)^{-2}$ in the cross-section favours the "small" values of Δ^2 . Small Δ^2 implies $\vec{Q}_n$ almost parallel to $\vec{p}_1$ and $\vec{Q}'_n$ almost parallel to $\vec{p}_2$ in the c.m. system: this effect, not observable on the individual particles at low energy for the processes with high multiplicity, becomes very important at cosmic ray energies, where the final particles appear actually divided into two jets. Therefore the one-particle exchange offers a simple explanation of the fire-ball model ⁴¹⁾. This connection has been discussed in one of the quoted papers by Salzman and Salzman ⁴²⁾. The formulae for the Δ^2 -distribution and for the total cross-section in the pole approximation were first calculated by Berestetski and Pomeranchuk ⁴³⁾. They can be deduced in a straightforward way by following the procedure outlined in sub-section 2.2. These formulae are

$$\frac{d\sigma}{d\Delta^2} = \frac{1}{16\pi^3 F^2} \frac{1}{(\Delta^2 + \mu^2)^2} \int d\omega^2 \int dv^2 F_1(v) \phi_1(v) F_2(\omega) \phi_2(\omega) \quad (3.2)$$

Here $F = k^B W$ is the initial flux factor,

$$\omega^2 = -Q_n^2, \quad v^2 = -Q_n'^2 \quad (3.3)$$

F_1 and F_2 are the flux factors relative to the collisions $k + p_1$ and $-k + p_2$ respectively ⁴⁴⁾ and are given by

$$F_1(v) = v P(v; M_1^2, \mu^2) \quad F_2(\omega) = \omega P(\omega; M_2^2, \mu^2) \quad (3.4)$$

$\sigma_1(v)$ is the total cross-section, at total c.m. energy v , for the collision $k + p_1$, and $\sigma_2(\omega)$ is the total cross-section at total c.m. energy ω , for the collision $-k + p_2$. The introduction of the total cross-sections, of course, implies a summation over all the possible values of n and n' compatible with the energy available, with the exclusion of single particle states. (This particular case can be treated separately and will be considered below). Furthermore, it must be remarked that (3.2) holds in the pole approximation, which is the only one of practical use in the present case. The double integration over v and ω has to be performed over the phase-space region indicated in Appendix 2, formulae (A2.8) or (A2.9), and (A2.3). In order to get the total cross-section, it is convenient to integrate over Δ^2 first. One gets

$$\sigma_{\text{tot}} = \frac{1}{16\pi^3 F^2} \int d\omega^2 \int dv^2 F_1(v) \sigma_1(v) F_2(\omega) \sigma_2(\omega) \cdot \frac{\Delta_{\text{max}}^2(\omega^2, v^2) - \Delta_{\text{min}}^2(\omega^2, v^2)}{(\Delta_{\text{max}}^2(\omega^2, v^2) + \mu^2)(\Delta_{\text{min}}^2(\omega^2, v^2) + \mu^2)} \quad (3.5)$$

between the limits

$$\begin{aligned} m_1' &\leq \omega \leq W - m_2' \\ m_2' &\leq v \leq W - \omega \end{aligned} \quad (3.6)$$

where m_1' and m_2' are given by the sum of the masses of the lightest system which can be produced in the upper and in the lower vertex respectively, with the exclusion of systems consisting of a single particle. The limiting values for Δ^2 , Δ_{max}^2 and Δ_{min}^2 , are given in Appendix 2, formulae (A2.4) or (A2.6).

In order to obtain a complete picture of the phenomenon, one should sum over the different charge states of the intermediate particle (this will only introduce an isotopic spin coefficient). One has not to worry about

the large multiplicity of diagrams which may originate from the exchange of identical particles : they are all contained in the sum performed over the number of particles coming out from the vertices. Of course, in this philosophy, all the interferences between such diagrams are neglected. This will always be done in this section. Formula (3.5), however, should not be taken too seriously, since the corrections in the vertices induced by the virtuality of the intermediate particle cannot be estimated, and are expected to be very important at high energy. In effect, Dremin and Cernavskij⁴⁵⁾ have calculated peripheral cross-sections at extremely high energy (10^{11} eV), and found values some orders of magnitude larger than the experimental ones observed in cosmic-ray events. On the other hand, G. Salzman⁴⁶⁾ pointed out that the peripheral model without corrections at very high energies may violate unitarity. Finally, according to the picture of the interaction at high energies suggested in the work by Amati et al.²²⁾, it follows that a formula of the type (3.5) is always expected to give an overestimate of the total cross-section. We would only mention here that at rather high energies (10-20 GeV) statistical theories seem to predict correctly some integral quantities relative to pion-nucleon and nucleon-nucleon reactions, such as pion multiplicities and c.m. energy distributions (averaged over all angles). This does not seem to contradict the hypothesis of a large contribution due to peripheral mechanism : when a large number of particles is produced at the same vertex, the characteristic effects of the one-pion exchange interaction cannot affect strongly the distributions of the individual particles ; these distributions are then expected to be close to statistical distributions. However, it should be remarked that the particles which are produced in smaller number (nucleons, antinucleons) show deviations from the statistical predictions also at these energies. In fact, it has been proposed⁴⁷⁾ to take into account the effects of the peripheral interaction also in a statistical picture, in order to obtain a better agreement with the experiments.

3.2 THE PARTICULAR CASE OF ONE OR TWO PARTICLES COMING OUT FROM ONE OF THE VERTICES

In the quite general picture discussed in sub-section 3.1 there are particular cases from which more detailed information can be extracted. These particular cases are shown in Figs. 5 and 6, where the partition of the final particles is such as to let only one or two particles come out from one vertex, and an arbitrary number from the other one

Fig. 5

Fig. 6

The labelling of the particles in the diagrams has been chosen for reasons of convenience, which will become apparent in the following developments. We will also give the formulae for the same graphs with p_1 and p_2 interchanged (Fig. 7 and Fig. 8) so as to exhaust all the possible basic diagrams of this type.

The interest of considering graphs of these particular types can be illustrated by the following arguments. These are processes in which one of the final particles cannot emerge from both vertices in the one-particle exchange picture. An example of this is given by π - $\bar{N}$ collisions described through the exchange of a single pion. As long as nucleon pair production is negligible, the final nucleon will always emerge from the lower vertex. If the multiplicity is not too large, one can reasonably expect that the bulk of

the contribution of all the possible peripheral diagrams consists essentially in the diagrams of the types '5' and '6'. Should also this not be the case, diagrams of the type '5' or '6' for the angular and energy distribution of the final nucleon give a much more characteristic spectrum than other possible mechanisms of interaction (including the more complicated peripheral graphs) ; the behaviour of the spectrum predicted by these graphs can be extracted from experiment also in the presence of some "background". For instance, for the above mentioned π - N reactions, diagram '5' predicts final nucleons preferentially backwards in the c.m.s. The latter considerations also apply to the case of a final particle which can actually emerge from both vertices. If one takes into account all the significant contributions of diagrams of the type '5' or '6', they will provide characteristic spectra at least in some regions of the phase-space, i.e., for low values of the "mass" of the intermediate particle. Finally, the formulae given will allow the calculation of all processes where 4 final particle are present, in the approximation in which all the interferences between different graphs are neglected. This approximation should be reasonable, according to the general considerations made in sub-section 2.5.

We are obviously interested in the distributions relative to particle q_2 for graph '5', and to particle q_1 for graph '6'.

The reaction described by the diagram of Fig. 5 is

$$p_1 + p_2 \rightarrow q_2 + Q_1 + Q_2 + \dots + Q_n \quad (3.7)$$

We put

$$Q = Q_1 + Q_2 + \dots + Q_n \quad (3.8)$$

$$\omega^2 = -Q^2 \quad (3.9)$$

We can easily obtain an obvious generalization of formulae (2.67) where all symbols previously defined keep their meaning

$$d\sigma = \frac{d^3q_2}{q_{20}} \frac{(2\pi)^{f'}}{F} q_{20} p_{20} |V(\Delta^2)|^2 |\Delta'_F(\Delta^2)|^2 \sum_n (2\pi)^{4-3n} \Lambda_n^2 \cdot \int |M^{(n)}(\omega^2, \Delta^2, \dots)|^2 \delta^4(p_f - p_i) \frac{d^3Q_1 \dots d^3Q_n}{Q_{10} \dots Q_{n0}} \quad (3.10)$$

Here

$$\Lambda_n^2 = 2^{-b'} \left[\prod_{i=1}^{f'} m_i \right]$$

where b' is the number of the bosons and f' the number of fermions present in the upper vertex (excluding the intermediate boson k), and $M^{(n)}(\omega^2, \Delta^2, \dots)$ is the off-shell M -matrix element for the reaction

$$k + p_i \rightarrow Q_1 + Q_2 + \dots + Q_n \quad (3.11)$$

The variables on which M depends are: $\Delta^2 = k^2$, the "mass" of the virtual incident particle, ω^2 , the c.m. energy, and all the internal degrees of freedom of the system of particles $Q_1, Q_2, \dots, Q_n$. In (3.10) the integration over $Q_1, \dots, Q_n$ is carried out because we are only interested in the distributions for particle q_2 .

