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REGGE TRAJECTORIES FOR A CLASS OF WEAK SHORT RANGE POTENTIALS

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A B S T R A C T

The movement of the poles in the S matrix for non-relativistic scattering by a potential of the form $U(r) = \alpha \mu^{1-\beta} \frac{e^{-\mu r}}{r^{1+\beta}}$ with small coupling constant is investigated to first order in α . It is found that for $\beta \neq 0$ a second kind of trajectory appears which is not present for the Yukawa potential. For $|K|^2 \ll \mu^2$ the picture of trajectories shows periodicity with respect to β .

*) Most part of this work has been performed during a stay of the author at the Institute of Theoretical and Experimental Physics, Moscow, USSR.

Recently the concept of Regge poles has been quite successful in relativistic theory explaining high energy phenomena in terms of low energy resonances ¹⁾. Nevertheless, there are still a lot of open questions and it is of interest to know as much as possible about the behaviour of trajectories in non-relativistic theory. A number of investigations of this subject has already been performed ²⁾ especially for Coulomb ³⁾, Yukawa ^{4),6),11),12)} and square well potentials ⁵⁾. Azimov, Anselm and Shekhter ⁶⁾ have shown that for a Yukawa potential with weak coupling, to first order in the coupling constant, it is possible to follow the trajectories without numerical calculations just using the properties of Legendre functions. In this paper their method is extended for the potential

$$U(r) = \alpha \mu^{1-\beta} \frac{e^{-\mu r}}{r^{1+\beta}}.$$

The method consists in constructing the trajectory in parts by solving the transcendental equation for the poles of the S matrix using suitable approximations according to the values of K^2 and ℓ . Then one builds up the whole trajectory using the fact that the validity of the different approximate equations overlaps in the intermediate regions.

The S matrix for potential scattering can be represented in terms of the wave function in the form

$$S_e = e^{2i\delta_e} = \frac{K \cos \ell\pi - e^{i(\ell+\frac{1}{2})\pi} \int_0^\infty j_\ell(Kr) U(r) \psi_e(r) dr + \int_0^\infty j_{-\ell-1}(Kr) U(r) \psi_e(r) dr}{K \cos \ell\pi - e^{-i(\ell+\frac{1}{2})\pi} \int_0^\infty j_\ell(Kr) U(r) \psi_e(r) dr + \int_0^\infty j_{-\ell-1}(Kr) U(r) \psi_e(r) dr} \quad (1)$$

$$j_\ell(x) = \sqrt{\frac{\pi x}{2}} J_{\ell+\frac{1}{2}}(x).$$

2.

For small potentials we can use first order perturbation theory

$$\psi_l(r) \approx j_l(kr). \quad (2)$$

Substituting (2) in Equ. (1) we find as a necessary condition for the poles of the S matrix

$$\begin{aligned} \frac{1}{k} \int_0^{\infty} j_l^2(kr) U(r) dr &= \frac{1}{k} e^{i(l+\frac{1}{2})\pi} \int_0^{\infty} j_{-l-1}(kr) j_l(kr) U(r) dr + \\ &+ e^{i(l+\frac{1}{2})\pi} \cos l\pi. \end{aligned} \quad (3)$$

We consider the potential

$$U(r) = 2mV(r) = \alpha \mu^{1-\beta} \frac{e^{-\mu r}}{r^{1+\beta}}. \quad (4)^*$$

In this case we can use the theory of Laplace transforms⁹⁾ and find inserting (4) into (3)

*) For the transition to the Coulomb potential it should be recalled that α is actually an abbreviation for $\alpha = \frac{2m}{\mu} \left(\frac{\mu}{m}\right)^\beta \cdot \alpha'$. $U(r)$ is the potential times the mass.

$$\begin{aligned}
& \frac{\alpha \sqrt{\pi}}{2^{1+\beta}} \frac{\Gamma(\ell+1-\frac{\beta}{2}) \Gamma(\ell+\frac{3}{2}-\frac{\beta}{2})}{[\Gamma(\ell+\frac{3}{2})]^2} \left(\frac{\mu}{K}\right)^{1-\beta} \left(\frac{K}{\mu+2iK}\right)^{2\ell+2-\beta} f_1(\ell, \beta, K) - \\
& - \frac{\alpha \pi}{2} e^{i\pi(\ell+\frac{1}{2})} \frac{\Gamma(1-\beta)}{\Gamma(\ell+\frac{3}{2}) \Gamma(\frac{1}{2}-\ell)} \left(1 + \frac{2iK}{\mu}\right)^{-1+\beta} f_2(\ell, \beta, K) = \\
& = e^{i(\ell+\frac{1}{2})\pi} \cos \ell \pi.
\end{aligned} \tag{5}$$

