Dynamical Tide in Solar-Type Binaries

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ABSTRACT

Circularization of late-type main-sequence binaries is usually attributed to turbulent convection, while that of early-type binaries is explained by resonant excitation of g modes. We show that the latter mechanism operates in solar-type stars also and is at least as effective as convection, despite inefficient damping of g modes in the radiative core. The maximum period at which this mechanism can circularize a binary composed of solar-type stars in $10^{10}$ yr is as low as 3 days, if the modes are damped by radiative diffusion only and g-mode resonances are fixed; or as high as 6 days, if one allows for evolution of the resonances and for nonlinear damping near inner turning points. Even the larger theoretical period falls short of the observed transition period by a factor two.

Subject headings: stars: binaries: close, spectroscopic—stars: oscillations

1. Introduction

Coeval spectroscopic binaries show an abrupt transition between circular orbits at short periods and eccentric orbits at longer periods. Furthermore it appears that $P_{\text{circ}}$ increases with age among main-sequence binaries with roughly solar-mass components (Mathieu et al. 1992).

These observations are explained by dissipative tidal interactions. In eccentric binaries, the tide felt by each star is time-dependent in a frame rotating with the fluid. This is also true when the stellar rotation is not synchronous with the orbit. Dissipation tends to damp the variable part of the tide at the expense of orbital or rotational energy, leading to circularization and synchronization. Since the tidal force depends strongly on the distance between the stars—and the tidal dissipation even more strongly—only short-period binaries synchronize and circularize.

Zahn and collaborators have extensively analyzed two plausible mechanisms for tidal dissipation (cf. Zahn 1992 for a review and more complete references). The “Theory of the Equilibrium Tide” assumes instantaneous hydrostatic equilibrium in the tidal potential and attributes dissipation to turbulence in convection zones (Zahn 1966, Zahn 1977). The theory
has been confirmed and roughly calibrated by comparison with binaries containing giant stars (Verbunt & Phinney 1996). An extension of the theory explains the small residual eccentricities of binary pulsars with evolved companions (Phinney 1992). The predicted transition period for solar-type binaries with ages $\sim 10^{10}$ yr is $P_{\text{circ}} \approx 2.2$ d (Goodman & Oh 1997, henceforth Paper I) if one allows for the fact that the largest eddies turn over more slowly than the binary orbit and hence do not contribute to tidal dissipation (Goldreich & Nicholson 1977, Goldreich & Keely 1977). Because of the latter effect, the empirical calibrations from giant stars cannot be applied directly to the main-sequence case, but even if the inefficiency of slow eddies is ignored, $P_{\text{circ}}$ rises only to 6 d (Paper I).

Notwithstanding theory, observation indicates that $P_{\text{circ}} \gtrsim 11 - 12$ d for local disk and halo dwarfs (Duquennoy & Mayor 1991, Latham et al. (1992)). Because of the very great sensitivity of tidal circularization mechanisms to the orbital period, this is a large discrepancy between observation and theory.

The second of Zahn’s mechanisms, the “Theory of the Dynamical Tide” involves excitation of g modes in the radiative zones of stars [Zahn 1970, Zahn 1975, Zahn 1977]. Since the orbital period is long compared to the dynamical time of the star, the g mode has a short radial wavelength, so that significant coupling between the mode and the tidal potential occurs only at the boundary between radiative and convective zones where the radial wavenumber of the g mode vanishes. This mechanism has been applied to circularization of early-type binaries and appears to be very satisfactory (Giuricin et al. 1984, Claret & Cunha 1997). These stars have convective cores and radiative envelopes. It is important for the theory that the thermal timescale of the envelope is short, so that the g modes are efficiently damped.

Having concluded in Paper I that the Theory of the Equilibrium Tide is inadequate for main-sequence solar-type binaries, we decided to adapt the Theory of the Dynamical Tide to late-type systems, in which the cores are radiative and the envelopes convective. This appears not to have been done before, perhaps because the damping of g modes in radiative cores is known to be very slow, which appears at first sight to reduce the efficiency of the mechanism sharply (§3).

Since our methods are similar to Zahn’s, we omit many details and emphasize those aspects that are peculiar to the late-type case. §2 treats the excitation of the dynamical tide as if damping causes the g-mode resonances to overlap. In fact the resonances do not overlap unless nonlinear damping is efficient, but the calculation of §2 is easy and provides an upper limit to $P_{\text{circ}}$ if the circularization mechanism depends upon the dynamical tide. §3 considers the (large) corrections that must be made for a finite star with discrete damped modes. Finally, §4 summarizes our results and discusses the prospects for resolving the apparent conflict between theory and observation concerning $P_{\text{circ}}$. 
2. Tidal forcing of g modes

In a frame corotating with the mean motion of the binary orbit, the time-dependent quadrupole potential exerted on star 1 by the epicyclic motion of star 2 is, to first order in the eccentricity ($e$),

$$
\delta \Phi_{\text{ext}}(t, r, \theta, \phi) = e \frac{G M_2}{a^3} r^2 \sqrt{\frac{3\pi}{10}} \times \left\{ Y_{2,2}(\theta, 0) \left[ 7 \cos(\omega t - 2\phi) - \cos(\omega t + 2\phi) \right] - \sqrt{6} Y_{2,0} \cos \omega t \right\},
$$

where $a$ is the semimajor axis of the binary orbit, $\omega \approx \sqrt{G (M_1 + M_2)/a^3}$ is the orbital frequency, and spherical polar coordinates have been chosen with the axis perpendicular to the orbital plane.