In the most part of the cases (cf. 31) q_2 and p_2 are fermions, and one can again apply formulae (2.63) and (2.64) and get

$$d\sigma = \frac{d^3q_2}{q_{20}} (2\pi)^{-3} \frac{1}{4F} G_r^2 K^2(\Delta^2) K'^2(\Delta^2) \frac{\Delta^2 + (M_2 \pm m_2)^2}{(\Delta^2 + \mu^2)^2} \cdot \sum_n (2\pi)^{4-3n} \Lambda_n^2 \int |M^{(n)}|^2 \delta^4(p_f - p_i) \frac{d^3Q_1 \dots d^3Q_n}{Q_{10} \dots Q_{n0}} \quad (3.12)$$

We can now define an "off-shell cross-section" for reaction (3.11) in a way analogous to the one which was introduced in sub-section 2.4 for the 2-particle case⁴⁸⁾

$$\sigma^{(n)}(\omega^2, \Delta^2) = (2\pi)^{4-3n} \frac{\Lambda_n^2}{2p_1^0 \omega} \int |M^{(n)}(\omega^2, \Delta^2, \dots)|^2 \delta^4(p_f - p_i) \cdot \frac{d^3 Q_1 \dots d^3 Q_n}{Q_{10} \dots Q_{n0}} \quad (3.13)$$

and then we get $(G^2 = G_r^2/4\pi, \sigma_1 = \sum_n \sigma^{(n)} = \text{total cross-section for the collision } k + p_1)$

$$d\sigma = (2\pi)^{-2} \frac{G^2}{F} K^2(\Delta^2) K_1^2(\Delta^2) \frac{\Delta^2 + (M_2 \pm m_2)^2}{(\Delta^2 + \mu^2)^2} k_1^0 \omega \cdot \sigma_1(\omega, \Delta^2) \frac{d^3 q_2}{q_{20}} \quad (3.14)$$

and in the pole approximation

$$d\sigma = (2\pi)^{-2} \frac{G^2}{F} \frac{\Delta^2 + (M_2 \pm m_2)^2}{(\Delta^2 + \mu^2)^2} k_1^0 \omega \sigma_1(\omega) \frac{d^3 q_2}{q_{20}} \quad (3.15)$$

Formulae (3.14) and (3.15) look very similar to the corresponding ones valid for the three particle final state case. The only difference consists in the different meaning of the cross-section σ_1 , which is now a total cross-section, whereas in the three-particle case it was a partial cross-section. Apart from this difference, all formulae from (2.94) to (2.109) giving the spectra of interest for particle q_2 can be applied without further changes. Also the kinematic limits stay unchanged; only the lower limit for ω^2 is different, and it is now given by the square of the mass of the lightest many-particle system which can emerge from the upper vertex.

If we consider the diagram of the type '5' with p_1 and p_2 interchanged (Fig. 7) we arrive at the same conclusion : all formulae listed in sub-section 2.5, from (2.128) to (2.137) can be still applied with the proper meaning of δ_1 .

Fig. 7

It is worth noting that in these sets of formulae the masses m_1 and m appear only in the lower limit for ω^2 , so that no ambiguities occur when their validity is extended to the many-particle case, provided the correct value for such lower limit is taken.

The reaction described by the graph of Fig. 6 is

$$p_1 + p_2 \rightarrow q_1 + q + Q_1 + Q_2 + \dots + Q_n. \quad (3.16)$$

We shall find that the distributions predicted for particle q_1 by this graph are given by a superposition of distributions for particle q_1 relevant to the case of three final particles, where the "particle" q_2 has variable mass.

We will now redefine in an obvious way some of the invariants (2.11) and (2.12) by putting

$$Q = Q_1 + Q_2 + \dots + Q_n \quad (3.17)$$

$$v^2 = -Q^2 \quad (3.18)$$

$$\left\{ \begin{array}{l} u^2 = -(Q+q) = -(q_1 - p_1 - p_2)^2 \\ \underline{\Delta}^2 = (Q - p_2)^2 = (q_1 + q - p_1)^2 \\ \bar{\Delta}^2 = (Q - p_1)^2 = (q_1 + q - p_2)^2 \end{array} \right. \quad (3.19)$$

Relation (2.13) will be replaced by

$$\bar{\Delta}^2 = W^2 - m_1^2 - m_2^2 - v^2 - \omega^2 - \Delta^2 \quad (3.20)$$

All the other relations of interest (including the kinematical definitions of the quantities relative to particle q_1) stay unchanged, as they do not contain m_2 ; whenever this quantity occurs, it has to be replaced by v . The S-matrix element for the diagram of Fig. 6 can now be written (for a given value of n in (3.16))

$$S_{fi}^{(n)} = \int (2\pi)^{3/2} \sqrt{2k_0} S_2^{(n)}(v^2, \Delta^2, \dots) \Delta_F'(\Delta^2) \cdot \\ \cdot (2\pi)^{3/2} \sqrt{2k_0} S_1(\omega^2, t^2; \Delta^2) d^4k \quad (3.21)$$

where S_2 and S_1 are the S-matrix elements for the processes

$$-k + p_2 \rightarrow Q_1 + Q_2 + \dots + Q_n \quad (3.22)$$

and

$$k + p_1 \rightarrow q + q_1 \quad (3.23)$$

respectively. k is the 4-momentum of the intermediate particle going from the lower to the upper vertex in Fig. 6⁴⁴⁾. The dependence of S_2 on the internal degrees of freedom of the system of particles $q_1, q_2, \dots, q_n$ has not been written explicitly.

We now insert $\mathcal{M}_1$ in place of S_1 through (2.9) and (2.10), use for $\Delta_F(\Delta^2)$ formula (2.64) and get

$$S_{fi}^{(n)} = -(2\pi)^{-3} \sqrt{2k_0} S_2^{(n)}(v^2, \Delta^2, \dots) \frac{\lambda}{(\Delta^2 + \mu^2)} K'(\Delta^2) \cdot [2q_0 q_{10} p_{10}]^{-1/2} m_1(\omega^2, t^2; \Delta^2) \quad (3.24)$$

We want to consider together all the diagrams in which the c.m. energy of the particles outgoing from the lower vertex has a given value v . Then we square (3.23) and multiply it by the factor

$$d^4Q \delta^4\left(\sum_{i=1}^n Q_i - Q\right) \delta(Q^2 + v^2) dv^2 \quad (3.25)$$

which does not affect the result of any integration, and simply expresses definitions (3.17) and (3.18). We put the expression obtained into (2.2) and then integrate over $q_1, q_2, \dots, q_n$ and sum over n . We get

$$d\sigma = (2\pi)^{-6} \frac{F_2(v, \Delta^2)}{F} |m_1(\omega^2, t^2; \Delta^2)|^2 \frac{\lambda^2}{(\Delta^2 + \mu^2)^2} K'^2(\Delta^2) \cdot \frac{d^3q d^3q_1}{q_0 q_{10}} \frac{d^3Q}{2Q_0} \delta^4(Q + q + q_1 - p_1 - p_2) dv^2 \sigma_2(v, \Delta^2) \quad (3.26)$$

Here we have put

$$F_2(v, \Delta^2) = v P(v; M_2^2, -\Delta^2). \quad (3.27)$$

This function represents an "off-shell flux factor" for reaction (3.22) when particle k is virtual. We will need also the on-shell limit of (3.27); such a function represents the flux factor for reaction (3.22) when k is real, and is given by

$$F_2(v) = v P(v; M_2^2, \mu^2). \quad (3.28)$$

We have also made use of the definition

$$\delta^4(Q + q + q_1 - p_1 - p_2) \sigma_2(v, \Delta^2) = (2\pi)^6 \frac{k_0 p_{20}}{F_2(v, \Delta^2)} \cdot \sum_n \frac{1}{TV} \int \delta^4\left(\sum_{i=1}^n Q_i - Q\right) |S_2^{(n)}(v, \Delta^2, \dots)|^2 d^3Q_1 \dots d^3Q_n. \quad (3.29)$$

This formula is the off-shell generalization of (2.2) of the same type of (3.13). In the r.h.s. the δ -function enters in the integral and in the sum only formally, since it reduces to $\delta^4(Q + q + q_1 - p_1 - p_2)$ for any n and for any choice of $Q_1, \dots, Q_n$, because of the δ -function contained in S_1 . σ_2 has the meaning of an off-shell total cross-section for the collision $-k + p_2$. Comparison of formula (3.26) with the analogous one in the 3-final particle case [i.e., (2.68)] shows a close correspondence. The integrand has a somewhat different structure, since $K^2(\Delta^2)G^2[\Delta^2 + (m_2 \pm M_2)^2]$ is replaced by $\frac{1}{2\pi^2} F_2(v, \Delta^2) \sigma_2(v, \Delta^2)$: but the formula can still be presented as a three final particle distribution, where one of the particles, namely Q , has a variable mass $\mathcal{W}$, over which a sort of weighted integration has to be performed. It is then apparent that, with the addition of this new

integration, all the considerations made for the spectrum of particle q_1 in sub-sections 2.4 and 2.5 still hold. In particular, it will be true that all information needed is still contained in the cross-section $d^2\sigma/du^2dt^2$. The expression of such a cross-section will contain the integration over ν ; we think that it is practically convenient to perform the integration over this variable after all others. The limits of integration of this new variable will be discussed in Appendix 2. Now we develop Eq. (3.26) and give all the formulae useful for calculating the quantities of interest for the diagram of Fig. 6 as well as for the one of Fig. 8 (p_1 and p_2 interchanged).

By applying (2.82) calculated in the Q-system, we get

$$d\sigma = (2\pi)^{-4} \frac{2\omega^2 F_2(\nu, \Delta^2)}{F} \frac{p_1^Q}{q^Q} \frac{d\sigma_1(\omega^2, \cos\theta_1^Q; \Delta^2)}{d\Omega} \cdot \quad (3.30)$$

$$\cdot \frac{k'^2(\Delta^2)}{(\Delta^2 + \mu^2)^2} G_2(\nu, \Delta^2) \delta^4(Q + q + q_1 - p_1 - p_2) \frac{d^3q d^3q_1 d^3Q}{q_0 q_{10} Q_0} d\nu^2$$

and in the pole approximation

$$d\sigma = (2\pi)^{-4} \frac{2\omega^2 k_1^Q}{q^Q} \frac{F_2(\nu)}{F} \frac{d\sigma_1(\omega^2, \cos\theta_1^Q)}{d\Omega} G_2(\nu) \frac{1}{(\Delta^2 + \mu^2)^2} \cdot \quad (3.31)$$

$$\cdot \delta^4(Q + q + q_1 - p_1 - p_2) \frac{d^3q d^3q_1 d^3Q}{q_0 q_{10} Q_0} d\nu^2$$

Now we list the formulae of interest for the diagram of Fig. 6.