The functions f_1 and f_2 are the so-called Appell functions ¹⁰⁾.

$$f_1(\ell, \beta, K) = F_2\left(2\ell+2-\beta; \ell+1, \ell+1; 2(\ell+1), 2(\ell+1); \frac{1}{1-i\frac{\mu}{2K}}, \frac{1}{1-i\frac{\mu}{2K}}\right) \tag{6}$$

$$f_2(\ell, \beta, K) = F_2\left(1-\beta; \ell+1, -\ell; 2(\ell+1), -2\ell; \frac{1}{1-i\frac{\mu}{2K}}, \frac{1}{1-i\frac{\mu}{2K}}\right).$$

We see at once that we must exclude the singular potential which behaves like r^{-2} at the origin and restrict β to be smaller than one ($\beta < 1$). Using the reducibility of the Appell functions for particular values of the parameters ¹⁰⁾ one can easily show that the first term in Equ. (5) reduces for $\beta = 0$ to $\frac{\alpha \mu}{2} \frac{1}{K} Q_\ell \left(1 + \frac{\mu^2}{2K^2}\right)$ in accordance with Ref. ⁶⁾.

For $|K| \rightarrow \infty$ the functions f_1 and f_2 tend to constants which can be evaluated from an integral representation for the Appell functions. We see that for $\beta < 1$ due to the factor $(K/\mu)^{-1+\beta}$ the left-hand side of Equ. (5) will go to zero as $K^2 \rightarrow -\infty$ unless ℓ approaches simultaneously the poles of the Γ functions in the numerator of the first term. If the latter is the case, the second term on the left may be neglected and by developing the functions in the neighbourhood of their poles we find as solutions of Equ. (5) the two sets

$$\ell = -1 + \frac{\beta}{2} - n - \frac{\alpha}{\sqrt{\pi}} \frac{\Gamma^2(n + \frac{1}{2} - \frac{\beta}{2})}{\Gamma(n + \frac{1}{2}) n!} \cos \frac{\beta \pi}{2} \cdot 2^{2n-1-\beta} \cdot \left(\frac{\mu}{-iK}\right)^{1-\beta} (-1)^n \left(1 - i \frac{\mu}{2K}\right)^{2n} \cdot F_2\left(-2n; -n + \frac{\beta}{2}, -n + \frac{\beta}{2}; -2n + \beta, -2n + \beta; \frac{1}{1 - i \frac{\mu}{2K}}, \frac{1}{1 - i \frac{\mu}{2K}}\right) \quad (7) *$$

and

$$\ell = -\frac{3}{2} + \frac{\beta}{2} - n - \frac{\alpha}{\sqrt{\pi}} \frac{\Gamma^2(n + 1 - \frac{\beta}{2})}{\Gamma(n + \frac{3}{2}) n!} \sin \frac{\beta \pi}{2} \cdot 2^{2n-\beta} \cdot \left(\frac{\mu}{-iK}\right)^{1-\beta} (-1)^n \left(1 - i \frac{\mu}{2K}\right)^{2n+1} \cdot F_2\left(-2n-1; -n - \frac{1}{2} + \frac{\beta}{2}, -n - \frac{1}{2} + \frac{\beta}{2}; -2n-1 + \beta, -2n-1 + \beta; \frac{1}{1 - i \frac{\mu}{2K}}, \frac{1}{1 - i \frac{\mu}{2K}}\right) \quad (8) *$$

with $n = 0, 1, 2, \dots$

Clearly, if $\beta = 0$ or any negative integer number only one of the two sets of solutions exists. For instance, for $\beta = 0$ only the solution (7) exists and it can be shown ¹⁰⁾ to reduce to the simpler form

$$\ell = -1 - n + \frac{\alpha}{2} \frac{\mu}{iK} P_n\left(1 + \frac{\mu}{2K^2}\right)$$

which has been given in Ref. ⁶⁾. If $\beta = -m$ the poles lie at $K^2 = -\infty$ in all negative integer points starting from $-\frac{m}{2} - 1$ if m is even and from $-\frac{m+3}{2}$ if m is odd as has also been found in Ref. ⁷⁾.

