CM-P00056818

ONE-PHOTON-EXCHANGE PROCESSES AND RADIATIVE CORRECTIONS

V. Gorge^{*)}, M. Locher^{*)}

and

H. Rollnik

CERN - Geneva

ABSTRACT

The Weizsäcker-Williams method is discussed using modern techniques. It is shown how gauge invariance guarantees the vanishing of the longitudinal cross-section for zero momentum transfer. Explicit calculations are carried out for virtual Compton scattering and applied to determine the radiative corrections due to hard photons in muon-electron scattering.

^{*)} Institut für theoretische Physik der Universität Bern.

I. Introduction

The formalism used by many authors ¹⁾ to describe one-pion-exchange processes can be easily generalized to cover also electromagnetic processes, where an exchange of one photon is involved ²⁾. The only new feature arises from the fact that a virtual photon can have a longitudinal as well as a transverse polarization. The corresponding cross-sections are in general quite different and therefore - roughly speaking - the number of unknown functions, to be determined from experiment, is doubled ³⁾. This complication was already discussed in 1936 and led to the invention of the well-known Weizsäcker-Williams Method ⁴⁾ (W.W.M.). In fact as far as the basic ideas are concerned the theory of peripheral collisions only applies the W.W.M. to meson physics. But on the other hand the W.W.M. has gained by the new development. Using Feynman diagram techniques one no longer needs the semiclassical picture used in the papers of 1934 ⁴⁾, and moreover one has a better suited formalism to discuss the influence of the longitudinal part of the exchanged, virtual photon ⁵⁾.

Usually one argues that the Coulomb field of a fast moving particle behaves in a good approximation like a superposition of transverse light waves. But the Coulomb potential can be described by a time-like four vector ⁶⁾ while the transverse polarization vectors are space-like. So it is difficult to understand how they can be even approximately equal. In fact it will turn out with the help of modern covariant formulae that the Weizsäcker-Williams argument only applies to the (essentially nonclassical!) spin-flip part of the electromagnetic field. Therefore one must look for other reasons which allow one to neglect the effect of the longitudinal photons. They will be found in a general property of the photo-production amplitude (lower vertex for Fig. 1) : the longitudinal part of the photoproduction amplitude tends to zero, if the (imaginary) mass of the photon vanishes. This is a situation analogous to those in which mesons undergo peripheral collisions ¹⁾, where the structure of the production vertices is important. As a consequence the region of validity of the W.W.M. is determined by special properties of the production vertices. To

illustrate this point we have calculated the total Compton cross-section for a virtual incoming photon on electrons. This can be done by a straightforward perturbation computation and leads to a generalized Klein-Nishina formula. We have worked out the necessary formulae in detail for they can be used to calculate the radiative corrections due to hard quanta ⁷⁾ for muon-electron scattering. As it is known ⁸⁾ these corrections depend on the kinematical details of the experimental arrangement. Fortunately the muon-electron scattering experiment carried through lately at CERN ¹³⁾ was performed in such a way that our method gives by a simple one dimensional integration the desired corrections. This comes out because only the sum of electron and photon energy was measured by a large shower telescope, and therefore the total cross-section for the virtual Compton scattering is just the relevant quantity.

Now we outline the detailed contents of this paper. We deal in Sections 2 and 3 with a general inelastic process generated by a lepton say, via a one photon exchange, cf. Fig. 1. Section 2 starts with a discussion of the lepton vertex and it is demonstrated that only the spin-flip part of the current matrix element has a transverse character. Then the most general expression for the cross-section is written down obtained after an integration over all internal angles in the W-group ⁹⁾. Due to gauge invariance this expression depends only on two arbitrary functions of the photon mass $\sqrt{-\Delta^2}$ and the total energy W of the produced particles. In Section 3 we discuss the physical meaning of these functions introducing the polarization vectors of a vector particle with imaginary mass ¹⁰⁾ and give a proof of the vanishing of the longitudinal cross-section if the virtuality of the photon tends to zero. This proof is based on gauge invariance and on the assumption that the matrix elements (21) of the electromagnetic current have no singularities as long as all momenta involved correspond to physical particles. Since in electrodynamics one may doubt this assumption because of the infra-red divergences this problem will be discussed in some detail.

Sections 4 through 6 contain the formulae for our special example. Section 4 gives the necessary formulae for the virtual Compton scattering since the results of the trace calculations needed do not seem to be available in the literature. Section 5 contains the application to muon-electron scattering combining our results with the formula given recently by K.E. Eriksson¹¹⁾. We are left with one integration over a certain range of W^2 which could be done by hand. For the sake of convenience we did this integration numerically. The results for the μ -e scattering as well as for the virtual Compton effect are found in Section 6 where we also discuss the range of validity of the Weizsäcker-Williams method.