TOTAL CROSS-SECTION

In order to evaluate the total cross-section it is more convenient to put ourselves in the Q-system. We get

$$\sigma = \frac{1}{16\pi^3} \frac{1}{F^2} \iint d\omega^2 dv^2 \int_{\Delta_{\min}^2(\omega^2, v^2)}^{\Delta_{\max}^2(\omega^2, v^2)} d\Delta^2 \frac{1}{(\Delta^2 + \mu^2)^2} K'^2(\Delta^2) F_2(v, \Delta^2) \cdot F_1(\omega, \Delta^2) \sigma_2(v, \Delta^2) \sigma_1(\omega, \Delta^2) \quad (3.32)$$

where $\sigma_1(\omega, \Delta^2)$ is the off-shell cross-section for reaction (3.23); and, in the pole approximation

$$\sigma = \frac{1}{16\pi^3 F^2} \iint d\omega^2 dv^2 F_2(v) \sigma_2(v) F_1(\omega) \sigma_1(\omega) \cdot \frac{\Delta_{\max}^2(\omega^2, v^2) - \Delta_{\min}^2(\omega^2, v^2)}{[\Delta_{\max}^2(\omega^2, v^2) + \mu^2][\Delta_{\min}^2(\omega^2, v^2) + \mu^2]} \quad (3.33)$$

$\Delta_{\max}^2, \Delta_{\min}^2$ are given in Appendix 2, Eqs. (A2.4) or (A2.6) (cf. also subsection 3.1). The region of integration over ω, v is given by

$$\begin{aligned} m_1 + m &\leq \omega \leq W - v \\ m'_2 &\leq v \leq W - (m_1 + m) \end{aligned} \quad (3.34)$$

where m'_2 is the mass of the lightest many-particle system which can come out from the lower vertex. In analogy with the flux factors F_2 (defined by (3.27) and (3.28)) we have also introduced the off-shell and on-shell flux factors F_1 for reaction (3.23), given by

$$F_1(\omega, \Delta^2) = p_1^Q \omega \quad ; \quad F_1(\omega) = k_1^Q \omega = \omega P(\omega; m_1^2, \mu^2) \quad (3.35)$$

Equation (3.33) is again the formula of Berestetski and Pomeranchuk, (3.5), with the only difference that σ_1 is not a total cross-section, but the cross-section for a definite reaction. The same formulae can be applied also to a particular specification of the lower vertex (e.g., in the description of a 4 final particle process) : in this case also σ_2 will be considered as the cross-section for the particular process under consideration.

DISTRIBUTIONS FOR PARTICLE q_1

All formulae from (2.110) to (2.118) except the lower limit of u^2 which is given by $(m+m'_2)^2$, m'_2 being defined below (3.34), with the relevant considerations, stay unchanged. We only have to give here the expression for $d^2\sigma/du^2dt^2$ (see Footnote 37) for the φ^P -integration)

$$\frac{d^2\sigma}{du^2dt^2} = \frac{1}{16\pi^3} \frac{1}{F^2} \int_{m'^2_2}^{(u-m)^2} dv^2 \frac{q^P}{u} \int_{-1}^1 d\cos\theta_2^P F_2(v, \Delta^2) \sigma_2(v, \Delta^2) \cdot \quad (3.36)$$

$$\cdot \frac{K^{12}(\Delta^2)}{(\Delta^2 + \mu^2)^2} \int_0^{2\pi} d\varphi^P \omega^2 \frac{p_1^Q}{q^Q} \frac{d\sigma_1}{d\Omega} (\omega^2, \cos\theta_1^Q; \Delta^2)$$

and, in the pole approximation

$$\frac{d^2\sigma}{du^2dt^2} = \frac{1}{16\pi^3} \frac{1}{F^2} \int_{m'^2_2}^{(u-m)^2} dv^2 \frac{q^P}{u} F_2(v) \sigma_2(v) \cdot \quad (3.37)$$

$$\cdot \int_{-1}^1 \frac{d\cos\theta_2^P}{(\Delta^2 + \mu^2)^2} \int_0^{2\pi} d\varphi^P \omega^2 \frac{k_1^Q}{q^Q} \frac{d\sigma_1}{d\Omega} (\omega^2, \cos\theta_1^Q)$$

In Appendix 2 it is shown that the limits of $\mathcal{V}^2$ at fixed u^2 and t^2 do not depend on t^2 , but only on u . It is important to note that, since $\mathcal{V}$ plays the role of mass of Q , and corresponds to the mass m_2 in the 3-particle case, all expressions for the kinematical quantities in the P-system, given in sub-section 2.3, still apply here, with m_2^2 replaced by $\mathcal{V}^2$ everywhere [(2.51) to (2.58)]. We do not report them there, because they are changed only in this respect.

The expression of Δ^2 , ω^2 , and of all related quantities as functions of our variables of integration will be obtained in the same way already outlined in sub-section 2.4, by using the angles α^P and ϵ_2^P and Eqs. (2.51) to (2.58) together with (2.120). For the link between $\bar{\Delta}^2$, Δ^2 and ω^2 , (3.20) will be used instead of (2.13).

We want to remark that, when the distribution of interest contains a further integration over $d^2\zeta/dt^2 du^2$ to be performed, this can be carried out before the $\mathcal{V}^2$ -integration contained in (3.36) and (3.37), so as to leave the $\mathcal{V}^2$ -integration as the last one. This might be more convenient in some cases. The detailed discussion of this procedure and the limits which have to be used are contained in Appendix 2.

In order to conclude, we give the formulae analogous to (3.33), (3.34), (3.36) and (3.37) for the "crossed" graph represented in Fig. 8

Fig. 8

TOTAL CROSS-SECTION

$$\sigma = \frac{1}{16\pi^3} \frac{1}{F^2} \iint d\omega^2 dv^2 \int_{\bar{\Delta}_{\min}^2(\omega^2, v^2)}^{\bar{\Delta}_{\max}^2(\omega^2, v^2)} d\bar{\Delta}^2 \frac{1}{(\bar{\Delta}^2 + \mu^2)^2} K^{12}(\bar{\Delta}^2) \cdot$$

$$\cdot F_1(v, \bar{\Delta}^2) \sigma_1(v, \bar{\Delta}^2) F_2(\omega, \bar{\Delta}^2) \sigma_2(\omega, \bar{\Delta}^2) \quad (3.38)$$

and in the pole approximation

$$\sigma = \frac{1}{16\pi^3} \frac{1}{F^2} \iint d\omega^2 dv^2 F_1(v) \sigma_1(v) F_2(\omega) \sigma_2(\omega) \cdot$$

$$\frac{\bar{\Delta}_{\max}^2(\omega^2, v^2) - \bar{\Delta}_{\min}^2(\omega^2, v^2)}{[\bar{\Delta}_{\max}^2(\omega^2, v^2) + \mu^2][\bar{\Delta}_{\min}^2(\omega^2, v^2) + \mu^2]} \quad (3.39)$$

The limits on $\bar{\Delta}^2$ are given by (A2.10) or (A2.11); the limits on ω^2 and v^2 are again given by (3.34).

PARTIAL CROSS-SECTION $d^2\sigma/du^2 dt^2$

$$\frac{d^2\sigma}{du^2 dt^2} = \frac{1}{16\pi^3} \frac{1}{F^2} \int_{m_2'^2}^{(u-m)^2} dv^2 \frac{q^P}{u} \int_{-1}^1 d\cos\theta_2^P \int_0^{2\pi} d\varphi^P \frac{K^{12}(\bar{\Delta}^2)}{(\bar{\Delta}^2 + \mu^2)^2} \cdot$$

$$\cdot F_1(v, \bar{\Delta}^2) \sigma_1(v, \bar{\Delta}^2) \omega^2 \frac{p_2^Q}{q^Q} \frac{d\sigma_2}{d\Omega}(\omega^2, \cos\epsilon_1^Q; \bar{\Delta}^2) \quad (3.40)$$

and in the pole approximation

$$\frac{d^2\sigma}{du^2 dt^2} = \frac{1}{16\pi^3} \frac{1}{F^2} \int_{m_2'^2}^{(u-m)^2} dv^2 \frac{q^P}{u} F_1(v) \sigma_1(v) \int_{-1}^1 d\cos\theta_2^P \cdot$$

$$\cdot \int_0^{2\pi} d\varphi^P \frac{1}{(\bar{\Delta}^2 + \mu^2)^2} \omega^2 \frac{p_2^Q}{q^Q} \frac{d\sigma_2}{d\Omega}(\omega^2, \cos\epsilon_1^Q) \quad (3.41)$$

The flux factors $F_{1,2}$ are defined in the following way

$$F_1(v, \bar{\Delta}^2) = v P(v; M_1^2, -\bar{\Delta}^2) ; F_1(v) = v P(v; M_1^2, \mu^2) \quad (3.42)$$

$$F_2(\omega, \bar{\Delta}^2) = \omega P(\omega; M_2^2, -\bar{\Delta}^2) ; F_2(\omega) = \omega P(\omega; M_1^2, \mu^2) = \omega k_2^Q. \quad (3.43)$$

ACKNOWLEDGEMENTS

We are particularly indebted to Professor L. Van Hove who encouraged us to the achievement of this work and to its extension to the present size, and we thank him for illuminating discussions and suggestions. We thank also Professor G. Bernardini for useful critical remarks.

APPENDIX 1

In this Appendix we want to study the kinematical limitations to which are submitted the invariants used in the text for the description of the inelastic process (2.6) (three particles in the final state). There are in this process 5 four-vectors : the energy-momentum conservation and the mass conditions reduce to five the number of independent invariants which describe the process. Any point in the 5-dimensional space of these invariants is then such as to satisfy the energy-momentum conservation law and the mass-shell conditions. Not every point in this space is, however, physically acceptable. E.g., we could satisfy both energy-momentum conservation and mass-shell conditions with negative energies and imaginary momenta. We have thus to limit the set of physical points by imposing two requirements.

I) Requirement that the energies should lie in the physical region :

the energy of any particle, in any reference system, must be positive and larger than the corresponding mass. The moduli of the 3-momenta are then necessarily real.

II) Requirement that the angles should lie in the physical region :

the cosine of the angle between the 3-momenta of any pair of particles in any reference system must lie in the interval $(-1,1)$.

