CM-P00056867

A PHENOMENOLOGICAL DISCUSSION OF INELASTIC COLLISIONS AT HIGH ENERGIES

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A B S T R A C T

Taking advantage of the progress recently made in our knowledge of diffraction scattering, a phenomenological discussion is given of inelastic collisions of strongly interacting particles at very high energies. The method followed is to try out simple forms for the wave function of the inelastic final state and to determine under which conditions they are compatible through unitarity with the known or conjectured properties of diffraction scattering. With the exception of the slow shrinking of the peak, the properties of diffraction are found compatible with very simple inelastic final states having no correlation between secondaries (beyond the obvious one imposed by momentum-energy conservation). Compatibility requires, however, the longitudinal momentum distribution to tail off slowly toward large values of the longitudinal momentum, thereby imposing a slow increase of the average multiplicity with incident energy. This behaviour is in qualitative agreement with observation. The shrinking of the diffraction peak is found to require a certain amount of correlation between secondaries in inelastic collisions, and our discussion suggests which general type of correlation may be the inelastic counterpart of the shrinking phenomenon.

1. INTRODUCTION

Considerable progress has been made recently in the study of small angle elastic scattering of strongly interacting particles at high energies (laboratory energy $\gtrsim 5$ GeV). Experiment has revealed two important features. Firstly, the angular dependence is such that over an angular range where the differential cross-section $d\sigma_{el}/d\Omega$ decreases by more than an order of magnitude, it has the angular dependence

$$(1.1) \quad d\sigma_{el}/d\Omega \propto \exp(-\gamma k^2 \theta^2) = \exp[2\gamma k^2(1-\cos\theta)]$$

with γ independent of θ (k denotes the centre-of-mass momentum). Secondly, at least for proton-proton scattering between 5 and 30 GeV, γ increases slowly with energy. This increase also manifests itself through a slow decrease of σ_{el}/σ_{tot} ¹⁾. Simultaneously and independently, these features have been conjectured on the basis of theoretical considerations which describe the asymptotic form of the elastic scattering amplitude for high energy and low momentum transfer in terms of a Regge pole trajectory, the so-called Pomeranchuk trajectory²⁾.

Experimental work has progressed less on the study of inelastic collisions. Within the framework of field theory and dispersion relations, Amati, Fubini and Stanghellini have approached the problem by introducing a modification of the peripheral model of high energy inelastic collisions. This so-called multi-peripheral model is then found from unitarity to agree with the essential features of elastic scattering mentioned above³⁾.

The present paper discusses inelastic collisions in a more phenomenological way. It tries out very simple forms for the wave function of the final state and determines under which conditions they are compatible through unitarity with the known or conjectured properties of elastic scattering. In addition to experimentally known facts, like the multiplicity being large compared to one and the secondaries having small transverse momenta, our most radical simplification resides in the assumption that there are no correlations between secondaries except the weak and obvious correlation imposed by over-all energy-momentum

conservation. The angular dependence (1.1) is then seen to follow from the large multiplicity. The discussion of the width γ has been made under an additional assumption of factorization between transverse and longitudinal momentum distributions of the secondaries, keeping the transverse momentum distribution independent of longitudinal momentum and incident energy. One then finds that γ either decreases or remains constant for increasing incident energy. The latter alternative, which is closer to reality, only holds for longitudinal momentum distributions which tail off slowly toward large values of the longitudinal momentum, and for multiplicities increasing slowly with incident energy, two conditions which seem experimentally verified. The slow increase of γ detected experimentally can only be explained in our framework by correlations between secondaries. On the basis of two examples, one of which is the Amati-Fubini-Stanghellini model, we make a guess at the general type of correlation which may be effective in this respect.

After discussing in the next Section the relation between elastic and inelastic scattering, we carry out the main analysis in Sections 3 and 4. Section 5 gives an account of the conclusions.

2. ELASTIC SCATTERING AND OVERLAP FUNCTION OF INELASTIC FINAL STATES

Our purpose is to study certain aspects of the relation between elastic scattering and inelastic collisions of strongly interacting particles in the high energy region where transverse momenta are on the average much smaller than the longitudinal momenta in the c.m. (centre-of-mass) system. This relation is a well-known consequence of unitarity of the S-matrix. We neglect spin and statistics of the incident particles, which give rise only to very small effects at high energies. Scattering of the two incident particles is called elastic if it leads to a state which can interfere with the initial state, i.e., to a state consisting of the same two particles. Collisions leading to two other particles or to more than two particles are conveniently classified together as inelastic. The total wave function of the two incident particles in the c.m. system is

$$\psi = \exp(i\mathbf{k} \cdot \mathbf{r}) + \psi_{\text{el.sc.}}$$

with, for the elastically scattered wave $\psi_{\text{el.sc.}}$, the asymptotic form ($kr \gg 1$)

$$(2.1) \quad \psi_{\text{el.sc.}} = (i/2kr) \exp(ikr) \sum_{\ell=0}^{\infty} (2\ell+1) \eta_{\ell}(k) P_{\ell}(\cos\theta)$$

We put $\hbar=c=1$, $\mathbf{k}$ and $-\mathbf{k}$ are the incident momenta, $\mathbf{r}$ the relative coordinate and θ the scattering angle in the c.m. system. The ℓ -th partial wave amplitude has been written $i\eta_{\ell}(k)$. At high energies it is believed to be purely imaginary, (shadow or diffraction scattering), and $\eta_{\ell}(k)$ is then real ^{*)}.

^{*)} In the absence of inelastic collisions $\eta_{\ell}(k) = 1 - \exp(2i\delta_{\ell})$
 where δ_{ℓ} is a real phase shift.

4.