2. General considerations

Let us consider processes (cf. Fig. 1) where a lepton of mass μ interacts with a possibly strongly interacting particle by a one-photon-exchange (i.e., electromagnetic interaction in lowest order) thereby initiating an inelastic reaction. The corresponding production amplitude can be written in the form

$$T = \frac{ie}{(2\pi)^3} 2\mu \bar{u}(p') \gamma^\nu u(p) \frac{1}{\Delta^2} \langle W | j_\nu | q \rangle \quad (1)$$

where the notation of Fig. 1 has been used and

$$\Delta_\mu = p'_\mu - p_\mu ; \Delta^2 = \vec{\Delta}^2 - \Delta_0^2 ; W^2 = - (q + \Delta)^2 .$$

In (1) we have denoted the contribution of the lower vertex by $\langle W | j_\nu | q \rangle$ omitting for brevity the detailed specification of the final state; thus the bra-vector $\langle W |$ depends not only on the total energy W but also on the individual

momenta and spins of the "W-group". In the same way $|q\rangle$ contains also the spin of the incoming particle of mass m . Moreover a Lorentz invariant normalization of the Hilbert state vectors has been chosen.

For a calculation of $\langle W | j_\nu | q \rangle$ we have to look at the process of Fig. 2 where the virtual particle of mass $\sqrt{-\Delta^2}$ hits the particle q . To avoid the problems connected with the virtual character of the incoming photon the W.W.M. introduces two approximations:

- 1) in the transverse part of $\langle W | j_\nu | q \rangle$ the Δ^2 -dependence is neglected.
- 2) the contribution of the longitudinal polarization is dropped completely.

For a justification of the first approximation one can refer to the occurrence of the photon propagator $1/\Delta^2$ which makes the neighbourhood of the pole $\Delta^2 = 0$ most important. With regard to the second point one has argued ⁵⁾ that the current matrix element

$$\Gamma_{ss'}^\nu = i \bar{u}_s(p') g^\nu u_s(p) \quad (2)$$

is essentially transverse for small Δ^2 . But the current four vector of a particle without magnetization is time-like in classical physics. In accordance with this fact a simple calculation shows that (2) must contain longitudinal components, especially for small Δ^2 . If one evaluates

$$\Gamma_{\mu\nu} = \sum_{s,s'} \left(\bar{u}_{s'}(p') g_\mu u_s(p) \right)^* \left(\bar{u}_s(p') g_\nu u_{s'}(p) \right) \quad (3)$$

which is needed anyhow, one finds the well-known expression for the trace

$$\frac{1}{2} \Gamma_\mu^\mu = \frac{1}{2} \sum_{s,s'} |\Gamma_{ss'}|^2 = \frac{\Delta^2 - 2\mu^2}{2\mu^2} \quad (4)$$

Here the modulus should be understood as the invariant length of the four vectors (2). This formula tells us that for $\Delta^2 \approx 0$ at least one of the vectors (2) is time-like, i.e., longitudinal (cf. the further discussion of the polarization of a space-like photon below). On the contrary transverse polarization vectors are always space-like, even for unphysical photons.

To understand better why the expression (4) can change sign we write down in detail the vectors (2) in that special frame where the final lepton [momentum $p' = (\mu, \vec{0})$] is at rest:

$$\Gamma_{ss'}^\nu = \sqrt{\frac{p^0 + \mu}{2\mu}} \left(\begin{array}{c} \delta_{ss'} \\ \frac{1}{p^0 + \mu} [\vec{p} \delta_{ss'} - i \vec{\sigma}_{ss'} \times \vec{p}] \end{array} \right) \quad (5)$$

The spin-flip part of this expression is transverse, which can easily be understood from the fact that the photon has to carry **away** angular momentum. But this spin-flip matrix element vanishes for $\Delta^2 \rightarrow 0$. On the other hand the non-flip part of the current is longitudinal and has for all values of Δ^2 the invariant length $|\Gamma|^2 = -1$.

Therefore the leptonic vertex cannot be made responsible for the validity of approximation 2). The discussion of the structure of $\langle W | j_\mu | q \rangle$ alone can teach us why and to what extent we are allowed to neglect the contribution of the longitudinal quanta.

In order to avoid clumsy formulae let us pass from the production amplitude (1) to the cross-section integrating over all possible states of the lower ("W") group of particles keeping fixed W and Δ^2 . In other words we shall discuss directly:

$$\frac{\partial^2 \sigma}{\partial \Delta^2 \partial W^2} = N \Gamma_{\mu\nu} K^{\mu\nu} \frac{1}{\Delta^4} \quad (6)$$

6.

Here we have used (3) and introduced the tensor

$$K_{\mu\nu} = \frac{1}{(2\pi)^3} \int \prod_{i=1}^n \frac{d^3 q^{(i)}}{2 q^{(i)}} \delta^4(\Delta + q - \sum_{i=1}^n q^{(i)}) \langle W | j_\mu | q \rangle^* \langle W | j_\nu | q \rangle \quad (7)$$

where the integration is performed over all final momenta of Fig. 2. The factor N containing kinematical factors and the electric charge is given by

$$N = \frac{\pi}{4} \frac{\mu^2}{(U K_U)^2} \frac{e^2}{4\pi} \quad (8)$$

where U is the total energy in the over-all centre-of-mass frame and K_U is the corresponding CMS momentum.