Synchronization is faster than circularization because the moment of inertia of the individual stars is small compared to that of the orbit. Therefore we assume that the stars corotate with the mean motion of the orbit, but we neglect coriolis forces so that we may separate variables in spherical polar coordinates. The relative error in the dissipation rate caused by this neglect is expected to be of order unity at most. Consequently the error in $P_{\text{circ}}$ is expected to be small, since the dissipation rate varies as a high power of the period.

The fluid displacement in response to the potential (2) is conveniently divided into a hydrostatic “equilibrium tide” $\vec{\xi}_{\text{eq}}$, which is calculated as though $\omega = 0$, and a residual “dynamical tide” $\vec{\xi}_{\text{dyn}}$. In stably-stratified regions,

$$
\xi_{\text{eq}} = -\frac{\delta \Phi}{d\Phi/dr}, \quad \text{and} \quad \nabla \cdot \xi_{\text{eq}} = 0.
$$

Here $\Phi(r)$ is the potential of the unperturbed spherically-symmetric star, and $\delta \Phi = \delta \Phi_{\text{ext}} + \delta \Phi_{\text{self}}$, where $\delta \Phi_{\text{self}}(r, \theta, \phi)$ is the change in the potential of star 1 due to its distortion. Equations (3) are consequences of the following facts:

(i) In hydrostatic equilibrium, the density and pressure are constant on equipotentials, and hence the entropy is also constant on equipotentials.

(ii) In a stratified region, entropy serves as a lagrangian radial coordinate—that that is, it is conserved by fluid elements and differs from one mass shell to the next.

In unstratified regions (convection zones), the actual fluid displacement does not follow equations (3) in the limit $\omega \to 0$. Where the entropy and molecular weight are uniform, vorticity is conserved. For a nonrotating star, therefore, $\vec{\nabla} \times \vec{\xi} = 0$, whereas (in general) $\vec{\nabla} \times \xi_{\text{eq}} \neq 0$ if $\xi_{\text{eq}}$ obeys equations (3). We regard equations (3) as the definition of $\xi_{\text{eq}}$ and require $\xi_{\text{dyn}} \neq 0$ if a convection zone exists.

For our purposes, we may ignore the selfgravity of both the equilibrium and the dynamical tides. The selfgravity of the dynamical tide is negligible in the convection zone because the mass
of the convection zone of a solar-type star is only a few percent of the stellar mass; and in the radiative core, the dynamical tide takes the form of high-order, low-frequency g-modes, whose natural frequencies are very accurately reproduced in the Cowling approximation because of their short radial wavelength. For the equilibrium tide, one can use the apsidal-motion constant (Claret & Gimenez 1995) to estimate that the error in the tidal dissipation rate caused by neglecting $\delta \Phi_{\text{self}}^{\text{eq}}$ is no more than 10%.

One can show that throughout the star, $\xi_{\text{dyn}}$ approximately satisfies the inhomogeneous linear equation

$$\frac{\partial^2}{\partial r^2} (r^2 \xi_{\text{dyn}}) + \frac{\partial}{\partial r} \left( \frac{d\ln \rho}{dr} r^2 \xi_{\text{dyn}} \right) + \ell (\ell + 1) \left( \frac{N^2}{\omega^2} - 1 \right) \xi_{\text{eq}} = \ell (\ell + 1) \xi_{\text{eq}} - \frac{\partial^2}{\partial r^2} (r^2 \xi_{\text{eq}}).$$  \hspace{1cm} (4)

In addition to the approximations discussed earlier (neglect of Coriolis forces and $\delta \Phi_{\text{self}}$) we have discarded certain terms that are smaller than those displayed by factors of $\omega^2/\omega_{\ast}^2$, since the tidal frequencies of interest are small compared with the dynamical frequency of the star,

$$\omega_{\ast} \equiv \left( \frac{GM_1}{R_1^3} \right)^{1/2}. $$  \hspace{1cm} (5)

Thus equation (4) does not describe p modes, but only g modes. The righthand side of eq. (4) is essentially the curl of the equilibrium tide and serves as a source for the dynamical tide in this formulation.

The square of Brunt-Väisälä frequency,

$$N^2(r) \equiv -\frac{1}{\rho} \frac{dP}{dr} \ln \left( \frac{P^{1/T_1}}{\rho} \right),$$  \hspace{1cm} (6)

is positive and $\sim \omega_{\ast}^2$ throughout most of the radiative core. In the convection zone, we take $N^2$ to vanish, although properly it has a small negative value there, and hence $N^2(r) \to 0$ at the boundary between the core and the envelope, which occurs at $r_c = 0.712R_\odot$ in the solar model of Bahcall & Pinsonneault (1995) (henceforth BP). The run of $N(r)$ derived from their model is shown in Figure 1.