Consider the S-matrix, $S = 1 - T$, and separate from T the 4-dimensional delta function expressing momentum-energy conservation

$$(2.2) \quad \begin{aligned} & \langle \text{final state} | T | \text{initial state} \rangle = \\ & = \prod_{\mu=0}^3 (P_{\text{fin}}^{\mu} - P_{\text{init}}^{\mu}) \langle \text{fin.st.} | t | \text{init.st.} \rangle . \end{aligned}$$

If we denote by $| \underline{k}, -\underline{k} \rangle$ the two-incident-particle state of momenta $\underline{k}, -\underline{k}$, the matrix element of t for elastic scattering is easily found to be for $\underline{k} \equiv | \underline{k} | = k' \equiv | \underline{k}' |$

$$(2.3) \quad \langle \underline{k}', -\underline{k}' | t | \underline{k}, -\underline{k} \rangle = [(\varepsilon_1 + \varepsilon_2)/4\pi k \varepsilon_1 \varepsilon_2] E(k, \theta)$$

with

$$(2.4) \quad E(k, \theta) = \sum_{\ell=0}^{\infty} (2\ell+1) \eta_{\ell}(k) P_{\ell}(\cos \theta)$$

$\varepsilon_1 = (k^2 + m_1^2)^{\frac{1}{2}}$ and $\varepsilon_2 = (k^2 + m_2^2)^{\frac{1}{2}}$ are the energies of the two particles, θ is the angle between $\underline{k}$ and $\underline{k}'$. We use for the two particle states the normalization

$$\langle \underline{k}'_1, \underline{k}'_2 | \underline{k}_1, \underline{k}_2 \rangle = \delta(\underline{k}'_1 - \underline{k}_1) \delta(\underline{k}'_2 - \underline{k}_2) .$$

Of the unitarity condition for the S-matrix

$$(2.5) \quad S^{\dagger} S = 1 \quad \text{or} \quad T + T^{\dagger} = T^{\dagger} T$$

we will now take the matrix element between the states $| \underline{k}, -\underline{k} \rangle$ and $| \underline{k}', -\underline{k}' \rangle$ with $k=k'$. Among the intermediate states occurring in $\langle \underline{k}', -\underline{k}' | T^{\dagger} T | \underline{k}, -\underline{k} \rangle$

we separate the elastic scattering states, i.e., the states $|k'', -k''\rangle$ consisting just of the two incident particles. Denoting by Q the projection operator on their linear manifold, we write (P^μ is the total four momentum operator and $p_0^\mu = \varepsilon_1 + \varepsilon_2, 0, 0, 0$ its value in the state $|k, -k\rangle$)

$$(2.6) \quad (1-Q)S |k, -k\rangle = (1-Q)T |k, -k\rangle = \prod_{\mu=0}^3 \delta(P^\mu - p_0^\mu) |f(k)\rangle$$

This is the inelastic part of the final state for an initial state $|k, -k\rangle$. The matrix element of the unitarity condition becomes after elimination of a four-dimensional momentum-energy delta function

$$(2.7) \quad 2\text{Re} \langle k', -k' | t | k, -k \rangle = \langle f(k') | \prod_{\mu=0}^3 \delta(P^\mu - p_0^\mu) | f(k) \rangle$$

$$+ \varepsilon_1 \varepsilon_2 k^{-1} (\varepsilon_1 + \varepsilon_2)^{-1} \int \langle k'', -k'' | t | k', -k' \rangle^* \langle k'', -k'' | t | k, -k \rangle \delta(k'' - k) d_3 k''$$

or in terms of the function E

$$(2.8) \quad 2\text{Re} E(k, \theta) = 2F(k, \theta) + (4\pi)^{-1} \int d\cos\theta_1 d\cos\theta_2$$

$$E^*(k, \theta_1) E(k, \theta_2) (1 - \cos^2\theta_1 - \cos^2\theta_2 - \cos^2\theta + 2\cos\theta_1 \cos\theta_2 \cos\theta)^{-\frac{1}{2}}$$

where the integration extends over the angles for which the argument under the square root is positive, and where we have put

$$(2.9) \quad F(k, \theta) = \left[2\pi k \varepsilon_1 \varepsilon_2 / (\varepsilon_1 + \varepsilon_2) \right] \langle f(k') | \prod_{\mu=0}^3 \delta(P^\mu - p_0^\mu) | f(k) \rangle$$

In (2.9) as before the vectors $\underline{k}$ and $\underline{k}'$ have the same length and form an angle θ . Eq. (2.8) can be rewritten in partial waves

$$(2.10) \quad \operatorname{Re} \eta_{\ell}(k) - \frac{1}{2} \left| \eta_{\ell}(k) \right|^2 = f_{\ell}$$

with

$$(2.11) \quad F(k, \theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_{\ell} P_{\ell}(\cos \theta)$$

We further note that unitarity of the S-matrix also implies inequalities for $\eta_{\ell}(k)$ and f_{ℓ} . The expectation value of S for the two-incident-particle state of orbital angular momentum ℓ is $1 - \eta_{\ell}(k)$, and consequently

$$(2.12) \quad \left| 1 - \eta_{\ell}(k) \right| \leq 1.$$

In view of (2.10) this implies

$$(2.13) \quad 0 \leq f_{\ell} \leq \frac{1}{2}.$$

We end these formal considerations by giving the well-known expressions for differential elastic, integrated elastic, integrated inelastic and total cross-sections

$$(2.14) \quad \frac{d\sigma_{el}}{d\Omega} = (4k^2)^{-1} \left| \sum (2\ell+1) \eta_{\ell}(k) P_{\ell}(\cos \theta) \right|^2$$

$$(2.15) \quad \sigma_{el} = (\pi/k^2) \sum (2\ell+1) \left| \eta_{\ell}(k) \right|^2$$

$$(2.16) \quad \sigma_{in} = (2\pi/k^2) \sum (2\ell+1) f_{\ell} = (2\pi/k^2) F(k, 0)$$

$$(2.17) \quad \sigma_{tot} = (2\pi/k^2) \sum (2\ell+1) \operatorname{Re} \eta_{\ell}(k) = (2\pi/k^2) \operatorname{Re} E(k, 0)$$

Except for unimportant coefficients, the function F defined in (2.9) has a very simple physical significance. It is the overlap integral for the wave functions of the final states in two inelastic collisions with initial states $| \underline{k}, -\underline{k} \rangle$ and $| \underline{k}', -\underline{k}' \rangle$. Clearly, one of these wave functions is obtained from the other by an over-all rotation of angle Θ . We shall call F the overlap function. Because of its direct physical meaning, one can relate its gross features to the gross properties of inelastic scattering; this will be attempted in the next Section. We first discuss its Θ dependence from what we know about elastic scattering, using (2.8) or (2.10). Here and in the following we limit ourselves to high energies.