Equation (6) is the exact analogue of expressions which have been written down frequently for the exchange of one boson with vanishing spin¹⁾. We have only to study the complications due to the tensorial character of $K_{\mu\nu}$. Evidently $K_{\mu\nu}$ is hermitian and can depend only on the momenta q_μ and Δ_μ , so the most general expression for $K_{\mu\nu}$ is given by⁹⁾

$$K_{\mu\nu} = c_1 g_{\mu\nu} + c_2 q_\mu q_\nu + c_3 \Delta_\mu \Delta_\nu + c_4 q_\mu \Delta_\nu + c_4^* q_\nu \Delta_\mu \quad (9)$$

The coefficients c_i are functions of the invariants

$$\Delta^2 \text{ and } W^2 = M^2 - \Delta^2 - 2q \cdot \Delta$$

and $c_{1,2,3}$ are real. We have assumed invariance under space reflection thus rejecting a term of the form

$$\epsilon_{\mu\nu\sigma\tau} \Delta^\sigma q^\tau.$$

Important restrictions result from gauge invariance:

$$\Delta^\mu K_{\mu\nu} = K_{\mu\nu} \Delta^\nu = 0. \quad (10)$$

These equations impose the following conditions on (9)

$$c_4 = \text{real}$$

and

$$\left. \begin{aligned} c_1 + c_3 \Delta^2 + c_4 (\Delta q) &= 0 \\ c_2 (\Delta q) + c_4 \Delta^2 &= 0 \end{aligned} \right\} \quad (11)$$

Therefore we are left with only two arbitrary functions of W and Δ^2 which are related to the total cross-sections of the transverse and longitudinal photons.

We shall take c_1 and c_3 as independent functions. The reason for this choice is the possibility to choose c_1 and c_3 freely also in the case $\Delta^2 = 0$ of a real photon without introducing singularities in some of the functions c_i . Now from (9) and (11) it follows that

$$\begin{aligned} K_{\mu\nu} = & c_1 g_{\mu\nu} + \frac{\Delta^2}{(\Delta q)^2} [c_1 + c_3 \Delta^2] q_\mu q_\nu + \\ & + c_3 \Delta_\mu \Delta_\nu - \frac{1}{\Delta q} [c_1 + c_3 \Delta^2] (\Delta_\mu q_\nu + \Delta_\nu q_\mu). \end{aligned} \quad (12)$$

Only the first two terms contribute to (6) and in general to all quantities which are gauge invariant and hence of physical significance. But we shall show in detail that it is not appropriate to choose the coefficient of $q_\mu q_\nu$ in (12) as an independent function.

We conclude this section by stating two simple definiteness properties of the functions c_1 and c_3 . Because of (7) the tensor $K_{\mu\nu}$ is positive definite, more precisely the inequality

$$h^\mu K_{\mu\nu} h^\nu \geq 0$$

holds for all vectors n_μ space-like or time-like. Necessary and sufficient conditions for $K_{\mu\nu}$ to fulfil this condition are given by

$$C_1(w^2, \Delta^2) \geq 0 \quad (13a)$$

$$m^2 C_1(w^2, \Delta^2) + [(\Delta q)^2 + m^2 \Delta^2] C_3(w^2, \Delta^2) \geq 0. \quad (13b)$$

The last condition is obtained by putting n_μ equal to q_μ .

3. Transverse and longitudinal cross-sections

In order to give a physical interpretation of the relations derived in the last sections we introduce the polarization vectors of a virtual, space-like photon ¹⁰⁾. These are defined as the three mutually orthogonal four vectors

$e_\mu^{(\lambda)}(\Delta)$ which obey the Lorentz condition

$$e_\mu^{(\lambda)}(\Delta) \Delta^\mu = 0 \quad \text{and} \quad e^{(\lambda)} \cdot e^{(\lambda')} = 0 \quad \text{for } \lambda \neq \lambda'. \quad (14)$$

Because of $\Delta^2 > 0$ (space-like) only two of the $e^{(\lambda)}$ can be space-like, while the third must be time-like. The space-like polarizations can be chosen orthogonal to $\vec{\Delta}$, for brevity we use the symbol $e_{\perp}$ for both of them.