Throughout most of the core, the term $(N/\omega)^2$ on the lefthand side of eq. (4) is very large, since we are concerned with tidal frequencies $\omega \ll \omega_{\ast} \sim N$. Consequently, $\xi_{\text{dyn}}$ has a wavelength $\ll r$ and is accurately described by a WKBJ approximation. The radial wavenumber is

$$k_r \equiv \left[ \ell (\ell + 1) \frac{r^2}{r^2} \left( \frac{N^2(r)}{\omega^2} - 1 \right) \right]^{1/2}.$$  \hspace{1cm} (7)

Near $r = r_c$, the wavenumber vanishes, the mode has a turning point, and $\xi_{\text{dyn}}(r)$ can be approximated by a solution of Airy’s equation,

$$\frac{d^2y}{dx^2} - xy \approx 0,$$  \hspace{1cm} (8)

where

$$y \approx y_0 \left[ 1 - \frac{2}{\pi} \arctan \left( \frac{x - x_0}{a} \right) \right],$$  \hspace{1cm} (9)

with $a = 1/2$. The turning point is at $x = x_0$, and $y_0$ is a constant. A similar discussion applies to the g-type modes, with $\xi_{\text{eq}}$ being replaced by $\xi_{\text{neq}}$. The functions $y_0$ and $x_0$ are determined by matching at the core-envelope boundary and for $r = r_0$, where $r_0$ is the radius of the outer convection zone.
\[
\begin{align*}
  y & \equiv \rho^{1/2} r^2 \xi_r^\text{dyn}, \\
  x & \equiv \frac{r - r_c}{\lambda},
\end{align*}
\]  
(8)

and the radial lengthscale
\[
\lambda \equiv \left| \frac{\ell(\ell + 1) dN^2}{\omega^2 r^2} \right|^{-1/3}_{r = r_c} \approx 0.025 P_d^{-2/3} R_\odot.
\]  
(9)

The numerical value is derived from BP’s solar model (Fig. 1), where \( P_d \) is the orbital period \( 2\pi/\omega \) measured in days, and the subscript \( c \) denotes evaluation at \( r = r_c \). On the scale \( \lambda \), the righthand side of Eq. (4) is of order \((\lambda^2/r^2)\) compared to the lefthand side, so the Airy equation (8) has been made homogeneous for \( x < 0 \) (i.e. \( r < r_c \)).

In the convective envelope \((r_c \leq r \leq R_1)\), one must solve the full inhomogenous equation, because \( N^2(r) \approx 0 \) so that \( \xi_r^\text{dyn} \) varies on scales \( \sim r \). In fact \( \xi_r^\text{dyn} \) has a constant sign except near the boundaries:
\[
\begin{align*}
  \xi_r & \approx 0 \text{ at } r = R_1; \\
  \xi_r & \approx 0 \text{ at } r = r_c.
\end{align*}
\]  
(10)

The first of these conditions reflects the requirement that the surface be nearly an equipotential, because deviations from an equipotential have a characteristic frequency \( \sim \ell^{1/2}/\omega \gg \omega \). The second condition reflects the requirement that the solution in the convection zone match smoothly at \( r_c \) onto the solution in the radiation zone, for which
\[
\xi_r^\text{dyn} \sim \lambda \frac{\partial \xi_r^\text{dyn}}{\partial r} \ll r \frac{\partial \xi_r^\text{dyn}}{\partial r}.
\]

The strongest tidal components are quadrupolar \((\ell = 2)\), and they have the form
\[
\delta \Phi_{\text{ext}} = \text{Real} \left[ Q r^2 Y_{2,0}(\theta, \phi) e^{-i\omega t} \right],
\]
where \( Q \) is a constant with dimensions of \((\text{frequency})^2\). The solution of eqs. (4) and (10) in the convection zone yields
\[
\frac{\partial \xi_r^\text{dyn}}{\partial r}(r_c) \approx \sigma_c \frac{Q R_1^3}{GM_1},
\]  
(11)

where \( \sigma_c \) is a dimensionless constant that depends mainly on the thickness of the convection zone, and the \((\theta, \phi, t)\) dependence is implicit but understood to be the same as that of \( \delta \Phi_{\text{ext}} \).

Equations (11) and (8) determine the amplitude of the wave that is launched into the radiative core from \( r = r_c \). This wave propagates inward until it is reflected from a turning point near \( r = 0 \) and returns to \( r = r_c \) with a phase that depends sensitively on the tidal forcing frequency \( \omega \). If the waves were entirely undamped in the core, their energy would increase secularly only when \( \omega \) coincided with a global g-mode eigenfrequency \( \omega_{\ell, n} \); otherwise the energy would saturate after a
time \sim (\Delta \omega)^{-1} determined by the distance \Delta \omega to the nearest resonance. Thus without damping, there is no secular effect on the binary eccentricity except at exact resonance.