At high energies the elastic scattering is strongly peaked forward. A striking feature of the experimental data is the linear behaviour of $\ln(d\sigma/d\Omega)$ in $\cos\Theta$ over a considerable range of variation of $d\sigma/d\Omega$ starting from the forward direction ¹⁾. This suggests the following representation for the function $E(k, \Theta)$

$$(2.18) \quad E(k, \Theta) \simeq E(k, 0) \exp \left[(\cos\Theta - 1) / \delta_k^2 \right] \simeq E(k, 0) \exp(-\Theta^2 / 2 \delta_k^2) .$$

It assumes, as is likely to be approximately true, that $E(k, \Theta)$ is real in the small angular range where it is appreciable. Then, from (2.17),

$$(2.19) \quad E(k, 0) = (k^2 / 2\pi) \sigma_{\text{tot}}$$

From experiment, σ_{tot} is constant for laboratory energies above 5 GeV. As for the width δ_k , it is very small, $\delta_k \ll 1$, and appears to have the form

$$(2.20) \quad \delta_k = (k R_k)^{-1}$$

where the length R_k , which can be regarded as an effective radius for scattering at centre-of-mass momentum k , varies only very slowly with k . The latest experiments on proton-proton scattering have revealed a slow increase of R_k with k ¹⁾. In addition, validity of (2.18), (2.19), (2.20) with constant σ_{tot} and logarithmically increasing R_k^2 is predicted by the very interesting speculations describing high energy scattering by means of Regge pole trajectories, especially the so-called Pomeranchuk trajectory ²⁾.

The form (2.18) corresponds to the following partial wave amplitudes

$$(2.21) \quad \gamma_{\ell}(k) = \frac{1}{2} E(k, 0) \delta_k^2 \exp(-\frac{1}{2} \ell^2 \delta_k^2)$$

This equation is valid for $\ell \gg 1$, $\ell \delta_k$ of order 1. It is obtained by using the equations

$$(2.22) \quad \lim_{\ell \rightarrow \infty} P_{\ell}(1 - u^2/2\ell^2) = J_0(u),$$

$$(2.23) \quad \int_0^{\infty} \exp(-x^2/2) J_0(x\xi) x dx = \exp(-\xi^2/2).$$

From (2.8) or (2.10) one finds the corresponding overlap function

$$(2.24) \quad F(k, \theta) = E(k, 0) \left[\exp(-\theta^2/2 \delta_k^2) - \frac{1}{8} E(k, \theta) \delta_k^2 \exp(-\theta^2/4 \delta_k^2) \right].$$

It contains the dimensionless quantity

$$(2.25) \quad \frac{1}{8} E(k, 0) \delta_k^2 = \sigma_{\text{tot}}/16\pi R_k^2 = \sigma_{\text{el}}/\sigma_{\text{tot}}$$

The last equality results from integrating the differential cross-section over angles, using (2.18).

For proton-proton scattering from 5 to 30 GeV laboratory energy, one has $\sigma_{\text{tot}} \sim 40$ mb, $\sigma_{\text{el}} \sim 10$ mb and $R_k \sim 1.8 \times 10^{-13}$ cm. Hence (2.25) is of order $\frac{1}{4}$. It decreases slowly for increasing energy (σ_{tot} remains constant, R_k^{-1} and σ_{el} decrease slowly). While the last term in (2.24) is only a 25 % correction for $\theta = 0$, its width is larger by a factor $\sqrt{2}$. Our formulae, however, are not to be trusted for too large angles. Actually, from Fig. 1 of Ref. ¹⁾, the validity of (2.18) extends perhaps to angles where

$$(d\sigma/d\Omega)/(d\sigma/d\Omega)_{\theta=0} = \exp(-\theta^2/\delta_k^2) \sim 1/25$$

i.e., where

$$E(k,\theta)/E(k,0) \sim 1/5$$

The range of validity of (2.24) is necessarily smaller and the angular dependence of the last term should therefore not be regarded as well founded, so that this formula is probably only of rather limited significance for energies up to 30 GeV.

If, however, we extrapolate the behaviour of diffraction cross-sections to higher energies by following the predictions of the Regge pole considerations, the range of values of $E(k,\theta)$ over which the angular dependence (2.18) is valid becomes ever larger, and the ratio σ_{el}/σ_{tot} tends to zero (σ_{tot} constant, $\sigma_{el} \rightarrow 0$). At very high energy we then have from (2.24), (2.25), with a very good approximation,

$$(2.26) \quad F(k,\theta) \simeq E(k,\theta) \simeq (k^2 \sigma_{tot}/2\pi) \exp(-\theta^2/2\delta_k^2)$$

and this simple angular dependence of the overlap function, namely linearity of $\log F$ in $\cos\theta$, holds over an angular range in which $F(k,\theta)$ decreases by several orders of magnitude. Furthermore, the width δ_k has the form (2.20), R_k^2 increasing logarithmically with energy. We now proceed to discuss the relation of these remarkable properties with the gross features of inelastic collisions.

3. OVERLAP FUNCTION AND PROPERTIES OF INELASTIC COLLISIONS

Our object is to discuss how the overlap function (2.9) is affected by the main properties of inelastic collisions. We have little to say on the value of $F(k,0)$, which is directly related to the norm of the inelastic final state (2.6); from (2.16) this relation is simply the well-known expression of the inelastic cross-section in terms of the inelastic S-matrix elements. We therefore concentrate on the angular variation of $F(k,\theta)$ which must depend in an essential way on the multiplicity and momentum distribution of the particles in the final state. This dependence being very difficult to discuss in its generality, we shall proceed by studying highly simplified cases, which, while reproducing some of the essential features of the real situation, allow an explicit algebraic discussion of F . The essential features taken into account are the following :

- i) the multiplicities are on the average large compared to one;
- ii) the transverse momentum of the secondaries is on the average small compared to their longitudinal momentum.