*) It should be mentioned that an expansion of $F_2(-2n, \dots, \frac{1}{1 - i \frac{\mu}{2K}})$ in powers of $\frac{\mu}{iK}$ contains only even powers, while the corresponding $F_2(-2n-1, \dots, \frac{1}{1 - i \frac{\mu}{2K}})$ contains only odd powers. This means that the set of trajectories (7) behaves asymptotically as $(\frac{\mu}{-iK})^{1-\beta}$ while the set (8) behaves as $(\frac{\mu}{-iK})^{2-\beta}$.

Also for general β the K dependence of the trajectory at large K^2 will be in the form of a polynomial similar to the case $\beta = 0$, since F_2 reduces to a simple polynomial of the order $m - 1$ when its first parameter is a negative number $-m$ and its arguments are equal.

$$\left(1 - i \frac{\mu}{2K}\right)^{2n} F_2\left(-2n, \dots, \frac{1}{1 - i \frac{\mu}{2K}}\right) = \left(1 - i \frac{\mu}{2K}\right) \mathcal{P}_{2n-1}\left(1 - i \frac{\mu}{2K}\right)$$

and

$$\left(1 - i \frac{\mu}{2K}\right)^{2n+1} F_2\left(-2n-1, \dots, \frac{1}{1 - i \frac{\mu}{2K}}\right) = \left(1 - i \frac{\mu}{2K}\right) \mathcal{P}'_{2n}\left(1 - i \frac{\mu}{2K}\right)$$

are polynomials in $\left(1 - i \frac{\mu}{2K}\right)$ of the order $2n$ and $2n + 1$ respectively with the corresponding number of zeros in the interval $+\infty > -K^2 > 0$. For instance, one of these zeros lies at $K = i \frac{\mu}{2}$. Equations (7) and (8) are valid as long as the deviations from the limiting points at $K^2 = -\infty$, $-1 + \frac{\beta}{2} - n$ and $-\frac{3}{2} - \frac{\beta}{2} - n$ respectively, are sufficiently small ^{*)}. For small $|\alpha| \ll 1$, as in our case, this is fulfilled until K approaches $i|\alpha|\mu$.

Hence Eqs. (7) and (8) simply say that for K^2 in the interval $+\infty > -K^2 > \alpha^2 \mu^2$ the trajectories oscillate along the real axis around their limiting points with an amplitude of the order $|\alpha|$.

*) However, as has been remarked by Dr. W.M. Shekhter, in second order in α there appears yet another set of solutions at the points $\ell = -\frac{3}{2} + \beta - \frac{n}{2}$ and in m -th order in α at the points $\ell = -\frac{1}{2} + m \frac{\beta-1}{2} - \frac{n}{2}$. This is connected with the fact that iterating the integral equation for $\Psi_\ell(r)$ each succeeding approximation will contain terms which differ by $r^{1-\beta}$ from the former ones. An investigation of the higher orders is being performed by W.A. Kolkunov (private communication).

As K^2 goes to zero the polynomials increase rapidly and hence the Eqs. (7) and (8) will no longer be valid. For $K^2 \approx 0$ we must use another approximation to solve Equ. (5). We can still learn from (7) and (8) the direction of motion of the pole after it has passed the limiting point for the last time. For sufficiently small K the leading power will dominate and we find that the deviation from the limiting points is proportional to

$$- \alpha (-1)^n \left(\frac{\mu}{-iK} \right)^{2n+1-\beta} \cos \frac{\beta\pi}{2}$$

and

$$- \alpha (-1)^{n+1} \left(\frac{\mu}{-iK} \right)^{2n+2-\beta} \sin \frac{\beta\pi}{2}$$

This means that corresponding poles of the two sets belonging to the same n move in opposite directions. For n odd ($n = 1, 3, \dots$) they move towards each other along the real axis, while they go apart for even n ($n = 0, 2, \dots$) if $\alpha < 0$ and vice versa if $\alpha > 0$. (Compare Fig. I and II.)