If such requirements are satisfied in a particular reference frame, they are satisfied in any reference frame as a consequence of the Lorentz transformation. Thus we can consider a suitable fixed reference frame and see which limitations on the independent invariants are implied by conditions I) and II). These limitations, involving only invariant quantities, will then ensure that conditions I) and II) are satisfied in any inertial system. Let us consider

the over-all barycentric system (B-system) and choose here a system of polar co-ordinates with $\vec{p}_1^B$ as polar axis and the plane $(\vec{p}_1, \vec{q}_1)$ as azimuthal plane ($\varphi = 0$). In this system conditions I) and II) are automatically satisfied for the initial particles p_1 and q_1 if $W > M_1 + M_2$. Thus we need only to consider the three final particles. For each of them an energy condition can be written

$$q_{10}^B \geq m_1 \quad ; \quad q_{20}^B \geq m_2 \quad ; \quad q_0^B \geq m \quad . \quad (A1.1)$$

From inspection of Eq. (2.32), (2.33) and (2.35) one finds immediately

$$\omega^2 \leq (W - m_2)^2 \quad (A1.2)$$

$$u^2 \leq (W - m_1)^2 \quad (A1.3)$$

$$u^2 + \omega^2 \leq m_1^2 + m_2^2 + 2W\mu \quad (A1.4)$$

It is useful also to obtain the lower limits for ω^2 and u^2 . They can be obtained immediately by applying condition I) to q_{20} in the P-system ($q_{20}^P \geq m_2$) and to q_{10} in the Q-system ($q_{10}^Q \geq m_1$). We get from (2.44) and (2.54)

$$\omega^2 \geq (m_1 + m)^2 \quad (A1.5)$$

$$u^2 \geq (m_2 + m)^2 \quad (A1.6)$$

and then the complete limitations on ω^2 and u^2 are ⁴⁹⁾

$$(m_1 + m)^2 \leq \omega^2 \leq (W - m_2)^2 \quad (A1.7)$$

$$(m_2 + m)^2 \leq u^2 \leq (W - m_1)^2 \quad (A1.8)$$

For the application of condition II), since $\vec{p}_1$ and $\vec{p}_2$ have opposite directions, we get again three independent conditions. We can choose the following ones : first take as "physical" the angles between $\vec{p}_1$ and $\vec{q}_1$ and $\vec{p}_1$ and $\vec{q}_2$:

$$|\cos \theta_1^B| \leq 1 \quad (\text{A1.9})$$

$$|\cos \theta_2^B| \leq 1 \quad (\text{A1.10})$$

The angle λ^B between $\vec{p}_2$ and $\vec{q}_2$ will be related to the above-mentioned angles and to the azimuthal angle φ of $\vec{q}_2$ by the trigonometric relation (Fig. 9)

Fig. 9

$$\cos \lambda^B = \cos \theta_1^B \cos \theta_2^B + \sin \theta_1^B \sin \theta_2^B \cos \varphi \quad (\text{A1.11})$$

If we impose that φ be "physical", we get

$$\cos(\theta_1^B + \theta_2^B) \leq \cos \lambda^B \leq \cos(\theta_1^B - \theta_2^B) \quad (\text{A1.12})$$

which implies also λ^B to obey

$$-1 \leq \cos \lambda^B \leq 1 \quad (\text{A1.13})$$

if (A1.9) and (A1.10) are satisfied.

Let us summarize the limitations obtained. (A1.7) and (A1.8) give absolute limits (i.e., limits dependent only on W and masses) for ω^2 and u^2 . By looking at (2.38) and (2.39) we see that (A1.9) and (A1.10) give limits for Δ^2 dependent on ω^2 , and for t^2 dependent on u^2 . These limitations are

$$-m_2^2 - M_2^2 + 2p_{20}^B q_{20}^B - 2k^B q_2^B \leq \Delta^2 \leq -m_2^2 - M_2^2 + 2p_{20}^B q_{20}^B + 2k^B q_2^B \quad (\text{A1.14})$$

$$-m_1^2 - M_1^2 + 2p_{10}^B q_{10}^B - 2k^B q_1^B \leq t^2 \leq -m_1^2 - M_1^2 + 2p_{10}^B q_{10}^B + 2k^B q_1^B \quad (\text{A1.15})$$

From these expressions, the absolute limits for Δ^2 and t^2 can be deduced. One has to minimize the lower limit, and to maximize the upper limit with respect to ω^2 and u^2 respectively. For the more frequent case of reactions in which the sum of the masses of the final particles is larger than the sum of the masses of the initial particles, if $m_2 \geq M_2$ the absolute limits for Δ^2 are given by limits (A1.14) calculated for $\omega = m_1 + m$; if $m_1 \geq M_1$ the absolute limits for t^2 are given by limits (A1.15) calculated for $u = m_2 + m$. If the "final mass" is smaller than the "initial mass", or if the conditions $m_2 \geq M_2$, $m_1 \geq M_1$ are not verified, the minimum value of Δ^2 or t^2 may occur for a value of ω^2 or u^2 which is not the minimum (i.e., for the value, if any, which allows the lab. kinetic energy of particle q_2 or q_1 to be zero). These particular cases will be discussed in Appendix 3: we disregard them here. By inverting Eqs. (A1.14) and (A1.15) one can obtain limits for ω^2 depending on Δ^2 , and for u^2 depending on t^2 . 50)

$$(m_1+m)^2 \leq \omega^2 \leq W^2 + m_2^2 - \frac{W p_{20}^B}{M_2^2} (\Delta^2 + M_2^2 + m_2^2) + \frac{k^B W}{M_2^2} \left\{ [\Delta^2 + (m_2 - M_2)^2] [\Delta^2 + (m_2 + M_2)^2] \right\}^{1/2} \quad (A1.16)$$

$$(m_2+m)^2 \leq u^2 \leq W^2 + m_1^2 - \frac{W p_{10}^B}{M_1^2} (\bar{t}^2 + m_1^2 + M_1^2) + \frac{k^B W}{M_1^2} \left\{ [\bar{t}^2 + (m_1 - M_1)^2] [\bar{t}^2 + (m_1 + M_1)^2] \right\}^{1/2} \quad (A1.17)$$

If one wants to obtain the corresponding limits for the barred invariants, $\bar{\Delta}^2$, $\bar{t}^2$, one has simply to exchange the role of p_1 and p_2 in Eqs. (A1.14) to (A1.17). In particular, one gets

$$-m_2^2 - M_1^2 + 2 p_{10}^B q_{20}^B - 2 k q_2^B \leq \bar{\Delta}^2 \leq -m_2^2 - M_1^2 + 2 p_{10}^B q_{20}^B + 2 k q_2^B \quad (A1.18)$$

$$-m_1^2 - M_2^2 + 2 p_{20}^B q_{10}^B - 2 k q_1^B \leq \bar{t}^2 \leq -m_1^2 - M_2^2 + 2 p_{20}^B q_{10}^B + 2 k q_1^B \quad (A1.19)$$

Absolute limits are obtained by calculating (A1.19) and (A1.20) at the minimum value of ω^2 and u^2 , unless $m_2 < M_1$ or $m_1 < M_2$.

By inverting (A1.18) and (A1.19) one gets ⁵⁰⁾

$$(m_1+m)^2 \leq \omega^2 \leq W^2 + m_2^2 - \frac{W p_{10}^B}{M_1^2} (\bar{\Delta}^2 + M_1^2 + m_2^2) + \frac{k^B W}{M_1^2} \left\{ [\bar{\Delta}^2 + (m_2 - M_1)^2] [\bar{\Delta}^2 + (m_2 + M_1)^2] \right\}^{1/2} \quad (A1.20)$$

$$(m_2+m)^2 \leq u^2 \leq W^2 + m_1^2 - \frac{W p_{20}^B}{M_2^2} (\bar{t}^2 + m_1^2 + M_2^2) + \frac{k^B W}{M_2^2} \left\{ [\bar{t}^2 + (m_1 - M_2)^2] [\bar{t}^2 + (m_1 + M_2)^2] \right\}^{1/2} \quad (A1.21)$$

Limitation (A1.13) gives a further condition on ω^2 and u^2 , which cannot simultaneously be varied freely in the allowed domains given by (A1.7) and (A1.8). Limitation (A1.4) is included in it. We will not write these limits explicitly but they can be easily derived by means of (2.40). Limitation (A1.12), finally, is the most stringent one. If all other limitations listed up to now are expected, it gives the criterion for establishing whether a point in the 4-dimensional phase space $(\omega^2, u^2, \Delta^2, t^2)$ is outside or inside the physical region.

It is better to rewrite (A1.12) in the following form, which is completely symmetrical in the three angles θ_1^B, θ_2^B and λ^B

$$1 + 2\cos\theta_1^B \cos\theta_2^B \cos\lambda^B - \cos^2\theta_1^B - \cos^2\theta_2^B - \cos^2\lambda^B \geq 0 \quad (\text{A1.22})$$

From this inequation it is possible to get the limits of any of the five independent invariants once the other four have fixed values. If the proper expressions for the cosines are inserted in (A1.22), and the inequation is multiplied by the non-negative quantity $4q_1^{B^2} q_2^{B^2} k^{B^2}$, the l.h.s. of (A1.22) reduces to a polynomial of 2nd degree in any of the four invariants $\omega^2, u^2, t^2, \Delta^2$, for fixed values of the other three. From it one can derive a polynomial of 2nd degree in any linear function of the invariant chosen (like $\bar{t}^2, \bar{\Delta}^2, \cos\theta_{1,q_{10}}^Q$, etc.) for fixed values of the others. Such a polynomial must be non-negative, and the coefficient of the quadratic term is always ≤ 0 ; one can obtain the required limits by elementary algebra. We note that (A1.28) can be used also for a Monte Carlo type machine integration over the phase space: one can integrate any function of the four invariants without taking into account restriction (A1.22) (i.e., by using freely (A1.7), (A1.8), (A1.14) and (A1.15)), provided the integrand is put equal to zero at all the points of the phase space for which the l.h.s. of (A1.22) takes a negative value.

The problem of deriving the limits of one invariant when one or two only of the others are fixed is in general not simple. We have solved it in the text only for the cases of interest. However, this problem can be solved by choosing one or more suitable kinematic formulae among the ones given in sub-section 2.3 and in this Appendix, and by applying condition I) or II) to them.

APPENDIX 2

Let us discuss the kinematical limitations to which is subjected the invariant ν^2 which comes into play for the many final particle states. We consider first the case discussed in sub-section 3.1, Fig. 4, and assume arbitrary values for n and n' (compatible with energy conservation). We call again $m_1'^2$ and $m_2'^2$ the lower limit on ω^2 and ν^2 ; they are given by the mass of the lightest many-particle systems which can emerge from the vertices. At fixed W , the reciprocal limitations on the invariants ω^2 and ν^2 are obtained by an obvious generalization of (A1.7) (substantially, $m_2 \rightarrow \nu$)

$$m_2'^2 \leq \nu^2 \leq (W - \omega)^2 \quad (\text{A2.1})$$

$$m_1'^2 \leq \omega^2 \leq (W - \nu)^2 \quad (\text{A2.2})$$

The corresponding phase-space diagram is shown in Fig. 10, in which ω and ν are plotted, in place of their squares.