By secondaries, we shall understand all particles in the final state. The transverse and longitudinal components of momentum are of course defined relative to the momenta $\underline{k}$, $-\underline{k}$ of the incident particles. The simplified cases to be considered are characterized by the absence of correlations between secondaries, except for the inevitable correlation imposed by energy-momentum conservation. The role and possible nature of further correlation effects is discussed afterwards.

We consider first a very simple case in which we imagine the final state to contain $N \gg 1$ secondaries, all in the same single-particle state, the only correlation between them being the one imposed by total momentum-energy conservation. We assume the secondaries to be identical spinless bosons (to consider different kinds of secondaries would affect very little the subsequent treatment and leave our conclusions unchanged) and write accordingly

$$(3.1) \quad |f(\underline{k})\rangle = \gamma(N!)^{-\frac{1}{2}} \left[\int d\underline{q} \varphi(\underline{q}) A_{\underline{q}}^* \right]^N |0\rangle$$

γ is a constant, $\varphi(\underline{q})$ the single-particle wave function in momentum space, $|0\rangle$ the no-particle state and $A_{\underline{q}}^*$ the creation operator normalized to

$$(3.2) \quad A_{\underline{q}} A_{\underline{q}'}^* - A_{\underline{q}'}^* A_{\underline{q}} = \delta(\underline{q}' - \underline{q}) .$$

As required by the symmetry of the initial state, the function $\varphi(\underline{q})$ is supposed to have cylindrical symmetry around $\underline{k}$. An easy calculation gives for the overlap function (2.9)

$$(3.3) \quad F(\underline{k}, \Theta) = c \int d\underline{q}_1 \dots d\underline{q}_N \left[\prod_{i=1}^N \varphi'^*(\underline{q}_i) \varphi(\underline{q}_i) \right] \prod_{\mu=0}^3 \delta\left(\sum_{i=1}^N \underline{q}_i^\mu - \underline{p}_0^\mu\right) ,$$

which we must study as a function of Θ . c is a coefficient independent of Θ , $\varphi'(\underline{q})$ is defined as the function $\varphi(\underline{q})$ rotated in $\underline{q}$ space by an angle Θ around an arbitrary axis perpendicular to $\underline{k}$. q_i^0 is taken equal to the energy $(|\underline{q}_i|^2 + m^2)^{\frac{1}{2}}$, m being the mass of the secondaries. To allow a separate discussion of the correlation effects imposed by energy-momentum conservation, we shall need the following quantities

$$(3.4) \quad \int d\underline{q} \varphi'^*(\underline{q}) \varphi(\underline{q}) = \phi_\Theta$$

$$(3.5) \quad \phi_\Theta \int \varphi(\underline{q}) = \varphi'^*(\underline{q}) \varphi(\underline{q}) .$$

Then we have

$$(3.6) \quad F(\underline{k}, \Theta) = c \phi_\Theta^N I_\Theta$$

where the correlation effects mentioned above are all contained in

$$(3.7) \quad I_{\Theta} = \int d_{\underline{z}} q_1 \dots d_{\underline{z}} q_N \prod_i \rho_{\Theta}(q_i) \prod_{\mu} \delta \left(\sum_i q_i^{\mu} - p_0^{\mu} \right).$$

In accordance with the cylindrical symmetry of φ we write

$$(3.8) \quad \varphi(\underline{q}) = \varphi(q_t, q_{\ell})$$

where q_{ℓ} is the component of $\underline{q}$ parallel to $\underline{k}$, and $q_t = (|\underline{q}|^2 - q_{\ell}^2)^{\frac{1}{2}}$. We further normalize $\varphi(\underline{q})$ to one, so that $\phi_0 = 1$. One then finds for small angle Θ

$$(3.9) \quad \phi_{\Theta} = 1 - a\Theta^2$$

with

$$(3.10) \quad a = (\pi/2) \int |q_{\ell} (\partial\varphi/\partial q_t) + q_t (\partial\varphi/\partial q_{\ell})|^2 q_t dq_t dq_{\ell}.$$

Hence

$$(3.11) \quad \phi_{\Theta}^N = (1 - a\Theta^2)^N = \exp(-aN\Theta^2)$$

for N large and Θ small such that $aN\Theta^2$ be of order one or smaller. It is interesting that, as a direct consequence of the large multiplicity, the factor ϕ_{Θ}^N in the expression (3.6) of $F(k, \Theta)$ is found to have the exponential dependence on Θ^2 which, according to (2.26), should hold for $F(k, \Theta)$ itself. We now proceed to show that this dependence is not affected by the factor I_{Θ} in (3.6). This can be done under the natural assumption that the single-particle wave function $\varphi(\underline{q})$ leads to correct values for the average energy-momentum

$$(3.12) \quad N \int q_0 |\varphi(\underline{q})|^2 d_{\underline{z}} q = p_0^0 = \mathcal{E}_1 + \mathcal{E}_2, \quad \int \underline{q} |\varphi(\underline{q})|^2 d_{\underline{z}} q = \underline{p}_0 = 0.$$

We also assume for simplicity that for Θ small $\rho_{\Theta}(\underline{q})$ is real positive at all $\underline{q}$ where it is appreciably large. The argument consists in considering N

stochastic four-vectors $q_1^\mu, \dots, q_N^\mu$ with the normalized probability distribution

$$\prod_i \rho_{\Theta}(q_i) \delta \left[q_i^0 - (|\underline{q}_i|^2 + m^2)^{\frac{1}{2}} \right],$$

and in realizing that the probability distribution of $Q^\mu = \sum_i q_i^\mu$ then becomes gaussian for large N . We thus write for this distribution

$$(3.13) \quad P_{\Theta}(Q^\mu) = N^{-2} \alpha_{\Theta} \exp \left[-N^{-1} \sum_{\mu, \nu} \sigma_{\mu\nu}^{\Theta} (Q^\mu - Nq_{\Theta}^\mu)(Q^\nu - Nq_{\Theta}^\nu) \right]$$

with (putting again $q^0 = (|\underline{q}|^2 + m^2)^{\frac{1}{2}}$)

$$(3.14) \quad q_{\Theta}^\mu = \int q^\mu \rho_{\Theta}(q) d_3 q$$

$$(3.15) \quad 2 \sum_{\mu'} \sigma_{\mu\mu'}^{\Theta} \int (q^{\mu'} - q_{\Theta}^{\mu'})(q^\nu - q_{\Theta}^\nu) \rho_{\Theta}(q) d_3 q = d_{\mu\nu}^{\Theta}$$

$$(3.16) \quad \alpha_{\Theta} = \pi^{-2} (\det \sigma_{\mu\nu}^{\Theta})^{\frac{1}{2}}.$$