$$e_{\perp}^2 = +1 \quad (\text{transverse polarizations}). \quad (15a)$$

As a consequence the time-like polarization vector must be longitudinal so that

$$e_{\parallel}^2 = -1 \quad (\text{longitudinal polarization}). \quad (15b)$$

In addition the orthogonality properties

$$\vec{e}_{\perp} \cdot \vec{\Delta} = 0 \quad ; \quad \vec{e}_{\parallel} \parallel \vec{\Delta} \quad (15c)$$

are invariant as long as we restrict ourselves to Lorentz transformations which do not change the direction of $\vec{\Delta}$. Therefore we can use (15c) for Lab- as well as for the CMS-system. For further use we write down $e_{\parallel}$ explicitly as it follows from (14) and (15b, c):

$$e_{\parallel} = \frac{1}{\sqrt{\Delta^2}} \left(\Delta^0 \frac{\vec{\Delta}}{|\vec{\Delta}|} , |\vec{\Delta}| \right). \quad (15d)$$

With the help of these notions the total cross-section for the process of Fig. 2 is given by

$$\sigma_{\lambda}(W^2, \Delta^2) = \frac{\pi^2}{KW} e_{(\lambda)}^{\mu} k_{\mu\nu} e_{(\lambda)}^{\nu} \quad (16)$$

if the incoming virtual photon has the polarization $e_{(\lambda)}$. In (16) a flux factor appropriate to our Lorentz invariant normalization has been used. K is the incoming CMS-momentum corresponding to the total energy W and the photon mass $\sqrt{-\Delta^2}$. Equation (16) gives the physical interpretation of the coefficients c_i in (9), because from (15c) it is proportional to the transverse cross-section

$$C_1(W^2, \Delta^2) = \frac{kW}{\pi^2} \sigma_{\perp}(W^2, \Delta^2). \quad (17)$$

By the assumed parity conservation the cross-sections for both transverse polarizations are equal. For the longitudinal cross-section one gets

$$\frac{kW}{\pi^2} \sigma_{\parallel}(W^2, \Delta^2) = C_2(W^2, \Delta^2) \beta - C_1(W^2, \Delta^2) \quad (18)$$

where

$$\beta = (e_{\parallel} \cdot q)^2 = \frac{1}{\Delta^2} [(\Delta q)^2 + m^2 \Delta^2] = \frac{k^2 W^2}{\Delta^2}.$$

Note the minus sign in (18) which is a consequence of (15b). In view of (17) and (18) the inequalities (13a, b) only contain the trivial statement that the transverse and longitudinal cross-sections must be positive.

Let us re-write (18) in terms of c_1 and c_3 analogous to (12).

$$\frac{kW}{\pi^2} \sigma_{\parallel}(W^2, \Delta^2) = \frac{\Delta^2}{(\Delta q)^2} [m^2 C_1(W^2, \Delta^2) + k^2 W^2 C_3(W^2, \Delta^2)]. \quad (19)$$

We claim that one is allowed to conclude from this equation that the longitudinal cross-section tends to zero with vanishing Δ^2 , i.e.:

$$\sigma_{\parallel}(W^2, \Delta^2) \rightarrow 0 \quad \text{for} \quad \Delta^2 \rightarrow 0 \quad \text{and} \quad \Delta q \neq 0. \quad (20)$$

The condition $\Delta q \neq 0$ only states that the total energy W must remain different from m for $\Delta^2 = 0$.

To prove (20) we have only to convince ourselves that c_1 and c_3 do not contain a pole $1/\Delta^2$ which could compensate the factor Δ^2 in front. For c_1 this follows directly from (17): in the limit $\Delta^2 \rightarrow 0$ c_1 is proportional to the photoproduction cross-section for real photons and thus has to be finite.

For c_3 we have to remember the connection of $K_{\mu\nu}$ with the current matrix element corresponding to Fig. 2. We write it now in a more detailed form:

$$\langle W ; q_1, \dots, q_n \text{ out} \mid j_\mu \mid q \rangle . \quad (21)$$

Here we have to understand j_μ as the renormalized current operator. Now it is generally assumed that the matrix element (21) makes sense for all physical, incoming or outgoing Hilbert states. Moreover this can be proved in any renormalizable perturbation theory. (For an exception see below !). But one can certainly find a set of momenta $q, q_1 \dots q_n$ such that

$$(q_1 + \dots + q_n - q)^2 = \Delta^2 = 0.$$

Therefore (21) must be finite for $\Delta^2 = 0$ and hence also the tensor $K_{\mu\nu}$ and its coefficient functions c_i ($i = 1, \dots, 4$). The fact that the c_i 's approach a finite value for $\Delta^2 \rightarrow 0$ does not necessarily mean that they have no singularity at this point. They can very well have branch points due to the inset of new channels.

Unfortunately this consideration does not apply to quantum electrodynamics. Because of the infra-red divergences matrix elements like (21) are not meaningful even after renormalization of the ultra-violet divergences ^{*)}. To overcome this difficulty one can use the thorough work of several authors ¹²⁾. According to them the infra-red part of (21) can be factored out in an explicit way. To make it disappear completely from $K_{\mu\nu}$ we only need to add to the process of Fig. 2 all those processes which have in addition photons of low energy.