Fig. 10

Absolute limits are obviously given by

$$m_2'^2 \leq v^2 \leq (W - m_1')^2; \quad m_1'^2 \leq \omega^2 \leq (W - m_2')^2. \quad (\text{A2.3})$$

The limits on Δ^2 at fixed ω^2 and v^2 can be obtained by generalization of (A1.14) ($v \leftrightarrow m_2$)

$$-v^2 - M_2^2 + 2p_{20}^B Q_0^B - 2k^B Q^B \leq \Delta^2 \leq -v^2 - M_2^2 + 2p_{20}^B Q_0^B + 2k^B Q^B \quad (\text{A2.4})$$

where $Q_0^B = E(W; v^2, \omega^2)$ and $Q^B = P(W; v^2, \omega^2)$ (A2.5)

(A2.4) can be rewritten in a slightly different form

$$-\omega^2 - M_1^2 + 2p_{10}^B Q_0'^B - 2k^B Q^B \leq \Delta^2 \leq -\omega^2 - M_1^2 + 2p_{10}^B Q_0'^B + 2k^B Q^B \quad (\text{A2.6})$$

where

$$Q_0'^B = E(W; \omega^2, v^2). \quad (\text{A2.7})$$

By starting from (A2.4) and (A2.6) it is possible to get limits on ω^2 at fixed v^2 and Δ^2 , and limits on v^2 at fixed ω^2 and Δ^2 , which can be obtained by generalization of (A1.16) and are ⁵¹⁾

$$m_1'^2 \leq \omega^2 \leq W^2 + v^2 - \frac{W p_{20}^B}{M_2^2} (\Delta^2 + M_2^2 + v^2) + \frac{k^B W}{M_2^2} \left\{ [\Delta^2 + (v - M_2)^2] [\Delta^2 + (v + M_2)^2] \right\}^{1/2} \quad (\text{A2.8})$$

$$m_2'^2 \leq v^2 \leq W^2 + \omega^2 - \frac{W p_{10}^B}{M_1^2} (\Delta^2 + \omega^2 + M_1^2) + \frac{k^B W}{M_1^2} \left\{ [\Delta^2 + (\omega - M_1)^2] [\Delta^2 + (\omega + M_1)^2] \right\}^{1/2}. \quad (\text{A2.9})$$

We now consider the case of reaction (3.16) which has been discussed in the second part of sub-section 3.2. For the calculation of the total cross-section Eqs. (3.34) are substantially obtained from (A2.1) and (A2.2) (now $m_1' = m_1 + m$); the limits on Δ^2 as a function of ω^2 and ν^2 are still given by (A2.4) or (A2.6). We need further the limits on $\bar{\Delta}^2$ as a function of ω^2 and ν^2 : they can be obtained by exchanging p_1 and p_2 in (A2.4) or (A2.6):

$$-\nu^2 - M_1^2 + 2p_{10}^B Q_0^B - 2k^B Q^B \leq \bar{\Delta}^2 \leq -\nu^2 - M_1^2 + 2p_{10}^B Q_0^B + 2k^B Q^B \quad (\text{A2.10})$$

and

$$-\omega^2 - M_2^2 + 2p_{20}^B Q_0^{1/B} - 2k^B Q^B \leq \bar{\Delta}^2 \leq -\omega^2 - M_2^2 + 2p_{20}^B Q_0^{1/B} + 2k^B Q^B \quad (\text{A2.11})$$

We need also the limits of ν^2 when u^2 and t^2 are fixed. For the $u^2 - \nu^2$ reciprocal limitation, we can obtain them from the generalization $m_2 \rightarrow \nu$ of (A1.9)

$$(\nu + m)^2 \leq u^2 \leq (W - m_1)^2 \quad (\text{A2.12})$$

which implies

$$m_2'^2 \leq \nu^2 \leq (u - m)^2 \quad (\text{A2.13})$$

The $u - \nu$ phase space is shown in Fig. 11.

Fig. 11

When also t^2 (or $\bar{t}^2$) is fixed, one can easily convince oneself (e.g., considering the reaction in the P-system) that this restricts the interval of variation of v only through the modification of the maximum value which u can assume. This fact is a consequence of (A1.17) and (A1.21) : the maximum of u when t^2 (or $\bar{t}^2$) is fixed does not involve m_2 , but only the other masses.

This can be visualized by considering the phase space of Fig. 11. In the case of fixed t^2 (or $\bar{t}^2$) the allowed region has still the same form as in Fig. 11 but its extension is reduced (Fig. 12)

Fig. 12

In order to conclude : when t^2 (or $\bar{t}^2$) is fixed, the u - v integration has to be performed in the region of Fig. 12. If the v -integration is performed first, limit (A2.13) still hold. If the u -integration is performed first, as it is clear from Fig. 12, the succession of the limits is the following

$$\begin{cases} (v+m)^2 \leq u^2 \leq u_{\max}^2(t^2) \\ m_2^2 \leq v^2 \leq u_{\max}^2(t^2) - m^2 \end{cases} \quad (\text{A2.14})$$

If $\bar{t}^2$ is fixed, in (A2.14) $u_{\max}^2(t^2)$ is replaced by $u_{\max}^2(\bar{t}^2)$. These two functions are given by (A1.17) and (A1.21).

If a t^2 - v^2 integration has to be performed at fixed u^2 , since the limits for both variables depend only on u the order of integration is immaterial.

A P P E N D I X 3

In this Appendix we discuss the evaluation of the limits for the kinematical quantities which generally are the object of experimental measurements : i.e., energies and angles in the lab. and c.m. system. These limits do not enter in the formulae given in the text, but their determination is of interest for the practical calculations of the spectra.

In this connection we shall discuss the particular cases mentioned but not considered in Appendix 1. We will deal only with quantities relevant to particle q_2 , the limits for particle q_1 being obtained with a straightforward relabelling of invariants and masses.

Let us consider first the question of the absolute limits for the lab. kinetic energy of particle q_2 . Because of (2.25) these limits will be obtained directly from the absolute limits of Δ^2 , i.e.,

$$(T_2^L)_{\max} = \frac{1}{2M_2} \left[\Delta_{\max}^2 + (M_2 - m_2)^2 \right] \quad (A3.1)$$

$$(T_2^L)_{\min} = \frac{1}{2M_2} \left[\Delta_{\min}^2 + (M_2 - m_2)^2 \right] \quad (A3.2)$$

We recall that m_2 is the mass of the particle under consideration (q_2) and M_2 is the mass of the target particle (p_2). In the following formulae it will be made free use of the kinematical quantity containing W only (see Table I), since W is a well-determined function (obtainable from (2.21)) of the lab. energy T of the incident particle.

The calculation of $\Delta_{\max}^2$ and $\Delta_{\min}^2$ has been already performed in Appendix 1, under certain conditions on the masses. It has been found that [see (A1.14) for $\Delta_{\max}^2(\omega^2)$ and $\Delta_{\min}^2(\omega^2)$] ⁵²⁾

$$\Delta_{\max}^2 = \Delta_{\max}^2(\omega^2) \quad \text{calculated for } \omega^2 = \omega_{\min}^2 \quad (\text{A3.3})$$

$$\Delta_{\min}^2 = \Delta_{\min}^2(\omega^2) \quad \text{calculated for } \omega^2 = \omega_{\min}^2 \quad (\text{A3.4})$$

The conditions imposed in Appendix 1 are $m_2 \geq M_2$ and $M_f > M_i$, where $M_i = M_1 + M_2$ is the total mass of the initial particles, and $M_f = \omega_{\min} + m_2$ is the total mass of the final products. If any of these two conditions is not satisfied, it may happen that particle q_2 can be emitted at rest in the lab. system; i.e., one can have $T_2^L = 0$, that is $\Delta^2 = -(M_2 - m_2)^2$. (We see that here Δ^2 can assume negative values). We will refer to this case as to the "anomalous case", the other being called the "normal case". The condition for having the anomalous case is

$$\omega_{\min}^2 < \omega_0^2 = \frac{1}{M_2} \left[M_1^2 m_2 + (W^2 - M_1 m_2)(M_2 - m_2) \right] \quad (\text{A3.5})$$

In fact ω_0^2 is the value of ω^2 which allows T_2^L to vanish. For such a value of ω^2 the velocities of q_2 and p_2 in the c.m. system are equal in magnitude: the same condition can be expressed, e.g., by $p_{20}^B m_2 = q_{20}^B M_2$ or equivalent relations involving the c.m. momenta. It has to be noted that backward scattering angles in the lab. are allowed only for $\omega^2 < \omega_0^2$.

If $m_2 \geq M_2$ and $M_f > M_i$, (A3.5) can never be satisfied. If one of these two conditions is dropped, (A3.5) transforms into a condition on W^2 . If both these conditions are dropped, (A3.5) is always satisfied. We write here the condition on W equivalent to (A3.5):

$$\text{if } M_f > M_i, m_2 < M_2 : \quad W^2 > W_0^2 \quad (\text{A3.6})$$

$$\text{if } M_f < M_i, m_2 \geq M_2 : \quad W^2 < W_0^2 \quad (\text{A3.7})$$

where

$$W_0^2 = \frac{M_2 \omega_{\min}^2 - M_1^2 m_2}{M_2 - m_2} + M_2 m_2 \quad (A3.8)$$

When the anomalous case occurs, (A3.4) has to be replaced with

$$\Delta_{\min}^2 = -(M_2 - m_2)^2 \quad (A3.9)$$

Furthermore, the lower limit in (A1.16) is valid only for $\Delta^2 > \Delta_{\min}^2(\omega_{\min}^2)$. For smaller values of Δ^2 the lower limit in (A1.16) is a function of Δ^2 which can be obtained from the upper limit by changing the sign in front of the square root.