Note that (3.12) implies $Nq_0^\mu = p_0^\mu$. On the other hand, we have from (3.7)

$$(3.17) \quad I_{\Theta} = P_{\Theta}(p_0^\mu).$$

We thus get from (3.13)

$$(3.18) \quad I_{\Theta} = I_0 (\alpha_{\Theta} / \alpha_0) \exp \left[-N \sum_{\mu, \nu} \sigma_{\mu\nu}^{\Theta} (q_0^\mu - q_{\Theta}^\mu)(q_0^\nu - q_{\Theta}^\nu) \right].$$

Since $q_\Theta - q_0$ behaves as Θ^2 for small Θ , we conclude

$$(3.19) \quad I_\Theta/I_0 = (\alpha_\Theta/\alpha_0) \exp[-bN\Theta^4],$$

b being independent of Θ and N . In the main region of variation of (3.11), which is characterized by the conditions $N \gg 1$, $\Theta \ll \pi$, $N\Theta^2$ of order 1, we can replace (3.19) by one, which means that the correlation effect imposed by momentum-energy conservation has a negligible influence on the angular variation of F . Our analysis thus leads to the simple result ^{*)}

$$(3.20) \quad F(k, \Theta) = F(k, 0) \exp(-aN\Theta^2)$$

As mentioned above, we obtain automatically the correct angular dependence of F for small Θ . We now discuss its width. Comparing (3.20) with (2.26) one obtains for the width

$$(3.21) \quad \int k^2 = (2aN)^{-1}$$

or with (2.20)

$$(3.22) \quad R_k^2 = 2aN/k^2.$$

The next problem is therefore to study the order of magnitude of the coefficient a given by (3.10) and to find out in particular if and when a slow increase of R_k^2 with k can be obtained in the framework of our very simple ansatz for the inelastic final state. It is sufficient for this discussion to consider special, convenient forms for the single-particle wave function $\varphi(\underline{q})$, and we adopt the product form

$$(3.23) \quad \varphi(\underline{q}) = \varphi(\underline{q}_t, \underline{q}_\ell) = \varphi_t(\underline{q}_t) \varphi_\ell(\underline{q}_\ell).$$

*) If we had not imposed (3.12) we would have found for I_Θ/I_0 a dependence in $\exp(-b'N\Theta^2)$, i.e., a modification of the coefficient a in (3.20) by energy-momentum conservation.

We also take into account the empirical fact that most secondaries have longitudinal momenta much larger than their transverse momentum. Barring the unnatural case that φ would have violent oscillations, we can then approximate (3.10) by

$$(3.24) \quad a = (\pi/2) \int q_\ell^2 \left| \partial \varphi / \partial q_t \right|^2 q_t dq_t dq_\ell$$

or with (3.23)

$$(3.25) \quad a = \overline{q_\ell^2} / 8 \kappa^2$$

where we have defined

$$(3.26) \quad \overline{q_\ell^2} = \int_0^\infty q_\ell^2 w(q_\ell) dq_\ell$$

$$(3.27) \quad w(q_\ell) = \left[|\varphi_\ell(q_\ell)|^2 + |\varphi_\ell(-q_\ell)|^2 \right] / \int |\varphi_\ell(q'_\ell)|^2 dq'_\ell$$

$$(3.28) \quad \kappa^{-2} = 2 \int \left| \partial \varphi_t / \partial q_t \right|^2 q_t dq_t / \int |\varphi_t|^2 q_t dq_t .$$

Whereas (3.26) is of course the mean square longitudinal momentum of the secondaries, κ is related to their average transverse momentum in a way which strongly depends on the shape of φ_t . We also define the average magnitude of q_ℓ

$$(3.29) \quad \overline{q}_\ell = \int_0^\infty q_\ell w(q_\ell) dq_\ell .$$

For very high energies $\overline{q}_\ell$ is much larger than the mass of the secondaries and we can approximate the energy conservation condition (3.12) by

$$(3.30) \quad N \overline{q}_\ell = 2k .$$

Substituting (3.25) and (3.30) into (3.22) we obtain for R_k , the effective radius for scattering, the following very simple expression

$$(3.31) \quad R_k^2 = N^{-1} (\overline{q_\ell^2} / q_\ell^2) \kappa^{-2}.$$

As will appear in the next Section, the same type of equation also results from other forms of inelastic final state. It is therefore worthwhile to discuss its general implications. All factors entering (3.31) can in principle vary with incident energy, i.e., with k . Let us consider some rather natural possibilities among the many which present themselves.

i) Assume first that the shape of the longitudinal momentum distribution $w(q_\ell)$ is independent of k , i.e.,

$$(3.32) \quad w(q_\ell) = W(c_k q_\ell)$$

$W(x)$ being independent of k and c_k varying (presumably decreasing) with k . The ratio $f^2 = \overline{q_\ell^2} / q_\ell^2$ is then independent of k . Rewrite (3.31) as

$$\kappa = f N^{-\frac{1}{2}} R_k^{-1}.$$

Using the cosmic ray evidence that at high energies N seems to be proportional to $k^{\frac{1}{2}}$, we find for the transverse momentum parameter the energy dependence

$$(3.33) \quad \kappa \propto k^{-\frac{1}{4}} R_k^{-1} \propto k^{-\frac{1}{4}} (\ln k)^{-\frac{1}{2}}$$

(we have used the logarithmic increase of R_k^2 predicted by the Regge trajectory assumption). This decrease of κ is probably too fast to be reconciled with the cosmic ray evidence for a constant transverse momentum distribution. Also, on intuitive grounds, one would perhaps expect κR_k to remain constant or vary at most very slowly with k .