There is a broad regime of moderate wave damping in which energy is transferred steadily from the orbit to the star at a rate that is insensitive to the position of the resonances and independent of the actual damping rate, \( \gamma \). On the one hand, if \( \gamma \ll \omega \), then damping is too small to affect the tidal excitation of the ingoing wave. On the other hand, if \( \gamma \) exceeds the reciprocal of the round-trip travel time between the turning points,

\[
t_{\text{group}} = 2 \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{v_{\text{group}}} \approx 2 \int_0^{r_c} \left| \frac{\partial k_r}{\partial \omega} \right| dr,
\]

then the outgoing wave returning to \( r_c \) can be neglected. The frequency difference between neighboring resonances is \( \Delta \omega = 2\pi t_{\text{group}}^{-1} \), so the condition \( \gamma^{-1} \ll 2\pi t_{\text{group}}^{-1} \) ensures that the resonances are broad enough to overlap. Hence this regime can be summarized as

\[
2\pi t_{\text{group}}^{-1} \ll \gamma \ll \omega \quad \text{(moderate damping)}.
\]

Under the conditions (13), the solution (10)-(11) in the convection zone must match onto a solution of the homogeneous Airy equation (8) corresponding to an inward-propagating wave,

\[
y(x) \propto \text{Bi}(x) - i\text{Ai}(x), \quad x \leq 0.
\]

The mechanical power carried by this wave is

\[
\dot{E}_0 = \frac{3^{2/3}}{8\pi} R^2(1/3) [\ell(\ell + 1)]^{-4/3} \omega^{11/3} \left[ \rho \nu^5 \left| \frac{dN^2}{d\ln r} \right|^{-1/3} \left| \frac{\partial \epsilon_{\text{dyn}}}{\partial r} \right| \right]^{2}_{r=r_c}.
\]

With moderate damping, the quotient of \( \dot{E}_0 \) by the energy in the epicyclic motion of the relative orbit,

\[
E_{\text{epi}} = \frac{1}{2} M_1 M_2 e^2 \omega^2 a^2 + O(e^4),
\]

yields the rate of decay of the epicyclic energy. Evaluating \( \dot{E}_0 \) from equations (11) and (14), and adding the contributions of distinct quadrupolar spherical harmonics \( Y_{2,m} \) in quadrature, we find

\[
t_{\text{circ},0}^{-1} \equiv -\frac{1}{e} \frac{de}{dt} = \frac{\dot{E}_0}{2E_{\text{epi}}} \approx 2.88\sigma_c^2 \left[ \rho v^5 \left| \frac{dN^2}{d\ln r} \right|^{-1/3} \right]_{r=r_c} \omega_s^{5/3} \frac{M_2}{M_1^2 R_1^2} \left( \frac{M_1 + M_2}{M_1} \right)^{11/6} \left( \frac{\omega}{\omega_s} \right)^7.
\]

We then specialize to the case of solar-type stars and double the circularization rate to allow for dissipation in both stars. In BP’s solar model, \( dN^2/d\ln r \approx -50\omega_s^2 \) at \( r = r_c \), and by numerical
integration of eqs. (4) & (10), we find $\sigma_c \approx -0.64$ [cf. eq. (11)]. The result for the circularization time is

$$t_{\text{circ},0} \approx 8.0 \times 10^3 P_d^2 \text{ yr.}$$

(17)

The subscript “0” serves as a reminder that this pertains to the moderate damping regime.

In the simple quasilinear theory of this paper, the eccentricity decays exponentially but never entirely vanishes. For ease of reference, we define the transition period $P_{\text{circ}}$ by

$$t_{\text{circ}}(P_{\text{circ}}) \equiv t/3,$$

(18)

where $t$ is the age of the system, so that systems with periods $\leq P_{\text{circ}}$ and initial eccentricities $e_i \sim 0.3$ have reached $e \lesssim 0.015$.

According to eqs. (17) & (18), the circularization period of the oldest solar-type main-sequence stars would be $P_{\text{circ},0} \approx 6.4 \text{ d}$ in the moderately damped regime. As we show in the next section, however, solar-type binaries with comparable periods are not in the moderately damped regime unless nonlinear damping is efficient.

3. Discrete Modes and Damping

The derivation of the circularization rate (16) treated the g-mode excited at the edge of the core as though it were purely an ingoing wave. In fact, the tidally-forced disturbance is a superposition of discrete global modes—standing rather than traveling waves—whose frequencies are quantized: $\omega_{\text{mode}} \in \{\omega_{\ell,n}\}$, where $n \in \{1,2,\ldots\}$ indexes the number of radial nodes. The damping rate of these modes by radiative diffusion (which is the dominant linear damping mechanism) is small compared to the difference $\Delta \omega \equiv \omega_{\ell,n+1} - \omega_{\ell,n}$ between neighboring eigenfrequencies. This is quantified by the parameter $\alpha \ll 1$ estimated in equation (22) below.

The damping is not in the “moderate” regime defined above—it is much weaker, at least if we consider only linear damping mechanisms.

So it may fairly be asked why one should bother with the estimates of the previous section, rather than work with discrete modes from the outset. Part of the answer is that the tidal power $\dot{E_0}$ [eq. (14)] and the corresponding circularization rate $t_{\text{circ},0}$ [eq. (16)] are equal to the averages of the true quantities $\dot{E}(\omega)$ and $t_{\text{circ}}(\omega)$ over the discrete modes, if the average is taken over a range $\Delta \omega$ but $\ll \omega$ itself, and if the average is uniformly weighted in frequency. This is an example of a “sum rule.” It is true because $\dot{E}$ is derived from linear theory and because if the star is subject to a short-lived tidal disturbance with a lifetime $\Delta t \ll (\Delta \omega)^{-1} = t_{\text{group}}/2\pi$, then many discrete modes are excited, but the total dissipation is given correctly by the methods of §2 since the disturbance ends before the wave completes a round trip between the turning points.