*) One of the authors (H.R.) is indebted to Professor B. Stech, Heidelberg, for stressing the importance of the infra-red problem.

One ends up with a new $\tilde{K}_{\mu\nu}$ which depends on an energy resolution ΔE but contains no infinities any longer. To the thus gained coefficients $\tilde{c}_i$ our previous remarks apply. We conclude that relation (20) is based on firm grounds. It has moreover a very simple interpretation: for Δ^2 going to zero the longitudinal polarization vector $e_{||}^\mu$ gets more and more parallel to Δ^μ (in a four-dimensional sense) and therefore due to the continuity equation (10)

$$e_{||}^\mu K_{\mu\nu} e_{||}^\nu$$

tends to zero.

We conclude this section with the explicit formulation of the two assumptions of the Weizsäcker-Williams method.

$$1) \quad \sigma(W^2, \Delta^2) \approx \sigma(W^2, 0)$$

$$2) \quad \sigma_{||}(W^2, \Delta^2) \approx 0.$$

Therefore the cross-section for the process of Fig. 1 is given by

$$\frac{\partial^2 \sigma}{\partial W^2 \partial \Delta^2} = \frac{1}{\pi^2} \frac{N}{\Delta^4} \frac{\sigma_1(W^2, 0)}{2\mu^2} \left\{ KW(\Delta^2 - 2\mu^2) + \frac{\Delta^2}{KW} \left[2(pq)(p'q') - \frac{1}{2}m^2\Delta^2 \right] \right\}. \quad (22)$$

In this expression the Δ^2 -dependence is known explicitly and one can easily integrate over this variable if necessary. Such an integration corresponds to the integration over the classical impact parameter $b \sim 1/\sqrt{\Delta^2}$ well known in the original formulation of the W.W.M. ⁴⁾. But contrary to the b -integration the range of the Δ^2 -integration is now uniquely defined by the kinematics of the process in question.

However the problem of the range of validity for this approximation remains. This can only be answered by a detailed analysis of the vertex (21). We treat a simple example in the next two sections.

4. The virtual Compton scattering

In this section we evaluate the tensor $K_{\mu\nu}$ for the presumably simplest non-trivial case: namely Compton scattering of virtual photons limited to second order perturbation theory. Unfortunately the results of the necessary trace calculations cannot be found in the literature. For this reason we had to repeat the calculation by taking this more general situation into account.

The two diagrams of Fig. 3 contribute to our process. We introduce the tensor $J_{\mu\nu}$ which is just $K_{\mu\nu}$ before the integration over the final momenta is carried through:

$$J_{\mu\nu} = \frac{1}{2} \sum_{\text{spins}} \langle q', k | j_\mu | q \rangle \langle q', k | j_\nu | q \rangle^* \quad (23)$$

In (23) the summations over the spins indicate a summation over both the electron spins and the polarization of the final (real!) photon. $J_{\mu\nu}$ can easily be expressed in terms of the following two traces

$$Q_{\mu\nu} = \text{Tr} \{ \gamma_\mu (i \not{q}' - m + i\epsilon) \gamma_5 \Lambda(q') \gamma^\nu (i \not{q}' - m + i\epsilon) \gamma_5 \Lambda(q) \}$$

$$S_{\mu\nu} = \text{Tr} \{ \gamma_\mu (i \not{q}' - m + i\epsilon) \gamma_5 \Lambda(q') \gamma_\nu (i \not{q}' - m - i\epsilon) \gamma_5 \Lambda(q) \}$$

where as usual $\Lambda(q) = \frac{-i \not{q} + m}{2m}$.

Their evaluation gives the following result:

$$\begin{aligned} \frac{1}{4} Q_{\mu\nu} &= g_{\mu\nu} \left(m^2 + x - x' + q \cdot q' + \frac{x x'}{m^2} \right) - (q'_\mu q_\nu + q_\mu q'_\nu) + \left(\frac{x}{m^2} - 1 \right) (k_\mu q_\nu + q_\mu k_\nu); \\ \frac{m^2}{4} S_{\mu\nu} &= g_{\mu\nu} (q \cdot q') [x - x' - (q \cdot q' + m^2)] + q_\mu q_\nu x - q'_\mu q'_\nu x' + k_\mu k_\nu m^2 + \\ &\quad + q'_\mu q_\nu [q \cdot q' + x - x'] + q_\mu q'_\nu (q \cdot q') + \\ &\quad + q_\mu k_\nu (q \cdot q' + m^2) - q_\nu k_\mu m^2 + q'_\mu k_\nu m^2 - q'_\nu k_\mu (q \cdot q' + m^2); \end{aligned}$$

where $x \equiv -kq'$ and $x' \equiv -kq$.