ii) Our second assumption is therefore to take χR_k independent of k . The shape of the longitudinal momentum distribution must then vary with energy in such a way that

$$f^2 = \overline{q_\ell^2} / \overline{q_\ell}^2 = N R_k^2 \chi^2$$

be an increasing function of k , and that it be proportional to the multiplicity N . We shall now show that very simple distributions $w(q_\ell)$ give an increasing f^2 , and that, remarkably enough, they automatically imply proportionality between f^2 and N . The very fact that f^2 must tend to infinity as $k \rightarrow \infty$ suggests that $w(q_\ell)$ decreases for large q_ℓ with a low power of q_ℓ , thereby emphasizing large q_ℓ much more in $\overline{q_\ell^2}$ than in $\overline{q_\ell}^2$. We try accordingly the simple form

$$(3.34) \quad w(q_\ell) = \begin{cases} c(q_\ell + M)^{-\alpha} & \text{for } 0 \leq q_\ell \leq q_m \\ 0 & \text{for } q_\ell > q_m. \end{cases}$$

The parameter c is determined by the normalization condition

$$\int_0^{q_m} w(q_\ell) dq_\ell = 1.$$

We assume $q_m \rightarrow \infty$ as $k \rightarrow \infty$, experiment suggests to take q_m proportional to k . The parameter M , with the dimension of a mass, is taken independent of k . The actual form of $w(q_\ell)$ for $q_\ell \ll q_m$ is in fact quite irrelevant for our considerations, as long as it is not singular, since both $\overline{q_\ell}$ and $\overline{q_\ell^2}$ have to grow indefinitely with k , so that f^2 will be entirely controlled by the shape of $w(q_\ell)$ for large q_ℓ . An easy calculation gives the following results. From the requirements $\overline{q_\ell} \rightarrow \infty$, $f^2 \rightarrow \infty$ one concludes $1 \leq \alpha \leq 2$; further, neglecting corrections of order M/q_m ,

$$(3.35) \left\{ \begin{array}{ll} \bar{q}_\ell = q_m \left[\ln(q_m/M) \right]^{-1}, \quad f^2 = \frac{1}{2} \ln(q_m/M) & \text{for } \alpha = 1 \\ \bar{q}_\ell = \frac{\alpha-1}{2-\alpha} q_m^{2-\alpha} M^{\alpha-1}, \quad f^2 = \frac{(2-\alpha)^2}{(3-\alpha)(\alpha-1)} (q_m/M)^{\alpha-1} & \text{for } 1 < \alpha < 2 \\ \bar{q}_\ell = M \ln(q_m/M), \quad f^2 = (q_m/M) \left[\ln(q_m/M) \right]^{-2} & \text{for } \alpha = 2 \end{array} \right.$$

Using (3.30) and (3.31), we find for the corresponding multiplicities and effective radii

$$(3.36) \left\{ \begin{array}{ll} N = 2(k/q_m) \ln(q_m/M), \quad \chi^2 R_k^2 = \frac{1}{4} (q_m/k) & \text{for } \alpha = 1 \\ N = 2 \frac{2-\alpha}{\alpha-1} (k/q_m) (q_m/M)^{\alpha-1}, \quad \chi^2 R_k^2 = \frac{1}{2} \frac{2-\alpha}{3-\alpha} (q_m/k) & \text{for } 1 < \alpha < 2 \\ N = 2(k/M) \left[\ln(q_m/M) \right]^{-1}, \quad \chi^2 R_k^2 = \frac{1}{2} (q_m/k) \left[\ln(q_m/M) \right]^{-1} & \text{for } \alpha = 2 \end{array} \right.$$

As stated previously, experiment suggests to take the ratio q_m/k constant. We thus find for $1 \leq \alpha < 2$ increasing N and f^2 with exact compensation to give constant $\chi^2 R_k^2$, whereas for $\alpha = 2$ one has $N \propto k/\ln k$ and $\chi^2 R_k^2 \propto (\ln k)^{-1/2}$. It is amusing that the observed increase of N with k falls within the range here covered; it corresponds to $\alpha = 3/2$. Also the longitudinal momentum distribution (3.34) with α between 1 and 2 is probably compatible with present experimental evidence.

iii) One still has to investigate whether, within the framework here considered, longitudinal momentum distributions can be found for which $\chi^2 R_k^2$ increases with energy. The answer is negative. Indeed, from (3.30) and (3.31) we have

$$(3.37) \quad \kappa^2 R_k^2 = \frac{1}{2} \beta \bar{q}_\ell^2 / (\bar{q}_\ell q_m)$$

where β is defined as the ratio q_m/k . From momentum-energy conservation we have $\beta \leq 1$. Further

$$(3.38) \quad \bar{q}_\ell^2 = \int_0^{q_m} q_\ell^2 w(q_\ell) dq_\ell \leq q_m \int_0^{q_m} q_\ell w(q_\ell) dq_\ell = q_m \bar{q}_\ell.$$

Hence

$$(3.39) \quad \kappa^2 R_k^2 \leq \frac{1}{2}.$$

It thus seems likely that an increase of κR_k can only be produced by correlations between the secondaries. One theory of inelastic scattering where this happens is the field-theoretical model of Amati, Fubini and Stanghellini³⁾. In this model the transverse momentum distribution is independent of energy, while R_k^2 and the average multiplicity increase as $\ln k$. For all secondaries except the fastest ones in the c.m. system the energy distribution is in dq_0/q_0 , i.e., since most secondaries are relativistic, of the form (3.34) with $\alpha = 1$. While the multiplicity increase agrees with (3.36) for $\alpha = 1$, $\kappa^2 R_k^2$ increases as $\ln k$ and is therefore affected by the correlations. One can, of course, imagine many other forms of inelastic final state possessing correlations. A particularly simple type which also produces an increasing κR_k is obtained as follows. Suppose that the inelastic collisions produce a large number of unstable "first generation secondaries", which in turn decay in the observed secondaries. Assume the first generation secondaries to be uncorrelated, with an average transverse momentum κ_1 tending to zero for $k \rightarrow \infty$, and with a longitudinal momentum distribution of type (3.34), α satisfying $1 \leq \alpha < 2$ and q_m being proportional to k . As shown above, we have then $\kappa_1 R_k = \text{constant}$, hence $R_k \rightarrow \infty$ for $k \rightarrow \infty$. The average transverse momentum κ of the observed secondaries, however, is now of the order of the average momentum $\bar{p}$ of the decay

products of a first generation secondary in its rest system, if the energy is high enough to have $K_1 \ll \bar{p}$. Thus $K R_k \rightarrow \infty$ for $k \rightarrow \infty$, and this effect is due to the existence of the first generation secondaries which manifests itself by correlations between the observed secondaries.