The second part of the answer is that the estimates of §2 are secure, whereas the consequences of the discreteness of the modes are complex and somewhat uncertain. A uniformly-weighted
average of $t_{\text{circ}}^{-1}(\omega)$ with respect to $\omega$ (yielding $t_{\text{circ},0}^{-1}$) is not appropriate, since the former is extremely sensitive to the latter—varying by factors $\sim \alpha^{-2} \gg 1$ as $\omega$ varies by $\Delta\omega$—and since $\dot{\omega}$ is dominated by the tidal interaction itself when $\omega$ is close to a resonance. On the one hand, if the tidal interaction is the only cause of change in the distance $\min_{n} |\omega - \omega_{\ell,n}|$ from resonance, then one should average $t_{\text{circ}}$, not $t_{\text{circ}}^{-1}$, uniformly in $\omega$; this produces the “harmonic” mean (27), which is comparable to the maximum value of the instantaneous circularization time and is larger than $t_{\text{circ},0}$ by a factor $\alpha^{-1}$. On the other hand, the distance from resonance is changed by effects independent of the tide. One such effect is gravitational radiation; this is negligible for orbital periods of interest, but not by a very large factor. More important is the change in the mode frequencies $\{\omega_{\ell,n}\}$ as the star evolves and its mean density declines. Finally, there is good reason to expect that the damping rate should be enhanced by nonlinear processes, although these are difficult to estimate.

In this section, we discuss the effects of the last paragraph sequentially. First we consider linear damping by radiative diffusion and derive the appropriate (harmonic) mean circularization rate if the only influence on the resonance relations is the tidal interaction itself. Then we allow for changes in the resonant frequencies by stellar evolution. Lastly, we demonstrate that the dynamical tide is significantly nonlinear near the center of the star, and explain why this is likely to cause additional damping.

3.1. Damping by radiative diffusion

The density perturbations of a g mode are accompanied by temperature fluctuations. Radiative diffusion tends to smooth out the temperature variations and damp the mode at the local rate

$$\gamma(r) \approx \frac{1}{2} k_{r}^{2} \chi(r) \left[1 + \frac{\Gamma_{1} d\ln \mu/dr}{d\ln(P/\rho^{1/3})/dr}\right].$$

This formula assumes that $\omega \ll N(r)$ so that the radial wavenumber $k_{r} \gg r^{-1}$ [cf. eq. (7)]. The thermal diffusivity is given by

$$\chi \equiv \frac{4acT^{3}}{3k_{\rho}^{2}C_{P}} = \left(-\frac{dT}{dr} \frac{dm}{dr} \frac{5}{2} \frac{k_{B}}{\mu m_{H}}\right)^{-1} L_{r},$$

in which all symbols have their conventional meaning.

The stable stratification of the solar core is due mainly to entropy gradients ($dS/dr > 0$) but is reinforced by a gradient in molecular weight ($d\mu/dr < 0$), especially near the center. The bracketed factor in eq. (19) reflects the fact that temperature perturbations are associated only with the entropy gradient. This factor is less than unity in the core of BP’s solar model, but it is never negative (which would indicate instability to semiconvection).

The global damping rate is the average of the local rate weighted by the time spent by a wave
packet at each radius:

$$\gamma \approx \int_0^{r_c} \frac{\partial k_r}{\partial \omega} \gamma(r) dr / \int_0^{r_c} \frac{\partial k_r}{\partial \omega} dr.$$  

(21)

We have approximated the inner and outer turning points by 0 and $r_c$, respectively, which is appropriate for $\omega \ll \omega_*$ and $\ell = 2$.

We define the attenuation ($\alpha$) of the mode by $\gamma t_{\text{group}}$ [cf. eq. (12)]. During a roundtrip between the turning points, the wave amplitude is damped by $\exp(-\alpha)$. The attenuation determines whether the g-modes are effectively discrete or continuous: $\alpha/2\pi$ is the ratio between the half width at half maximum and the separation of neighboring g-mode resonances at a common $\ell$. Thus $\alpha \gtrsim 1$ is the condition for resonances to overlap. A numerical calculation in the solar model of BP yields

$$\alpha \approx \left( \frac{P}{11.6 \text{ d}} \right)^4.$$

(22)

The $P^4$ dependence occurs because the local damping rate $\gamma(r) \propto k_r^2 \propto P^2$ [eq (7)], and the roundtrip time $t_{\text{group}}$ is also $\propto P^2$ [eq. (12)]. Clearly $\alpha \ll 1$ and the resonances are well separated at the circularization period predicted in §2 (6.4 d), and this must be taken into account when one estimates the average circularization rate.

We can derive an approximate expression for the tidal dissipation rate from the requirements that

(i) $\dot{E}(\omega)$ should reduce to $\dot{E}_0$ when averaged over a range large compared to $\Delta \omega \equiv |\omega_{\ell,n+1} - \omega_{\ell,n}|$, but small compared to $\omega$ itself.

(ii) $\dot{E}(\omega)$ can be expressed as a sum of Breit-Wigner profiles centered at each resonance.

Of course $\dot{E}_0$ itself depends on $\omega$, but only as a power-law, whereas $\dot{E}(\omega)$ varies tremendously on the scale $\Delta \omega \ll \omega$. Therefore the average considered in point (i) makes sense only if $\omega \gg \Delta \omega$, or equivalently, if the modes with frequencies in the vicinity of $\omega$ are of large radial order $n$. In the solar model, $n \approx 100P_d$ for $\ell = 2$ modes (Provost & Berthomieu 1986), so this condition is well satisfied.