Now we consider the case of the limits of the kinetic energy in the lab. at fixed lab. scattering angle. In the normal case, the lab. scattering angle will be always smaller than a limiting value $< 90^\circ$; in the anomalous case it will assume all values from 0° to 180° . The limits to which these quantities are subjected can be easily determined by starting from the expression of T_2^L as a function of $\cos \theta_2^L$ and of the invariant ω^2 :

$$T_2^L = \frac{M_2 p_{20}^B q_{20}^B \pm m_2 k \cos \theta_2^L \left\{ \left(\frac{M_2}{m_2} \right)^2 q_{20}^B{}^2 - p_{20}^B{}^2 + k^B{}^2 \cos^2 \theta_2^L \right\}^{1/2}}{p_{20}^B{}^2 - k^B{}^2 \cos^2 \theta_2^L} - m_2 \quad (A3.10)$$

The ω^2 -dependence is contained in q_{20}^B , which ranges from $(q_{20}^B)_{\max}$ to m_2 as ω ranges from $\omega_{\min}$ to ω_{m_2} . $(q_{20}^B)_{\max}$ will be clearly given by

$$(q_{20}^B)_{\max} = \frac{1}{2W} (W^2 + m_2^2 - \omega_{\min}^2) \quad (A3.11)$$

In the normal case both signs in (A3.10) are allowed, and give two different values of T_2^L for the same $\cos \theta_2^L$ and q_{20}^B . In the anomalous case the same consideration applies for forward scattering angles and for $\omega > \omega_0$; for $\omega \leq \omega_0$ (which includes the case of backward scattering angles) only the + sign has to be kept. The condition that the argument of the square root must be positive gives the reciprocal limits of ω^2 and $\cos \theta_2^L$, and, in the normal case, a lower limit for the latter quantity. We get the following limit on $\cos \theta_2^L$ as a function of ω^2 , valid in the normal case and in the anomalous case if $\omega > \omega_0$ ⁵³⁾:

$$\cos \theta_2^L > \frac{(p_{20}^{B^2} m_2^2 - q_{20}^{B^2} M_2^2)^{1/2}}{m_2 k^B} \quad (A3.12)$$

In the anomalous case and for $\omega \leq \omega_0$ $\cos \theta_2^L$ is not limited in the interval $(-1, 1)$.

The absolute limit for $\cos \theta_2^L$ in the normal case is obtained by minimizing (A3.12) with respect to ω^2 :

$$\cos \theta_2^L \geq \frac{(p_{20}^{B^2} m_2^2 - (q_{20}^B)^2 m_{2x} M_2^2)^{1/2}}{m_2 k^B} \quad (A3.13)$$

The limits on ω^2 as a function of $\cos \theta_2^L$ are

$$\left\{ \begin{array}{l} \omega^2 \leq W^2 + m_2^2 - 2W \frac{m_2}{M_2} (p_{20}^{B^2} - k^{B^2} \cos^2 \theta_2^L)^{1/2} \quad (\text{forward scattering angles}) \\ \omega^2 \leq \omega_0^2 \end{array} \right. \quad (A3.14)$$

(backward scattering angles)

Finally, the limits of T_2^L as a function of θ_2^L can be obtained by (A3.10).

If we put

$$\zeta_1 = \frac{M_2 p_{20}^B (q_{20}^B)_{\max} - m_2 k^B \cos \theta_2^L \left\{ \left(\frac{M_2}{m_2} \right)^2 (q_{20}^B)^2 - p_{20}^{B^2} + k^{B^2} \cos^2 \theta_2^L \right\}^{1/2}}{p_{20}^{B^2} - k^{B^2} \cos^2 \theta_2^L} - m_2 \quad (\text{A3.15})$$

$$\zeta_2 = \frac{M_2 p_{20}^B (q_{20}^B)_{\max} + m_2 k^B \cos \theta_2^L \left\{ \left(\frac{M_2}{m_2} \right)^2 (q_{20}^B)^2 - p_{20}^{B^2} + k^{B^2} \cos^2 \theta_2^L \right\}^{1/2}}{p_{20}^{B^2} - k^{B^2} \cos^2 \theta_2^L} - m_2 \quad (\text{A3.16})$$

we get in the normal case

$$\zeta_1 \leq T_2^L \leq \zeta_2 \quad (\text{A3.17})$$

and in the anomalous case

$$0 \leq T_2^L \leq \zeta_2 \quad (\text{A3.18})$$

In the c.m. system, the situation is very simple : the variations of $\cos \theta_2^B$ and q_{20}^B are completely independent of each other : $\cos \theta_2^B$ ranges between -1 and 1, and q_{20}^B between m_2 and $(q_{20}^B)_{\max}$ given by (A3.11). We have already made use of this property for deriving some of the limits given in Appendix 1.

A P P E N D I X 4

In this Appendix we give a list of the existing works on peripheral interaction which contain numerical evaluations of cross-sections for physical processes. The assumptions made and the approximations used will be mentioned for each of the papers quoted. The papers will be grouped according to the reactions considered ⁵⁴⁾.

I Reaction $\pi + N \rightarrow \pi + \pi + N$

Only π^-p collisions have been considered up to now, and in particular the following two reactions

$$\pi^- + p \rightarrow \pi^- + \pi^+ + n \quad (A4.1)$$

$$\pi^- + p \rightarrow \pi^- + \pi^0 + p \quad (A4.2)$$

F. DONSIGNORI and F. SELLERI - Nuovo Cimento 15, 465 (1960).

In this paper the lab. energy distributions of the final nucleon in (A4.1) and (A4.2) are calculated at $T = 960$ MeV in the pole approximation, by leaving the average value of the $\pi - \pi$ cross-section as an adjustable parameter. Quantitative comparison with experiment is not made, but the discussion of the main features of the theoretical and experimental distributions is carried out extensively with particular emphasis on :

- a) the importance of the peripheral effect at low momentum transfers ;
- b) the possible occurrence of final state πN interactions.

I. DERADO - Nuovo Cimento 15, 853 (1960).

Reaction (A4.2) is investigated at 1 GeV. The lab. energy spectrum of the final proton is compared with the pole approximation formula.

J.C. RUSHBROOKE and D. RADOJCIC - Phys.Rev. 5, 567 (1960).

The same comparison as before is made at 960 MeV. In addition, experimental distributions in Δ^2 at fixed ω^2 are compared with the corresponding peripheral predictions.

A.R. ERWIN, R. MARCH, W.D. WALKER and E. WEST - Phys.Rev.Let. 6, 628 (1960).

The experimental data for reactions (A4.1) and (A4.2) together are selected and analyzed in order to construct the distribution in ω^2 integrated over Δ^2 up to $\Delta = 400$ MeV/c. From comparison with the peripheral prediction in the pole approximation, the mean value of the $\pi - \pi$ cross-section is inferred.

E. PICKUP, D.K. ROBINSON and E.O. SALANT - Phys.Rev.Let. 7, 192 (1961).

Reactions (A4.1) and (A4.2) are studied at 1.25 GeV. The nucleon lab. kinematical energy distributions are compared with the predictions of the peripheral model in the pole approximation and for constant $\pi - \pi$ cross-section. The behaviour of the latter cross-section with energy is determined in the same way as in the preceding paper ; the upper limit used for Δ^2 is $10 \mu^2$.

II Reaction $N + N \rightarrow N + N + \pi$

The work performed up to now refers only to proton-proton collisions, and, at low energies, mainly to reactions

$$p + p \rightarrow p + n + \pi^+ \quad (A4.3)$$

$$p + p \rightarrow p + p + \pi^0 \quad (A4.4)$$

T. KOBAYASHI - Prog.Theor.Phys. 18, 318 (1957).

In this paper the total cross-section for reaction (A4.3) is calculated in the pole approximation in the range from threshold up to ~ 2 GeV. The calculations are underestimated by a factor ~ 2 because only diagram '1' is taken into account.

F. BONSIGNORI and F. SELLERI - loc. cit.

The lab. energy spectrum of the final neutron for reaction (A4.3) is calculated at $T = 970$ MeV by considering only diagram '1', which is the most important one in the region of small momentum transfers.

J. IIZUKA and A. KLEIN - Phys.Rev. 123, 669 (1961).

In this paper are calculated (in the pole approximation) total cross-sections and c.m.s. distributions of the nucleons for reaction (A4.3), which are compared with experiment from 800 to 1500 MeV. The calculation of the spectra for particle q_1 is made in an approximate way: the "mixed" interference terms (see sub-section 2.5 and DA PRATO's work below) are neglected. The spectra are underestimated by a factor 2, due to an error of normalization.

F. SELLERI - Phys.Rev.Let. 6, 64 (1961).

In this paper the lab. energy distribution of the neutron for reaction (A4.3) is calculated at 970 MeV in the pole approximation. Only diagrams '1' and '2' and their interference are retained (the other terms are about 9 times smaller). Comparison with experiment is made. The calculated spectrum contains a spurious factor $\frac{1}{2}$ due to an error of normalization.

G. DA PRATO - Nuovo Cimento 22, 123 (1961).

This paper contains a full calculation of the peripheral effects for reactions (A4.3) and (A4.4) in the pole approximation. The final energy lab. spectra of the nucleons are calculated and compared with experiment. Also the interferences between diagrams '1' and '3', '2' and '4' are calculated and proved to be small. The effect of the higher resonances ($T = \frac{1}{2}$) in the $\pi - \sqrt{N}$ scattering is taken into account.

E. FERRARI and F. SELLERI - Phys.Rev.Let. 7, 387 (1961).

In this paper the lab. energy spectra of the final nucleons for reactions (A4.3) and (A4.4) are calculated at 970 MeV and 2850 MeV and compared

with the experiment. The effects due to the virtuality of the intermediate pion are taken into account. They are summarized by a cut-off Δ^2 -dependent function, which is shown to be justified by the theory. From this function an evaluation of the "pionic form factor" $K^2(\Delta^2)K'(\Delta^2)$ (see sub-section 2.4) is tried. The mixed interference terms are neglected in the calculation.

E. FERRARI and F. SELLERI - to be published.

In this work the lab. energy spectra of the final nucleons are calculated and compared with experiment for those events having a Q-value of the π^+p system less than 350 MeV. This allows a more accurate application of the complete theory and a better determination of K^2K' . The spectra considered are again the ones at 970 MeV for reactions (A4.3) and (A4.4) and the ones at 2850 MeV for reaction (A4.3) only. Also total cross-sections are calculated in the energy range from threshold to 3 GeV.

NGUYEN CHI HIEN - Carnegie Institute of Technology - Report NYO 9285 (May 1961).

The high-energy part of the energy spectrum of the final protons from reactions (A4.3) and (A4.4) together is calculated in the lab. system at fixed lab. angle in the pole approximation. The isobar approximation with isotropic isobar decay (see Footnote ³⁸) is used for the spectra of particle q_1 . Comparison with experiment is made for $T \sim 1350-1380$ MeV and $\theta_p^L = 2^\circ 37', 3^\circ 38', 4^\circ 30'$. The mixed interferences are neglected and the higher πN resonances are taken into account.