For small K^2 the functions f_1 and f_2 tend to one and we can again neglect the second term in Equ. (5) as long as $\ell < -\frac{1}{2}$. We arrive at the equation ^{*)}

$$\xi \ell \left(\frac{K^2}{\mu^2} \right)^{\ell + \frac{1}{2}} = \frac{\cos(2\ell + \frac{1}{2} - \beta)\pi}{\cos \ell \pi} e^{i(\ell + \frac{1}{2})\pi} \quad (9)$$

*) This equation can also be derived inserting the expansion of the Bessel functions for small K directly in Equ. (3).

$$\xi_\ell = -\alpha \frac{\sqrt{\pi}}{2^\beta} \frac{[\Gamma(-\ell - \frac{1}{2})]^2}{\Gamma(-\ell + \frac{\beta}{2}) \Gamma(-\ell - \frac{1}{2} + \frac{\beta}{2})}$$

For negative K^2 we may replace $-K^2$ by κ^2 and rewrite Equ. (9)

$$\xi_\ell \left(\frac{\kappa^2}{\mu^2} \right)^{\ell + \frac{1}{2}} = \frac{\cos(2\ell + \frac{1}{2} - \beta)\pi}{\cos \ell \pi} \quad (9')$$

For ℓ sufficiently large and negative we may approximate ξ_ℓ by a constant. Since α is very small ξ_ℓ is also a small quantity. For $\kappa^2 \sim \mu^2$ the whole left-hand side of Equ. (9') is small and the equation can be solved for ℓ near to the zeros of the right-hand side $\ell = -\frac{1}{2} + \frac{\beta}{2} - \frac{m}{2}$ ($m = 1, 2, \dots$). As κ^2 decreases the left-hand side increases ($\alpha < 0$) and hence ℓ must vary in such a way that the right-hand side is positive and also increases. This gives just the same behaviour as we found above from Eqs. (7) and (8). Hence, we see that we can follow the trajectories continuously as K^2 varies. Let us first consider the case of attraction $\alpha < 0$. For repulsion $\alpha > 0$ only the points with even and odd n interchange. As κ^2 decreases further the poles with odd n move towards each other until they collide. Then they proceed on complex conjugate paths toward $\ell = -\frac{1}{2}$ in accordance with the general result in Ref. ⁸⁾. It is obvious from (9) that ℓ has to become complex as κ^2 decreases since the right-hand side is bounded for real ℓ unless ℓ approaches a half integer point. The paths in complex plane can be found from Equ. (9') approximating the cosine by the corresponding exponential, taking the logarithm of the equation, separating real and imaginary part and eliminating κ^2 . They are found to be approximately the circles

$$\left(\operatorname{Im} \ell \mp \frac{\ln \xi_e}{2\pi}\right)^2 + \left(\operatorname{Re} \ell + \frac{1}{2} - \frac{\beta}{2} + p\right)^2 = \frac{\ln^2 \xi_e}{4\pi^2} + \left(p - \frac{\beta}{2}\right)^2 \quad (10)$$

Here the $\mp$ signs refer to the upper and lower half plane respectively. p indicates the different branches of the logarithm and one easily verifies $2p = n+1$. The circles intersect the real axis at $\ell = -2p - \frac{1}{2} + \beta$ and $\ell = -\frac{1}{2}$. To find the exact point of collision between two poles one has to look for the double points of Equ. (9'). At these points (9') and its derivative with respect to ℓ must be fulfilled simultaneously. As shown in Ref. ¹¹⁾ for the case of a Yukawa potential the collision points between two trajectories give rise to branch points in the function $\ell = \ell(K^2)$ in complex energy plane. The corresponding branch cuts prevent the trajectory from being a closed loop as K^2 varies along a path in the complex energy plane proceeding from $-\infty$ to $+\infty$ along the real axis and then returning to the starting point along the half circle. For $\alpha > 0$ all trajectories have collision points in the physical sheet. As α becomes negative at first the branch point of the first trajectory goes into an unphysical sheet and it becomes a closed loop. (Compare Figs. I and II.) Then, as $-\alpha$ increases the same will also happen for the succeeding trajectories. For $K^2 > 0$ we can again approximate the trajectories in complex ℓ plane starting from Equ. (9) and using the same procedure as for the derivation of (10). In the upper half plane this approximation gives the straight lines

$$\operatorname{Im} \ell = \frac{2\pi(p + \beta/2)}{\ln \xi_e} \left(\operatorname{Re} \ell + \frac{1}{2}\right) \quad (11)$$

which indicate that the trajectories go to infinity as K^2 increases. In the lower half plane we find circles of just half the radius we had in Equ. (10)

$$\left(\text{Im } \ell + \frac{\ln \xi_\ell}{4\pi}\right)^2 + \left(\text{Re } \ell + \frac{1}{2} - \frac{\beta}{2} + \frac{p}{2}\right)^2 = \frac{1}{4} \left\{ \frac{\ln^2 \xi_\ell}{4\pi^2} + \left(p - \frac{\beta}{2}\right)^2 \right\} \quad (12)$$