From the two conditions cited, it follows that

$$\dot{E}(\omega) \approx \dot{E}_0 \sum_n \frac{\gamma \Delta \omega / \pi}{(\omega - \omega_{\ell,n})^2 + \gamma^2}$$

where $\gamma$ and $\Delta \omega$ vary slowly with $n$ but can be treated as constants over ranges $\Delta n \ll n$. In the same approximation, the infinite sum can be expressed in closed form as

$$\dot{E}(\omega) \approx \frac{\dot{E}_0 \sinh \alpha}{2 \left[ \sin^2 x + \sinh^2(\alpha/2) \right]}$$

(24)

$$x \equiv \left( \frac{\pi}{\Delta \omega} \right) \min_n |\omega - \omega_{\ell,n}|.$$  

(25)
The tidal frequency is not constant during the history of a close binary. The circularization process itself removes the epicyclic energy (15) from the orbit, causing $\omega$ to increase at the rate

$$\dot{\omega}_{\text{circ}} = \frac{3e^2}{\tau_{\text{circ}}(\omega)}.$$  \hspace{1cm} (26)

Therefore the time spent in the interval $(\omega,\omega + d\omega)$ of orbital frequency is proportional to $d\omega/\dot{E}(\omega)$, and it follows that the average of $\dot{E}(\omega)$ with respect to time is equivalent to its harmonic average with respect to frequency:

$$\langle \dot{E} \rangle_h = \int \frac{d\omega}{\dot{E}(\omega)} \dot{E}(\omega) / \int \frac{d\omega}{\dot{E}(\omega)} = \dot{E}_0 \tanh(\alpha).$$  \hspace{1cm} (27)

The corresponding circularization time of solar-type binaries is

$$t_{\text{circ},h} \approx \frac{\alpha^{-1} t_{\text{circ},0}}{1.5 \times 10^8 P^3_d \text{ yr}} (P \approx 11.6 \text{ d}),$$

and the transition period for systems $10^{10}$ yr old is $P_{\text{circ}} = 2.8$ d. This is very much less than the value 6.4 d obtained in §2 by assuming that the resonances overlap. The situation is summarized graphically by the solid lines in Figure 2.

### 3.2. Effects of stellar evolution

Equations (25) and (27) imply that the harmonic average of the dissipation rate is just twice dissipation rate halfway between resonances (i.e. at $x = \pi/2$): $\dot{E}_{\text{min}} \approx (\alpha/2)\dot{E}_0$. By contrast, the instantaneous dissipation rate at exact resonance ($x \approx 0$) is $\dot{E}_{\text{max}} \approx (2/\alpha)\dot{E}_0$. To the extent that $\dot{\omega} \propto \dot{E}(\omega)$, the time required to circularize is clearly dominated by the minimum value of $|\dot{\omega}|$ due to the tide, which we denote $\dot{\omega}_{\text{min}}$. Any other process that causes secular changes $|\dot{\omega}| > \dot{\omega}_{\text{min}}$—or even a nonsecular process that slowly modulates $\omega$ by more than $\Delta \omega/2$—can substantially influence the mean circularization rate in the regime $\alpha \ll 1$.

For example, gravitational radiation causes

$$\left(\frac{\dot{\omega}}{\omega}\right)_{\text{GR}} = \frac{1}{1.7 \times 10^{11} \text{ yr}} \left(\frac{M_1}{M_\odot}\right)^{5/3} P^{-8/3}_d,$$  \hspace{1cm} (29)

if the stars have equal masses and $e \ll 1$. Comparing this with equations (26) and (28), one sees that gravitational radiation would begin to influence the mean circularization rate when $e < 0.024 P^{1/6}_d$.

The ratio of the instantaneous dissipation rate to its time average depends on $\omega$ through the combination $x \propto \omega - \omega_{\ell,n}$. Thus changes in the resonant frequencies can be as important for $\langle \dot{E} \rangle$ as changes in $\omega$ (Savonije & Papaloizou 1983). During the main sequence phase, solar-type stars expand slightly: for example, Sackman et al. (1993) estimate that the Sun’s radius has increased
by about 11.5% since the zero-age main sequence. If the individual resonant frequencies \( \omega_{\ell n} \) scale with the mean density of the star, then

\[
\left( \frac{\dot{\omega}_{\ell n}}{\omega_{\ell n}} \right)_{ev} \approx -\frac{1}{3 \times 10^{10}} \text{ yr}.
\]  (30)

Clearly this effect contributes more strongly to \( \dot{x} \) than gravitational waves, and with the same sign. It exceeds the minimum value \( \dot{x}_{\text{min}} \) due to tidal dissipation alone if \( e < 0.06 P_{d}^{3/2}/P_{\text{circ}} \), or equivalently if \( P > 2.9 e_{0.3}^{2/3} \) d, where \( e_{0.3} \equiv (e/0.3) \sim 1 \) is representative of the initial eccentricity (Duquennoy & Mayor 1991). At substantially smaller eccentricities or longer periods, the mean dissipation rate exceeds the harmonic value (27) by a factor \( \approx (\dot{E}(\omega))/4 \dot{E}(\omega)_{0} \). This leads to

\[
t_{\text{circ,ev}} \approx 4.8 \times 10^{9} e P_{d}^{3/2} \text{ yr}.
\]  (31)

This value of \( t_{\text{circ}} \) is valid only if it is less than the harmonic mean (28), else evolutionary effects on the resonances are negligible, and greater than the moderately-damped value (17), because as \( \dot{\omega}_{\ell n} \to \infty \), averages over time and frequency are equivalent, and \( \langle \dot{E}(\omega) \rangle \to \dot{E}_{0} \). Once the eccentricity has been sufficiently reduced so that the prediction (31) is less than (17), then the latter should be used.