In terms of these traces the tensor $J_{\mu\nu}$ is given by

$$4 J_{\mu\nu} = \frac{1}{x^2} Q_{\mu\nu} + \frac{1}{x'^2} R_{\mu\nu} - \frac{1}{xx'} (S'_{\mu\nu} + S_{\nu\mu})$$

where $R_{\mu\nu}$ can be got from $Q_{\mu\nu}$ by the following replacement

$$R_{\mu\nu} = Q_{\mu\nu} \begin{pmatrix} q \leftrightarrow q' \\ k \leftrightarrow -k \end{pmatrix}.$$

To determine the tensor $K_{\mu\nu}$ an integration over the angles between the final momenta q' and k is necessary. The calculation has been carried through after contraction of $J_{\mu\nu}$ with $g_{\mu\nu}$ respectively $q_\mu q_\nu$. Then we obtained the two expressions

$$I_1 = K_{\mu\nu} g^{\mu\nu} \quad I_2 = K_{\mu\nu} q^\mu q^\nu$$

with the following results

$$\begin{aligned} I_1 &= 4 \frac{e^2}{4\pi^2} b \left\{ \frac{W^2 - m^2}{2W^4} + \frac{2(2m^2 - \Delta^2)}{b(W^2 - m^2)} + \frac{L}{2} \left[\frac{W^2 - m^2}{2b} - \frac{2(2m^2 - \Delta^2)}{W^2 - m^2} \right] \right\} \\ I_2 &= 4 \frac{e^2}{4\pi^2} b \left\{ \frac{W^2 - m^2}{4W^2} + \frac{m^2(\Delta^2 - 4b)}{b(W^2 - m^2)} - \frac{L}{2} \left[\frac{m^2(W^2 - m^2)}{4b} + \frac{m^2(\Delta^2 - 4b)}{W^2 - m^2} \right] \right\}. \end{aligned} \quad (24)$$

In (24) we have used the abbreviations

$$b \equiv m^2 - \Delta \cdot q$$

and

$$= \frac{1}{KW} \ln \frac{b + KW}{b - KW} .$$

Comparing this result with equation (9) we arrive at the following formulae for c_1 and c_2

$$\begin{aligned} c_1 &= \frac{1}{2} \left(I_1 + \frac{1}{\beta} I_2 \right) \\ c_2 &= \frac{1}{2\beta} \left(I_1 + \frac{3}{\beta} I_2 \right) \end{aligned} \quad (25)$$

with $\beta = \frac{K^2 W^2}{\Delta^2}$ [compare (18)!]

A discussion of these results can be found in Section 6.

5. Application to the radiative corrections in μ -e-scattering

Fortuitously the formulae given in the previous sections are most convenient to calculate the radiative corrections ⁷⁾ for a muon-electron scattering experiment performed recently at CERN ¹³⁾. In this experiment the sum

$$E = (k_o + q_o')_{\text{LAB}}$$

of the laboratory energies of photon and recoil electron has been measured while the experimental set-up itself integrated over all angles between both particles.

By using energy-momentum conservation one can easily express E in terms of W and Δ^2

$$2mE = W^2 + \Delta^2 + m^2.$$

Thus to get the differential cross-section

$$\frac{d\sigma_{\mu e \rightarrow \mu e \gamma}}{dE}$$

we have only to integrate the expression (6) along a straight line in the $W^2 - \Delta^2$ - plane:

$$\frac{d\sigma_{e\gamma}}{dE} = \pi \frac{e^2}{4\pi} \frac{m\mu^2}{(k_u u)^2} \int_{W^2=m^2}^{W_{\max}^2} \frac{\Gamma^{\rho\nu} k_{\rho\nu}}{\Delta^4} dW^2. \quad (26)$$

The upper limit of the W^2 -integration corresponds to the smallest possible mass Δ^2 of the photon, i.e., the smallest invariant momentum transfer of the muon.

Thus $W_{\max}$ occurs if the muon does not change its direction in the collision, hence

$$W_{\max}^2 = 2mE - m^2 + 2\left[\mu^2 - E_{\mu} E'_{\mu} + \sqrt{(E_{\mu}^2 - \mu^2)(E'^2_{\mu} - \mu^2)}\right]$$

where E_{μ} is the initial and $E'_{\mu} = E_{\mu} + m - E$ the final Lab-energy of the muon.

Because of the pole at $W = m$ in the integrand of (26) [compare eqs. (25) and (24)], the expression (26) is infra-red divergent, as it must be in order to cancel the infra-red divergence of the virtual photons. To arrive at such a cancellation we first extract from (26) the contribution of the pole term, i.e., an expression of the form

$$\int \frac{\text{Residue}}{W^2 - m^2} dW^2$$

and combine it with the results of K. Eriksson¹¹⁾. To do **this** properly we must not forget that we have neglected the possibility that the muon radiates. In other words we have only evaluated the diagram of Fig. 10a) and not that of Fig. 10b). To be consistent we have only to take account of those terms in Eriksson's formulae which correspond to Fig. 10c). [This amounts essentially to a Schwinger correction.] Especially two photon exchange processes do not occur in our approximation and so we do not touch any problems possibly related to them. The result is the following finite contribution to the cross-section