S. DRELL and K. HIIDA - Phys.Rev.Let. 7, 199 (1961).

The high-energy part of the lab spectra of the final protons from reactions (A4.3) and (A4.4) together is calculated at energies of 16-25 GeV and fixed lab. angle. In the kinematical conditions under which comparison with experiment is made ($\theta_p^L = 40$ and 56 milliradians) the only spectrum retained is the one for particle q_1 in graph '1', since all others are shown to be much less important. In fact, a

diffraction cross-section (strongly peaked forward) is used in the $\pi - N$ vertex. A cut-off Δ^2 -dependent function is inserted in order to take into account the off-shell pion effects.

W.J. FICKINGER, E. PICKUP, D.K. ROBINSON, E.O. SALANT -

Preprint BNL 5725 (September 1961).

For reaction (A4.3) the c.m. angular distribution and the lab. energy distribution of the final neutrons are calculated in the pole approximation at 2 GeV. The formula used is the same as in SELLERI's work (Phys.Rev. Let. 6, 64 (1961)). Also the lab. spectrum of the events having a Q-value of the $\pi^+ - p$ system in the Δ -resonance region is calculated and compared with experiment⁵⁵).

V.E. BARNES, D.V. BUGG, W.P. DODD, J.B. KINSON, L. RIDDIFORD -

Phys.Rev.Let. 7, 288 (1961).

In this work reaction (A4.3) is considered at 970 MeV. The lab. spectrum of the neutron is compared with the peripheral formula in the pole approximation, as given by SELLERI (loc. cit.). Also the $\pi^+ - p$ Q-value distribution is compared with the peripheral prediction.

D.S. CHERNAVSKIJ, I.M. DREMIN, I.M. GRAMENITSKI, V.M. MAXIMENKO -

Lebedev Phys. Inst. Report (Moscow) A 27 (1960).

The inelastic proton-proton collisions at 9 GeV are investigated in terms of the peripheral model. The reaction is considered as a two stage reaction $p+p \rightarrow H+H' \rightarrow$ decay products, where H and H' are two "isobars" which are produced by the peripheral mechanism and then decay isotropically. By this way it is possible to deduce angular c.m. distributions and transverse lab. momentum distributions, which are compared with experiment.

D.S. CHERNAVSKIJ, I.M. DREMIN - Lebedev Phys. Inst. Report (Moscow)
A 28 (1960).

The values of the total cross-sections for nucleon-nucleon collisions at 10^{11} eV are calculated in the pole approximation in the region $\Delta^2 \lesssim \delta^2$, for various values of the parameter δ^2 .

III Reactions of strange particle production.

The reactions considered are mostly of the type

but also other types of reactions have been studied.

E. FERRARI - Nuovo Cimento 15, 652 (1960).

The total cross-section for the reaction $p + p \rightarrow p + \Lambda^0 + K^+$ is calculated at 3 GeV in the pole approximation, by considering the exchange of a single pion and a single K-meson separately. The interference term between the 2 possible diagrams is calculated in the single wave approximation (S wave : see sub-section 2.5) and has the wrong sign.

E. FERRARI - Phys.Rev. 120, 988 (1960).

The same type of calculation as in the preceding work is extended to all the four possible reactions of the type (A4.5). The numerical results are wrong by a factor $\frac{1}{2}$ due to incorrect normalization. The interference terms are again calculated in the S-wave approximation, and furthermore have been obtained on the basis of a prescription for the "pole approximation" different from the one given in the present text. The relative $\Sigma - \Lambda$ parity is supposed to be even, the $pK\Lambda$ parity odd.

T. YAO - Phys.Rev. 125, 1048 (1962).

Total cross-sections, angular and momentum c.m. distributions for the various reactions of the type (A4.5) are calculated in the model with the one-pion exchange, and compared with experiment at 2850 MeV. The interference terms are systematically neglected. Also the reactions of the type $p+p \rightarrow N+Y+K+\pi$ are considered. The off-shell effects are taken into account by cutting off the integrals at $\Delta^2 = M^2$ (M = nucleon mass) ; this amounts to assuming $K(\Delta^2)K'(\Delta^2) \approx \theta(M^2 - \Delta^2)$ (θ = step function). This procedure gives unrealistic discontinuities in the calculated c.m. angular distributions.

P.G.O. FREUND and A. FUJII - (to be published in Nuovo Cimento).

In this paper the total cross-section for the reaction $\Lambda^0 + p \rightarrow \Lambda^0 + p + \pi^0$ is calculated by the one-pion exchange model in the pole approximation at a total c.m. energy of 2.5 GeV, under the hypothesis of a S-wave resonance in the $\Lambda\pi$ system.

IV Reactions involving γ -rays.

The most part of the work on this subject concerns double photoproduction of charged pions

$$\gamma + p \rightarrow \pi^+ + \pi^- + p \quad (A4.6)$$

Two types of peripheral diagrams with the exchange of a single pion can describe this reaction. One of them exhibits a 4-particle vertex $(\pi p \pi p)$ and will be called the Drell diagram (see Ref. ¹⁸) the other one exhibits a 4-particle vertex $(\gamma \pi \pi \pi)$ and will be called the Boccaletti-Gualdi (BG) diagram (see Ref. ¹⁵).

D. BOCCALETTI and C. GUALDI - Nuovo Cimento 18, 895 (1960).

The lab. energy distribution of the protons at fixed lab. angle is calculated from the BG diagram at 900 MeV at various angles in the pole approximation. A resonance for the $\pi\pi$ system is assumed in the state $T=J=1$. The spectra contain a spurious peak at 15 MeV (due to an error of computation).

D. BOCCALETTI, G. DE FRANCESCHI and C. GUALDI - Nuovo Cimento 20, 375 (1961)

The lab. energy distribution at fixed angle, and the lab. angular distribution at fixed energy of the negative pions are calculated from the BG diagram in the pole approximation at 900 MeV and different pion energies and angles. Final state interaction between π^+ and p is taken into account. The competing effects of the spectra obtained in this way and the ones predicted by the Drell diagram are discussed.

D. BOCCALETTI and F. SELLERI - Nuovo Cimento 22, 1099 (1961).

The total cross-section for process (A4.6) at energies ≤ 1.3 GeV is calculated in the pole approximation from both the Drell and BG diagrams and compared with experiment.

C. GUALDI - Nuovo Cimento 22, 1313 (1961).

The lab. spectrum of the final protons in reaction (A4.6) from the Drell diagram is calculated at 900 MeV, at fixed lab. angle and in the pole approximation.

F. HADJIOANNOU - (to be published).

This work gives an extensive discussion of the problems of double and multiple photoproduction for which essentially the "generalized" Drell and BG diagrams (in the sense that many particles are allowed to come

out with the final nucleons) are assumed to be important. The calculations are performed in the pole approximation, and the relative importance of the various diagrams is discussed. Non-peripheral corrections (deduced by a sort of static model) are also calculated to bring the predictions in agreement with experiment at low energy. Also reactions involving virtual γ -rays (electroproduction of pions, electron pair production in pion nucleon collisions) are investigated in the peripheral approach, in order to attempt a determination of the electromagnetic form factor of the pion.

REFERENCES AND FOOTNOTES

- 1) See for example : E.M. Friedländer, Phys.Rev.Lett. 5, 212 (1960) ;
I.M. Dremin and D.S. Chernavskij, Soviet Phys. JETP 11, 167 (1960) ;
I. Derado, Nuovo Cimento 15, 853 (1960) ;
G. Da Prato, Nuovo Cimento 22, 123 (1961) ;
W.J. Fickinger, E. Pickup, D.K. Robinson, E.O. Salant,
Phys.Rev.Lett. 7, 196 (1961) ;
V. Barnes, D.V. Bugg, W.P. Dodd, J.B. Kinson and L. Riddiford,
Phys.Rev.Lett. 7, 285 (1961) ;
Nguyen Chi Hien, Inelastic proton-proton scattering at 1.3 BeV,
Carnegie Institute of Technology, Report NYO 9285 (May 1961).
- 2) We call "inelastic" any process in which more than four particles
(initial and final) are involved.
- 3) M.J. Moravcsik, Report UCRL 5317 T (August 1958) (unpublished) ;
L.B. Okun and I.Ya. Pomeranchuk, Nucl.Phys. 10, 492 (1959) ;
M.M. McGregor, M.J. Moravcsik, H.P. Stapp, Phys.Rev. 116, 1248 (1959) and
P.S. Signell, Phys.Rev.Lett. 5, 474 (1960).
- 4) L.G. Hyman, R. Ely, D.H. Frisch, M.A. Wahlig, Phys.Rev.Lett. 3, 93 (1959) ;
M. Jacob and J. Mathews, Phys.Rev. 117, 854 (1960) ;
G. Bernardini, A.O. Hanson, A.C. Odian, T. Yamagata, L.B. Auerbach and
I. Filosofo, Nuovo Cimento 18, 1203 (1960).
- 5) J.J. Sakurai, Nuovo Cimento 20, 1212 (1961).
- 6) G.F. Chew and F.E. Low, Phys.Rev. 113, 1640 (1959).
- 7) C. Goebel, Phys.Rev.Lett. 1, 337 (1958).
- 8) F. Bonsignori and F. Selleri, Nuovo Cimento 15, 465 (1960).
- 9) S.D. Drell, Phys.Rev.Lett. 5, 278 and 342 (1960) ;
F. Salzman and G. Salzman, Phys.Rev.Lett. 5, 377 (1960) ; Phys.Rev. 120,
599 (1960) and 121, 1541 (1961) ;
See also : Proceedings of the International Conference on Theoretical
Aspects of Very High-Energy Phenomena, CERN internal Report 61-22 ;
S.D. Drell, Rev.Mod.Phys. 33, 458 (1961) ;
C. Goebel, Proc. 1961 Midwest Conference on Theoretical Physics.