These circles intersect the real ℓ axis at the points $\ell = -\frac{1}{2}$ and $\ell = -\frac{1}{2} - p + \frac{\beta}{2}$. If $\alpha > 0$ (repulsion) one must only substitute p by $p + \frac{1}{2}$ and ξ_ℓ by $-\xi_\ell$ in (10), (11) and (12) to get the corresponding equations. The trajectories which return to the points $\ell = -1 + \frac{\beta}{2} - n$ will at $K^2 \rightarrow \infty$ again be described by Equ. (7). Near $\ell = -\frac{1}{2}$ the second term on the left-hand side of Equ. (5) becomes important and causes deviations from the above equations. Also ξ_ℓ is there no longer constant.

Let us now consider the second kind of trajectory with even n . These points meet on their way the negative half integer points where the right-hand side of Equ. (9) has a pole. Hence, these trajectories will go at threshold $K^2 = 0$ to the negative half integer points. Let us introduce the notation $\ell = -\frac{1}{2} - m + \lambda$ ($m = 1, 2, \dots$). In the immediate neighbourhood of the negative half integer points we find the equation

$$\text{Re } \lambda = -\frac{1}{\pi} \xi_\ell^{-1} \left(\frac{K^2}{\mu^2}\right)^m \sin \beta \pi \quad (13)$$

However, as K^2 increases the trajectory will go away into the upper half of the complex ℓ plane. A good approximation to Equ. (9) is achieved supposing $\text{Im } \lambda$ to be big and at the same time retaining the pole at $\lambda = 0$.

$$\xi \left(\frac{K^2}{\mu^2} \right)^{-m+\lambda} = \frac{e^{i\pi/\beta}}{1 - e^{2i\pi\lambda}} \quad (14)$$

From Equ. (14) we can immediately reproduce Equ. (13) apart from a factor 2 we lost neglecting the term with the decreasing exponential on the right-hand side of (14). We can solve the equation approximately by

$$\begin{aligned} \operatorname{Im} \lambda &= -2\pi \left(p - \frac{\beta}{2} \right) \ln^{-1} \frac{K^2}{\mu^2} \\ \xi \left(\frac{K^2}{\mu^2} \right)^{-m+\operatorname{Re} \lambda} &= \frac{1}{1 - e^{2i\pi\lambda}} \end{aligned} \quad (15)$$

sufficiently distant from the pole (15) simply becomes a set of straight lines similar to (11)

$$\operatorname{Im} \lambda = \frac{2\pi \left(p - \frac{\beta}{2} \right)}{\ln \xi} (\operatorname{Re} \lambda - m) \quad (16)$$

A remarkable fact is the periodicity of equation (9) with respect to β . It means that, if the potential changes its behaviour at the origin by one power of r , the figures of the trajectories for small K^2 remain the same; only the pictures for attraction and repulsion change roles and the trajectory which starts the furthest to the right disappears.

Using the above equations it is easy to draw the picture of trajectories as it is done for $\beta = \frac{1}{2}$ in Figs. I and II. The new feature in the case $\beta \neq 0$ is the trajectories which go at threshold to half integer points. These trajectories will, however, also be absent if β is any negative integer number.

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FIGURE CAPTIONS

Fig. I : Regge trajectories for the potential

$$U(r) = \alpha \sqrt{\mu} \frac{e^{-\mu r}}{r^{3/2}}$$

in first order in α for the case of attraction ($\alpha < 0$). The parts of the trajectories where $K^2 < 0$ is denoted by a solid line and the parts for $K^2 > 0$ by a broken line.

Fig. II : Regge trajectories for the potential

$$U(r) = \alpha \sqrt{\mu} \frac{e^{-\mu r}}{r^{3/2}}$$

in first order in α for repulsion ($\alpha > 0$). The parts of the trajectories for $K^2 < 0$ are denoted by solid lines and the parts for $K^2 > 0$ by broken lines.

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Fig.1