$$P = \frac{e^2}{4\pi^2} \left(\frac{d\sigma}{dE} \right)_{el} \cdot D \cdot \ln \frac{W_{max}^2 - m^2}{2k_u u} \quad (27)$$

where

$$\left(\frac{d\sigma}{dE} \right)_{el} = 2\pi \left(\frac{e^2}{4\pi} \right)^2 \frac{m}{(k_u u)^2} \left[\frac{4(p \cdot q)^2}{\Delta_{el}^4} + \frac{2(p \cdot q)}{\Delta_{el}^3} - \frac{m^2 + u^2}{\Delta_{el}^2} + \frac{1}{2} \right]$$

is the elastic muon-electron cross-section, $\Delta_{el}^2 = 2m(E - m)$ and D stands for

$$D = 2 \left(-1 + \frac{1 + \frac{t}{2}}{\sqrt{1+t}} \ln \frac{\sqrt{1+t} + 1}{\sqrt{1+t} - 1} \right) \quad \text{with} \quad t = \frac{4m^2}{\Delta_{el}^2}$$

Finally the cross-section

$$\frac{d\sigma_{e+\gamma}}{dE} \equiv -\delta(E) \left(\frac{d\sigma}{dE} \right)_{el}$$

is given by

$$\frac{d\sigma_{e+\gamma}}{dE} = P + \pi \frac{e^2}{4\pi} \frac{m\mu^2}{(uk_u)^2} \int_{m^2}^{W_{max}^2} \left\{ \frac{\Gamma^{\mu\nu} K_{\rho\nu}}{\Delta^4} - \frac{\text{Residue}}{W^2 - m^2} \right\} dW^2 \quad (28)$$

where the integrand is now a regular function.

As mentioned in the introduction the remaining integration could be done explicitly, but we preferred to perform it numerically by means of the electronic computer Bull Gamma AET of the University of Berne.

6. Results

a) The virtual Compton scattering

The invariant functions c_1 and c_2 given in (25) and (24) have been plotted in Figs. 4 and 5 over the $W^2 - \Delta^2$ - plane. For convenience c_1 has been given in units of $\frac{2}{\pi} \alpha^2$ while c_2 having the dimension $(\text{energy})^{-2}$ has been given in units of $\frac{2}{\pi} \frac{\alpha^2}{m^2} = \frac{2}{\pi} r_0^2$ (*). Except for a flux factor $c_1(W^2, \Delta^2 = 0)$ gives the cross-section for physical Compton scattering. Therefore $c_1(m^2, 0)$ is finite and proportional to the Thomson cross-section. For all other values of Δ^2 , c_1 exhibits an infinity if W approaches m , because of the presence of the pole in (24). More generally this pole determines the gross structure of the surfaces in Figs. 4 and 5. As a consequence $c_1(W^2, \Delta^2)$ differs more from $c_1(W^2, 0)$, as the total energy W becomes smaller. In Fig. 5 the vanishing of c_2 for vanishing photon mass discussed in Section 3 is evident. Both c_1 and c_2 are monotonic functions in W^2 and Δ^2 , only the curvature of c_1 is opposite to that of c_2 .

According to (17) and (18) the two functions are simply related to the transverse and longitudinal cross-section. These quantities are directly shown in Fig. 6 as functions of the energy W for different values of Δ^2 as a parameter. One clearly sees how the transverse cross-section approaches the physical Compton cross-section. Qualitatively one can understand the features of the curves in the same way as discussed for Figs. 4 and 5. For $\Delta^2 > 0$ and $W = m$ both cross-sections are infinite.

*) Here we use the fine structure constant $\alpha = \frac{e^2}{4\pi}$.

For the function c_1 it is possible to define something like a generalized form factor by

$$F_1 = \frac{c_1(W^2, \Delta^2)}{c_1(W^2, 0)}$$

treating now W^2 as a parameter. F_1 is according to Fig. 7 a rapidly increasing function of Δ^2 . Only for large W does F_1 behave for a while somewhat constant. This increasing character is due to the traces occurring in $K_{\mu\nu}$ (cf. Section 4). The range of the validity of the Weizsäcker-Williams approximation can be picked out very simply from Fig. 8. Here the lines through the origin of the $W^2 - \Delta^2$ - plane correspond to constant F_1 . In order to make an error smaller than 10% with the W.W.M. the kinematics must be such that one remains on the left of the line labelled with $F_1 = 1.1$.

b) Radiative corrections to μ -e-scattering

From Fig. 8 we can see that the W.W.M. does not apply to this problem. For that purpose we have drawn the lines of constant Lab-energy E which are labelled with E . The dash-dotted line gives the region needed for an incident muon energy of 8 GeV, which is evidently outside of the W.W.M. region.