- 10) J.G. Rushbrooke and D. Radojicic, Phys.Rev.Lett. 5, 567 (1960) ;
I. Derado (see Ref. 1)) ;
A.R. Erwin, R. March, W.D. Walker, E. West, Phys.Rev.Lett. 6, 628 (1961) ;
E. Pickup, D.K. Robinson and E.O. Salant, Phys.Rev.Lett. 7, 192 (1961) ;
- 11) T. Kobayashi, Prog.Theor.Phys. 18, 318 (1957) ;
F. Selleri, Phys.Rev.Lett. 6, 64 (1961) ;
G. Da Prato (see Ref. 1)) ;
J. Iizuka and A. Klein, Phys.Rev. 123, 669 (1961).
- 12) E. Ferrari and F. Selleri, Phys.Rev.Lett. 7, 387 (1961).
- 13) Strange particle production in proton-proton collisions has been discussed
by E. Ferrari, Nuovo Cimento 15, 652 (1960) and Phys.Rev. 120,
988 (1960) ;
T. Yao, Phys.Rev. 125, 1048 (1962).
- 14) For a full discussion of the pole approximation, see Ref. 16).
- 15) Double photoproduction in the peripheral approach has been discussed by
D. Boccaletti and C. Gualdi, Nuovo Cimento 18, 895 (1960) ;
D. Boccaletti, G. De Franceschi and C. Gualdi, Nuovo Cimento 20, 375 (1961) ;
D. Boccaletti and F. Selleri, Nuovo Cimento 22, 1099 (1961).
- 16) E. Ferrari and F. Selleri, Nuovo Cimento 21, 1028 (1961) ;
See also : J. Iizuka and A. Klein, Prog.Theo.Phys. 25, 1017 (1961).
- 17) F. Selleri and E. Ferrari, CERN Report 61-22, page 297.
- 18) S.D. Drell, Phys.Rev.Lett. 5, 278 (1960).
- 19) S.D. Drell and K. Hiida, Phys.Rev.Lett. 7, 199 (1961).
- 20) G. Cocconi, A.W. Diddens, E. Lillethun and A.M. Wetherell,
Phys.Rev.Lett. 6, 231 (1961).
- 21) F. Salzman and G. Salzman, CERN Report 61-22, page 283.
- 22) D. Amati, S. Fubini, A. Stanghellini and M. Tonin, Nuovo Cimento
22, 569 (1961).
- 23) We use the metric $p^2 = |\vec{p}|^2 - p_0^2$. In the following, the same symbol
will be used to denote a particle and its 4-momentum.

- 24) See J.M. Jauch and F. Rohrlich, Theory of Photons and Electrons, Addison-Wesley (Cambridge) (1955), p. 164.
- 25) The procedure outlined in this subsection will be similar to the one described in Ref. 6).
- 26) In fact the vertex function $\langle q_2 | j(0) | p_2 \rangle$ has to be intended as a proper vertex function, since the higher order corrections to the propagator are already contained in Δ'_F . A similar agreement applies to $\langle q_1 | j(0) | p_1 \rangle$. See subsection 2.4 for a further discussion on this subject.
- 27) The symbol $P(\vec{r}; -M_2^2, -m^2)$ has obviously to be read as :

$$\frac{1}{2\vec{r}} \left\{ \vec{r}^4 + 2\vec{r}^2(M_2^2 + m^2) + (M_2^2 - m^2)^2 \right\}^{\frac{1}{2}}$$

$$\text{Similarly, } E(\omega; M_1^2, -\Delta^2) = (\omega^2 + M_1^2 + \Delta^2) / 2\omega$$

- 28) When not explicitly stated, the scattering angles will always be considered with respect to $\vec{p}_1$.
- 29) Of course W satisfies the "threshold limitations" $W \geq m_1 + m_2 + m$, $W \geq M_1 + M_2$.
- 30) For a more detailed discussion on this subject, see for example M. Nauenberg and A. Pais, Phys.Rev. 123, 1958 (1961), Sect. II.
- 31) The only important cases which are not represented by a formula of the type (2.68) (or by its counterpart (2.124)) are the Drell term of the double photoproduction (Ref. 18, 15) and the peripheral diagrams in which the intermediate particle is not a spinless boson. Examples of "particles" of such a kind which can give rise to peripheral effects are the experimentally known systems of two or more bosons in a resonant state (bipion, tripion, K^*) which can be fairly well represented by a single spin 1 particle. These particular cases can be handled by use of the general formulae (2.67) or (2.123). The functions V and Δ'_F will look different from the expressions considered in this paper : in any case, they will always contain multiplicative unknown functions of the momentum transfer alone of the same type as K and K' .
- 32) We denote the cross-section for process (2.7) by σ_1 , because we use the notation σ for the cross-section for process (2.6).
- 33) The approximation $K(\Delta^2) = K(-\mu^2) = 1$ for small Δ^2 (and similarly for K') has the same meaning as the approximation on the partial waves for $\Delta^2 \sim \mu^2$. In fact, there are no particular reasons why the form factors should vary very rapidly in the neighbourhood of $\Delta^2 = -\mu^2$; therefore, for small Δ^2 the on-shell approximation should be considered a good one.

- 34) This covers all the possible cases, since we can label by " q_1 " any of the two particles outgoing from the upper vertex in Fig. 1.
- 35) It has to be pointed out that an additional factor $\frac{1}{2}$ has to multiply the integrals (2.96) and (2.99) if there are two identical final particles (unless such a factor is already contained in $\mathcal{G}_1$). This has to be done since one integrates over all final momenta independently and thus counts twice the number of final states. More generally when among the final particles there are M groups of identical particles (in number of $m_1, \dots, m_M$, say) the expression for the total cross-section has to be multiplied by $1/m_1! m_2! \dots m_M!$
- 36) Complicated computer work can be avoided or reduced by using some approximations which will be discussed under Ref. 38). Since the effect of these approximations cannot generally be evaluated, we think that, when possible, the exact integration should be performed.
- 37) Since the dependence on $\mathcal{Q}_2^P$ enters only through the cosine, the integration over $\mathcal{Q}_2^P$ can be performed only between 0 and π . The result must be then multiplied by 2.
- 38) We want to point out that some of the formulae given in this and the following subsection can be found in the papers quoted in Ref. 11), which consider the particular case of single pion production in nucleon-nucleon collisions. Iizuka and Klein give also an approximate evaluation of the spectra for particle q_1 , which is probably good at low energies, but may be less reliable at high energies, especially in the presence of resonances in $\mathcal{G}_1$. Another approximate way for evaluating the spectra for particle q_1 in the presence of a resonance in $\mathcal{G}_1$ has been presented by Nguyen Chi Hien (Ref. 1)) ; the system of the two particles q and q_1 is considered as an "isobar" having a distribution in the "mass" ω proportional to $\mathcal{G}_1(\omega)$. The whole process (2.6) is then described as the superposition of two effects :
- 1) peripheral production of the system "particle q_2 + isobar";
 - 2) decay of the isobar (generally supposed to be isotropic).
- We feel, however, that the non-isotropic isobar decay may considerably change the results obtained in the isotropic approximation, at least in some cases (lab. energy distribution at fixed angle). We suggest therefore that the calculations should be performed with our rigorous formula which are free from too drastic approximations.
- 39) The same question is discussed also in Sect. II of Iizuka and Klein's paper quoted in Ref. 11). Here the smallness of such terms is inferred by phase-space considerations, without referring to the presence of a resonance in $\mathcal{G}_1$.

- 40) Formulae (2.141), (2.146) and (2.149) refer to the case in which the intrinsic parity of the initial and final states in the 4-particle reaction (2.7) is the same ($\mathcal{M}_1$ has then the form (2.70)). For the opposite case ($\mathcal{M}_1$ of the form (2.71)), the same formulae hold with the exchange of the two coefficients

$$g^{(+)} \sqrt{(p_{10}^Q - M_1)(p_{20}^Q - M_2)} \longleftrightarrow g^{(-)} \sqrt{(p_{10}^Q + M_1)(p_{20}^Q + M_2)}$$

The expressions for $g^{(+)}$ and $g^{(-)}$ depend only on the coupling in the 3-particle vertex and stay unchanged.

- 41) G. Cocconi, Phys.Rev. 111, 1699 (1958).
- 42) F. Salzman and G. Salzman, Phys.Rev. 121, 1541 (1961).
- 43) V.B. Berstetski and I. Ya. Pomeranchuk, Soviet Physics, JETP 12, 752 (1961) and Nucl.Phys. 22, 629 (1961).
- 44) It is apparent that the intermediate particle of the graph shown in Fig. 4 has to be considered with opposite charge quantum numbers according to whether we consider it colliding on p_1 or colliding on p_2 . In this sense, the expression "incident particle with 4-momentum $-k$ " means "incident antiparticle with four momentum $+k$ ".
- 45) I. Dremin and D.S. Chernavskij, Lebedev Physical Institute, Academy of Sciences of the USSR (Moscow), Report A-28 (1960).
- 46) F. Salzman and G. Salzman, CERN Report 61-22, 283 (Aug. 1961).
- 47) R. Hagedorn, CERN Report 61-22, 183 (Aug. 1961).
- 48) Cf. e.g., (2.85). The Q -system is obviously defined as the system in which $\vec{Q}_1 + \vec{Q}_2 + \dots + \vec{Q}_n = 0$
- 49) The limitations (A1.5) and (A1.6) can be obtained also in the B-system, by a more complicated procedure involving also condition II. We prefer to obtain these results in the most straightforward way.
- 50) The lower limits are generally valid only if $m_2 \gg M_2$ and $m_1 \gg M_1$ respectively, and if the "final mass" is larger than the "initial mass". The same consideration applies also to (A1.20) and (A1.21) with the conditions $m_2 \geq M_1$, $m_1 \geq M_2$.

- 51) A simple approximate expression of the $\omega^2 - v^2$ limitation for low Δ^2 is found in Ref. 42).
- 52) In Appendix 1 the specification $\omega_{\min}^2 = (m_1 + m)^2$ has been made, because the particular case of three final particles was handled. Since the agreement can be applied also to many-particle final states, here we use a more general notation. Of course, $\omega_{\min}$ is given by the sum of the masses of all final particles except q_2 .
- 53) This formula looks meaningless for $m_2 = 0$. In fact it is easily seen from (A3.5) that, if $m_2 = 0$, $\omega_0 = W = \omega_{\max}$ and then (A3.12) never applies.
- 54) Since this Appendix has been added after completion of the text, there will be quoted also some more recent works which have not been mentioned in the Introduction.
- 55) Some results of this paper are reported also in W.J. Fickinger, E. Pickup, D.K. Robinson, E.O. Salant, Phys.Rev.Let. 7, 196 (1961).