It is interesting that the two examples of final state correlations just mentioned have the following general property in common. Take all the secondaries with longitudinal momentum q_ℓ in the interval $\xi q_m \leq q_\ell \leq \xi' q_m$, ξ and ξ' fixed satisfying $-1 < \xi < \xi' < 1$ (q_ℓ is now meant again as the algebraic component of $\underline{q}$ in the incident direction, which can be positive or negative); their total transverse momentum remains finite as $k \rightarrow \infty$. We have perhaps here an indication of what would be the general type of correlation effect responsible for the shrinking of the diffraction peak.

4. OTHER SIMPLE FORM FOR THE INELASTIC FINAL STATE

The discussion of the foregoing Section applies to a wider class of inelastic final states than was considered in (3.1). It is probably the class of states with no or very weak correlations between secondaries ^{*)}. Without claiming full generality, we shall characterize it by means of an ansatz of the following form

$$(4.1) \quad |f(\underline{k})\rangle = \prod_{i=1}^n g_i(A^*) |0\rangle.$$

The $g_i(A^*)$ are functions of the creation operators $A_{\underline{q}}^*$ with the following structure

$$(4.2) \quad g_i(A^*) = \alpha_i + \int_{D_i} d_{\underline{z}\underline{q}} \beta_i(\underline{q}) A_{\underline{q}}^* + \\ + \int_{D_i} \int_{D_i} d_{\underline{z}\underline{q}} d_{\underline{z}\underline{q}'} \gamma_i(\underline{q}\underline{q}') A_{\underline{q}}^* A_{\underline{q}'}^* + \dots$$

where we take for the integration region D_i a cylinder

$$(4.3) \quad q_i \leq q_{\ell} \leq q_{i+1}, \quad q_t \leq \kappa_m.$$

As in (3.8) q_{ℓ} is the component of $\underline{q}$ in the direction of $\underline{k}$, q_t the length of the component transverse to this direction. The q_i divide the over-all range of variation of q_{ℓ} in a large number n of intervals

$$(4.4) \quad -q_m = q_0 < q_1 < \dots < q_i < q_{i+1} < \dots < q_n = q_m,$$

q_m is the maximum of $|q_{\ell}|$ and κ_m the maximum of q_t for all secondaries.

*) We henceforth disregard the obvious correlation imposed by momentum-energy conservation, which was seen not to affect the overlap function in the case studied in Section 3.

All functions in (4.2) are supposed to have cylindrical symmetry around $\underline{k}$. It is the product form of the over-all operator in the right-hand side of (4.1) which expresses the absence of correlations. Note that the state (3.1) used in the previous Section is the N-particle component of

$$(4.5) \quad \exp \left[\int d\underline{q} \varphi(\underline{q}) \underline{A}_{\underline{q}}^* \right] \cdot |0\rangle$$

which is a special case of (4.1), (4.2).

Neglecting the effect of momentum-energy conservation on the overlap function F , and normalizing each g_i so that $g_i(A^*) |0\rangle$ has norm one, we can write

$$(4.6) \quad F(k, \theta) = F(k, 0) \prod_i \phi_i,$$

$$(4.7) \quad \phi_i = \langle 0 | [g_i'(A^*)]^* g_i(A^*) | 0 \rangle$$

where $g_i'(A^*)$ is obtained from (4.2) by a rotation of an angle θ around an arbitrary axis perpendicular to $\underline{k}$. As θ increases from 0 onward, ϕ_i decreases from 1 to $|\alpha_i|^2$. For small angles it can be written

$$(4.8) \quad \phi_i = 1 - a_i \theta^2.$$

This entails immediately the exponential form for F :

$$(4.9) \quad F(k, \theta) = F(k, 0) \exp(-a\theta^2),$$

$$(4.10) \quad a = n^{-1} \sum_i a_i$$

(4.9) holds for n large and Θ small such that $a_n \Theta^2$ be of order one or smaller ^{*}). The similarity of (4.9) and (3.20) is striking.

The angular width of (4.9) is determined by the a_i . From (4.7) and (4.8), assuming $q_{i+1} - q_i \ll |q_i|$, one sees that a_i must have the general form

$$(4.11) \quad a_i = q_i^2 \chi_i^{-2} W_i$$

where χ_i is a measure of the transverse momentum spread of the functions β_i , γ_i in (4.2), and W_i is proportional to the square of their amplitude while depending in a complicated way on their actual shape. The situation becomes simple if, as was done in (3.23), we suppose the longitudinal momentum dependence of the inelastic final state to factor out. If all intervals (q_i, q_{i+1}) have the same length, this amounts to factoring out the i dependence of β_i , γ_i in (4.2) :

$$(4.12) \quad \beta_i(q) = c_i \beta(q^{(i)}), \quad \gamma_i(q, q') = c_i \gamma(q^{(i)}, q'^{(i)})$$

where $q^{(i)}$, $q'^{(i)}$ result from q, q' through translation of $-q_i$ in the longitudinal direction. Through proper normalization we can take

$$(4.13) \quad |c_i|^2 = 1 - |\alpha_i|^2.$$

One now finds simple expressions for a as well as for the average multiplicity N and c.m. energy $2k$ of the inelastic final state

^{*}) The validity of (4.9) requires that the ratio $\sum a_i^\lambda / (\sum a_i)^\lambda$ tends to zero as $n \rightarrow \infty$, i.e., as $k \rightarrow \infty$, for $\lambda = 2, 3, \dots$. This is the case for the longitudinal momentum distributions (3.32), and also for the distributions (3.34) if $\alpha < 2$ and if n increases proportionally to q_m , i.e., if the length of the intervals (4.3) is kept constant as $k \rightarrow \infty$.

$$(4.14) \quad a = n^{-1} \chi^{-2} W \sum_i q_i^2 |c_i|^2$$

$$(4.15) \quad N = W' \sum_i |c_i|^2$$

$$(4.16) \quad 2k = W'' \sum_i |q_i| \cdot |c_i|^2$$

where the "average transverse momentum" χ and the dimensionless numbers W, W', W'' depend only on the functions β, γ . In (4.16) we have approximated by q_i the energy of a secondary created by (4.2).