Since the evolutionary changes in \( \{ \omega_{\ell n} \} \) begin to be important only at periods comparable with the \( P_{\text{circ}} \) obtained from the harmonic average (\( \approx 2.8 \) d), we expect that \( P_{\text{circ}} \) is essentially unaffected, except in systems having a small initial eccentricity. But the later stages of circularization will proceed on the timescale (17) rather than (28).

### 3.3. Nonlinearity

In linear theory, the turning points occur where \( N(r) = \omega \) [cf. eq. (7)]. Near \( r = 0 \), \( N(r) \) is approximately linear in \( r \), and so the inner turning point \( r_{\text{min}} \) scales as \( P^{-1} \). Using the solar model of BP, we estimate that

\[
r_{\text{min}} / R_{\odot} \approx 1.25 \times 10^{-3} P_{d}^{-1}.
\]  (32)

The smallness of the coefficient reflects the steep gradient of \( N(r) \) near \( r = 0 \) (Fig. 1).

In the WKBJ approximation, the energy flux of the wave is conserved, so that \( \rho v_{\text{group}} |\tilde{\xi}_{r}|^2 \approx \text{constant} \) between the turning points. (Actually a standing wave has zero net flux, since the fluxes of the ingoing and outgoing wave cancel. Nevertheless, this argument gives the correct scaling of the wave amplitude.) If \( \tilde{\xi}_{r} \) in this formula is the root-mean-square radial displacement averaged over time and angle at fixed radius, then the constant is \( \dot{E}_{0}/4\pi \), where \( \dot{E}_{0} \) is given by eq. (14) and is related to \( t_{\text{circ,0}} \) by equations (15)-(16). This allows us to estimate \( \tilde{\xi}_{r}(r) \). An appropriate dimensionless local measure of nonlinearity is

\[
k_{r} \tilde{\xi}_{r} \approx 2.8 \times 10^{-4} e \left( \frac{N}{\omega_{s}} \right)^{1/2} P_{d}^{-11/6} \left( \frac{r}{R} \right)^{-5/2}.
\]  (33)
At the inner turning point (32), this becomes

\[ k_r \xi_r(r_{\text{min}}) \approx 1.7 \times 10^3 e P_d^{1/6}. \]  

(34)

We conclude that the wave is strongly nonlinear near its inner turning point, unless \( e \ll 10^{-3} \).

Sufficiently large-amplitude waves are bound to damp. Where \( k_r \xi_r > 1 \), for example, the wave inverts the stratification during part of its cycle, so that denser (lower-entropy) material overlies less dense (higher-entropy) material. A Rayleigh-Taylor instability results provided that the inverted profile persists for more than a growth time, which translates to \( \omega < N \). The latter condition is not satisfied at \( r_{\text{min}} \) but it will be adequately satisfied at radii a few times larger, where the local nonlinearity is smaller than eq. (34) by a factor \( \approx (r_{\text{min}}/r)^{5/2} \). So to be sure of this instability, one may take \( k_r \xi_r(r_{\text{min}}) \gtrsim 10 \) (say). If instability does occur, then the energy required to overturn and mix the material must come at the expense of the wave. This particular instability may be self-limiting, because mixing will reduce the molecular-weight gradient, hence move \( r_{\text{min}} \) outward and reduce \( k_r \xi_r(r_{\text{min}}) \). Subtler parametric instabilities may arise at weaker nonlinearity \( (k_r \xi_r < 1) \), as discussed by Kumar & Goodman 1996, which are less likely to mix the fluid.

If these or other nonlinear damping mechanisms substantially reduce the wave reflected from the inner turning point, or even if they merely perturb its phase in a chaotic manner, then the narrow resonances pictured earlier will broaden and overlap. The dynamical tide will then effectively be in the moderate-damping regime (13), and in that case the more optimistic estimates of \( \S2 \) for \( t_{\text{circ}} \) and \( P_{\text{circ}} \) should apply.

4. Discussion

We have adapted Zahn’s Theory of the Dynamical Tide to late-type stars with radiative cores and convective envelopes. As in the case of the Theory of the Equilibrium Tide (Paper I), we have two sets of estimates for the circularization time \( (t_{\text{circ}}) \) and the transition period dividing circular from noncircular orbits \( (P_{\text{circ}}) \).