In addition the quantity D of (27) vanishes for $\Delta^2 = 0$, i.e., there is no infra-red divergence and no "soft photon" contribution corresponding to the behaviour of $c_1(W^2, 0)$. Evidently only a comparison of the W.W.M. with respect to the "hard photon" contribution makes sense.

Therefore we have to evaluate (28) exactly. The result is given in Fig. 9. The dashed-dotted line gives the (negative) radiative correction alone, while the total corrected cross-section is indicated by the dashed curve. The corrections are of the order of 10 - 20% for the whole energy range.

As mentioned before we have neglected a possible emission of the photon by the muon. A simple estimate using Bremsstrahlungs-formulae gives for the muon radiation 3% of the elastic cross-section. This astonishingly large contribution is due to the fact that the initially resting electron "dislikes" to radiate in comparison with the fast moving muon¹⁴⁾. In addition one must not forget the interference between the matrix elements for electron and muon radiation which could be of the order of 7% according to our estimates. Again these remarks have no bearings with respect to the difficulties possibly connected with two photon exchange processes.

ACKNOWLEDGEMENTS

One of the authors (H.R.) gratefully acknowledges many discussions with Drs. B.D. Hyams, G. Knop and U. Stierlin on details of the π^-e experiment which stimulated this calculation. He is also indebted to Professor S. Drell for a critical reading of the manuscript.

REFERENCES

- 1) See e.g., the review article by S. Drell, Rev. Mod. Phys. 23, 458 (1961).
The formalism used in our paper is most akin to that of F. and G. Salzman, Phys. Rev. 120, 599 (1960) and 121, 1541 (1961)
- 2) For instance in the theory of electroproduction of mesons one has written down the relevant formulae; see R.H. Dalitz and D.R. Yennie, Phys. Rev. 105, 1598 (1959); S. Fubini, Y. Nambu and V. Wataghin, Phys. Rev. 111, 329 (1958); P. Denner, Phys. Rev. 124, 2000 (1961)
- 3) Parity conservation ensures the equality of the cross-sections for both transverse polarizations.
- 4) C.F. von Weizsäcker, Z. Phys. 88, 612 (1934); E.J. Williams, Kgl. Dansk. Vid. Selsk. 13, No. 4 (1935). Cf. also W. Heitler, The Quantum Theory of Radiation, 3rd edition, app. 6
- 5) P. Kessler, Nuovo Cimento 17, 809 (1960) has already used the Feynman diagram technique in connection with the W.W.M.
- 6) This difficulty cannot be removed by a gauge transformation. See Section 2.
- 7) As usual nowadays we use the "radiative corrections" in a broader sense which includes the contribution of the real hard photons too.
- 8) See e.g. Y.S. Tsai, Nuovo Cimento 16, 370 (1960); Phys. Rev. 120, 296 (1960)
- 9) We must apologize for writing down some simple formulae which are already available in the literature, cf. M. Gourdin, Nuovo Cimento 21, 1094 (1961). We do it for convenience of the reader and to fix our notation.
- 10) Such quantities have been used frequently before, e.g. by Fubini et al. Ref. ²⁾.

- 11) K.E. Eriksson, Nuovo Cimento 19, 1029 (1961). See also A.I. Nikishov, JETP 12, 529 (1961)
- 12) D.R. Yennie, S.C. Frautschi and H. Suura, Ann. of Physics, 13, 379 (1961); K.E. Eriksson, Nuovo Cimento 19, 1010 (1961)
- 13) G. Backenstoss et al., Phys. Rev. Letters 6, 415 (1961)
- 14) Compare W. Heitler, Ref. ⁴⁾.

FIGURE CAPTIONS

- Fig. 1 General form of the one-photon exchange process discussed in this paper.
- Fig. 2 The photoproduction part of Fig. 1.
- Fig. 3 Compton scattering diagrams.
- Fig. 4 The invariant amplitude $c_1(W^2, \Delta^2)$ for virtual Compton scattering over the $W^2 - \Delta^2$ - plane in units of $2 \frac{\alpha^2}{\pi}$.
- Fig. 5 The amplitude $c_2(W^2, \Delta^2)$ in units of $2 \frac{r_0^2}{\pi} = \frac{2}{\pi} \frac{\alpha^2}{m^2}$.
- Fig. 6 The transverse and longitudinal cross-sections for virtual Compton scattering. The dashed line gives the real Compton cross-section.
- Fig. 7 The "form factor" $F_1 = \frac{c_1(W^2, \Delta^2)}{c_1(W^2, 0)}$ as a function of the momentum transfer.
- Fig. 8 Lines of constant F_1 values in the $W^2 - \Delta^2$ - plane.
- Fig. 9 Muon-electron scattering cross-section for a muon lab-energy of 8 GeV.
- Fig. 10 Diagrams for μ -e-scattering.

FIG. 1

FIG. 2

FIG. 3

Fig. 5

Fig. 6

Fig. 7

Fig. 8

Fig. 9

a.

b.

c.

FIG. 10