Clearly, the $|c_i|^2$ play now the role of an unnormalized longitudinal momentum distribution. We define accordingly

$$(4.17) \quad \bar{q}_\ell = \sum_i |q_i| \cdot |c_i|^2 / \sum_i |c_i|^2$$

$$(4.18) \quad \bar{q}_\ell^2 = \sum_i q_i^2 |c_i|^2 / \sum_i |c_i|^2$$

and find after simple substitutions

$$(4.19) \quad N \bar{q}_\ell = 2(W'/W'')k$$

$$(4.20) \quad R_k^2 = 2an/k^2 = (8WW'/W''^2) N^{-1} (\bar{q}_\ell^2 / \bar{q}_\ell^2) \chi^{-2}.$$

Except for the occurrence of W, W', W'' these equations have the same structure as (3.30), (3.31), and our discussion of various possibilities for the k -dependence

of the longitudinal momentum distribution, as presented in the previous Section, can therefore be extended to them. In the spirit of this discussion the only energy variations considered are those of χ and of the longitudinal momentum distribution, so that W, W', W'' should be regarded as constants. Obviously, this simplifying assumption and many others introduced previously could be abandoned, but the corresponding generalizations would be very difficult to discuss in any constructive way as long as one has no more than the scanty experimental information available at present on inelastic collisions.

5. CONCLUSIONS

Despite the very special nature of the inelastic final states we have considered, our analysis suggests a few general conclusions. Although none of them has been (or could at present be) proved with any rigour, they may be significant enough to warrant further consideration. They can be formulated as follows.

If the transverse momentum distribution of the secondaries is independent of their longitudinal momentum and of the incident energy, one can only obtain a diffraction peak of increasing or constant width ^{*)} when one assumes no correlation between secondaries beyond the trivial one imposed by energy-momentum conservation. For the width to be constant the distribution function of longitudinal momentum q_ℓ must be at large q_ℓ a slowly decreasing function of q_ℓ , behaving probably as $q_\ell^{-\alpha}$ with $1 \leq \alpha < 2$ or as some function having a similar rate of decrease. The average multiplicity then increases as $k^{\alpha-1}$ (or $\ln k$ if $\alpha = 1$).

Under the same assumption concerning transverse momenta, a shrinking of the diffraction peak (i.e., an increase of R_k with k) seems to require correlations between secondaries, and suitable such correlations can readily be found. We have mentioned two types (see the end of Section 3). The first is realized in the field theoretical model of Amati, Fubini and Stanghellini, the second occurs when the observed secondaries are the decay products of very short-lived first generation secondaries which are created with decreasing transverse momentum.

^{*)} i.e., R_k decreasing or constant for $k \rightarrow \infty$ in the notation of Section 2.

Experiment has revealed a very slow shrinking of the diffraction peak (for proton-proton scattering, the only case properly studied above 5 GeV laboratory energy), and the theoretical description of diffraction scattering by means of a Regge pole trajectory predicts logarithmic shrinking. The slow nature of the shrinking suggests that the correlations may be rather weak at not too extreme energies, so that the uncorrelated situation may still have significance in first approximation. This would entail as mentioned above the slowly decreasing nature of the longitudinal momentum distribution, roughly of the type $q_p^{-\alpha}$ ($1 \leq \alpha < 2$), and a multiplicity in $k^{\alpha-1}$ (or $\ln k$ for $\alpha = 1$). The correlation effects would come in addition, as a not too drastic modification. At extremely high energies, however, their role may increase and possibly become predominant.

It is interesting that both examples of correlations mentioned above reveal themselves through similar observable consequences on the total transverse momentum of those sets of secondaries which are contained in given intervals of longitudinal momentum : their total transverse momentum is on the average smaller than in the case of uncorrelated particles (see the end of Section 3). This general type of correlation effect seems to be an attractive candidate for the inelastic counterpart of the shrinking effect in diffraction.

The validity of the preceding considerations is amenable to experimental study. One would naturally concentrate on the most common secondaries produced in inelastic collisions, i.e. pions, and one would, for each inelastic collision, determine momentum and charge of every pion produced. In addition to the obvious task of determining the longitudinal momentum distribution, one would look for correlations. In particular one would analyze statistically the sets of pions found in given intervals of longitudinal momentum (these intervals must be taken large enough to contain on the average an appreciable number of pions). As mentioned above, one should study the vector sum of their transverse momenta, while it would of course also be useful to look at other quantities, like the sum of their charges. It is on the latter aspects that the statistical analysis of the data should go beyond the more conventional determination of transverse momentum, angular and multiplicity distributions which are not suitable for the detection of correlations. It may well be, however, that with the accelerators available at present the average multiplicities are still too low to allow very significant work to be done.

R E F E R E N C E S

- 1) For p-p scattering see : A.N. Diddens, E. Lillethun, G. Manning, A.E. Taylor, T.G. Walker and A.M. Wetherell, Phys. Rev. Let. 9, 111 (1962), especially Fig. 1.
- 2) V.N. Gribov, J. Exptl. Theoret. Phys. (USSR) 41, 667 (1961) ; English translation Soviet Physics J.E.T.P. 14, 478 (1962).
G.F. Chew and S. Frautschi, Phys. Rev. Let 8, 41 (1962).
S. Frautschi, M. Gell-Mann and F. Zachariasen, Phys. Rev. 126, 2204 (1962).
- 3) D. Amati, S. Fubini and A. Stanghellini, Nuovo Cimento