Allowing only for the very inefficient linear damping of the tidally excited g modes by radiative diffusion, we find that the (harmonic mean) circularization time is given by eq. (28). Setting \( t_{\text{circ}} \rightarrow 3.3 \) Gyr (one third the main sequence lifetime of sunlike stars), we obtain a theoretical circularization period of only

\[ P_{\text{circ},h} \approx 2.8 \left( \frac{t}{10^{10} \text{ yr}} \right)^{1/3} \text{ d}. \]  

(35)

Although well short of the observationally inferred values (cf. \S1 and below), this prediction is larger than the value \( \approx 2.2 \) d predicted by the Theory of the Equilibrium Tide when the latter is corrected for the slowness of the convective eddies relative to the tidal period (Paper I). Adding the two mechanisms raises the prediction only to \( \approx 3.3 \) d because both mechanisms depend so strongly on \( P \).
Because of the combined effects of stellar evolution and nonlinear damping, $P_{\text{circ}}$ is likely to be larger than eq. (35) but still smaller than the observed values. As shown in §3.3, the linear theory indicates a strongly nonlinear wave amplitude near the inner turning point as long as $e \gtrsim 10^{-3} P_{d}^{1/6}$. If nonlinear losses substantially reduce the amplitude or randomize the phase of the wave reflected from the inner turning point, then we recover the “moderately-damped” circularization rate (17) and the corresponding transition period

$$P_{\text{circ},0} = 6.4 \left( \frac{t}{10^{16} \, \text{yr}} \right)^{1/7} \, \text{d.}$$ (36)

On the other hand, once $e \lesssim 0.04$ then stellar evolution dominates the rate of change of the distance from resonance even at exact resonance (§3.2), and one again recovers the larger circularization rate (17) and transition period (36).

Mathieu et al. 1992 have concluded that $t_{\text{circ}} \propto P^{10/3}$ gives a good fit to the observed scaling of $P_{\text{circ}}$ with binary age. The exponent of $P$ in the harmonic average (28) is close to $10/3$. However, the normalization of $t_{\text{circ}}$ that we predict is much too small to explain the large values of $P_{\text{circ}}$ that Duquennoy & Mayor 1991 and Latham et al. (1992) find for old stars ($11 - 19$ d).

In summary, theory and observation appear to be in conflict regarding circularization of late-type main-sequence stars. The conflict must be resolved either by adjusting the theory, or by revising the conclusions drawn from the observations:

(i) Perhaps one has overlooked some more effective mechanism of circularizing main-sequence binaries at long periods (\(\gtrsim 10\) d). Tassoul and collaborators have proposed that rapid circularization and synchronization of binaries occurs by a mechanism analogous to Ekman pumping of fluids in rigid containers (Tassoul 1987, 1995; Tassoul & Tassoul 1996, and references therein). The reality of this mechanism is controversial on theoretical grounds (Rieutord 1992, Rieutord & Zahn 1997, Tassoul & Tassoul 1997). Empirically, Tassoul’s mechanism does not seem to be required to explain circularization of early-type binaries (Giuricin et al. 1984, Claret & Cunha 1997) or systems containing giants (Verbunt & Phinney 1996), but it contains a free parameter that Tassoul supposes to vary by orders of magnitude among spectral types. Other possibilities include tidally excited parametric instabilities (Kumar & Goodman 1996), and tidal excitation of inertial modes, whose existence depends essentially on coriolis forces (Savonije & Papaloizou 1997); but neither of these mechanisms has yet been applied to circularization of late-type binaries.

(ii) Alternatively, perhaps the empirical inference that circularization occurs on the main sequence is incorrect. The significance of the trend of $P_{\text{circ}}$ with age is not clear, given the small numbers of binaries with well-determined masses, ages, and evolutionary states. Circularization is very sensitive to the history of the mean density of the stars, which can decrease by orders of magnitude in non-main-sequence phases of evolution (e.g. Verbunt & Phinney 1996). There exist old double-lined spectroscopic systems whose members are almost certainly unevolved with circular orbits and periods above 10 d (cf. Paper I), but a small number of such systems might be
ascribed to chance. Furthermore, it is possible that circularization occurs before the main sequence (Zahn & Bouchet 1989). If the trend of $P_{\text{circ}}$ with age is real, it might result from a difference in mass or metallicity between the younger disk binaries and older halo systems that correlates with a difference in pre-main-sequence evolution. For example, protostellar disks may strongly affect orbital eccentricity (Artymowicz et al. 1991), and it is conceivable that the persistence of massive gaseous disks may correlate inversely with the abundance of dust and other metals in them.

Larger samples of close, well-measured binaries might reduce these uncertainties, for example by allowing one to compare systems of similar mass and metallicity but different age. Even more helpful would be large samples of pre-main-sequence binaries, which it is not unreasonable to expect in the near future (Mathieu 1994).

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Fig. 1.— Solid curve: Brunt-Väisälä frequency $N$ normalized to dynamical frequency $\omega_* \equiv (GM_\odot/R^3)^{1/2}$ vs. radius, based on solar model of BP. Dashed line: tidal frequency on same scale for $P = 1$ d, i.e. $\omega/\omega_* = 0.116$. 
Fig. 2.— Circularization time, averaged over resonances, versus binary period. Broken solid line: Fixed resonances and linear radiative damping, based on eqs. (17), (28), and (22). Dot-dashed line: extension of eq. (17) to $P < 11.6$ d given efficient nonlinear damping and/or relatively rapid evolution of resonant frequencies (§3.3, §3.2). Dashed lines: Effect of evolving resonant frequencies for eccentricities as marked; applicable only when between solid and dot-dashed lines [eq. (31)].