CM-P00056927

DOMINANCE OF EXCHANGE AMPLITUDES IN ELASTIC SCATTERINGD. Amati ^{*)}, L.L. Foldy ^{**)}, A. Stanghellini and L. Van Hove

CERN--Geneva

ABSTRACT

For any internal symmetry group one can express the amplitude for the reaction $A+B \rightarrow C+D$ (s channel) in terms of amplitudes associated with irreducible representations of the group for the reaction $A+\bar{C} \rightarrow \bar{B}+D$ (t channel) or the reaction $A+\bar{D} \rightarrow C+\bar{B}$ (u channel). Physically these relations correspond to the expression of an s channel amplitude in terms of amplitudes associated with the exchange of various systems between the reacting particles, these systems being identified by quantum numbers associated with irreducible representations of the group. Experimentally there appears to be evidence that for very high s channel energies, a single such amplitude in the t channel gives the dominant contribution. Using the general crossing relations referred to above for the case of forward elastic scattering ($A=C$, $B=D$) and the optical theorem, we show (1) the only amplitude which can dominate in the t channel for any group is that associated with the identity representation of the group (even if there is not invariance under the group), (2) in the case of the isospin group the only amplitude which can dominate in the u channel is that corresponding to the maximum possible isospin.

^{*)} and Istituto di Fisica dell'Università - Palermo (Italy).

^{**) National Science Foundation Senior Postdoctoral Fellow on leave from Case Institute of Technology, Cleveland, Ohio, U.S.A. (1963-1964).}

1. INTRODUCTION

Pomeranchuk and Okun¹⁾ suggested in 1956 that in forward scattering at very high energies exchange amplitudes, and in particular charge-exchange amplitudes, become negligible compared with the non-exchange amplitude. This hypothesis was justified in physical terms by arguing that exchange scattering is to be regarded as a special case of inelastic collision, which at high energy is in competition with all other inelastic processes. Non-exchange scattering $a + b \rightarrow a + b$, on the contrary, is truly elastic in the sense that each of the two incident particles a and b remains identical to itself, so that interference between incident and scattered waves is possible, and scattering is bound to occur as the shadow of all inelastic processes.

The Pomeranchuk-Okun rule on charge-exchange scattering can also be expressed by saying that exchange of any non-vanishing value of isospin I between a and b gives at high energy a negligible contribution compared with exchange of $I = 0$. While this rule is strongly supported by experiment, one does not know of any way to derive it from first principles. It is therefore of interest that a more limited result in this direction could be established as a rigorous consequence of the optical theorem^{2),3)}. The result in question is a theorem stating that if the imaginary part of the forward scattering amplitude is analyzed in contributions from exchange of various isospin values I , the contribution arising from $I = 0$ cannot be negligible compared with contributions of $I \neq 0$. In particular, if only one isospin exchange dominates, it must be $I = 0$. Thus the Pomeranchuk-Okun rule will hold if in forward scattering the contributions of exchange of the various I values have such different asymptotic behaviours at high energy that a single I value eventually dominates all others. The proof of this theorem was based on explicit consideration of the crossing matrix for particles of arbitrary isospin. The matrix elements are closely related to Racah coefficients, and the available information regarding the sign of the latter is sufficient to establish the desired conclusion.

The present paper extends the result just mentioned to general groups and shows that, once the theorem is formulated in sufficient generality, the proof that the contribution of the amplitude which relates identity representations in the t channel cannot be negligible compared to the others, becomes extremely simple and actually almost trivial. It also appears that the theorem even applies to groups for which the forward scattering matrix is not invariant ; all what is needed is that particles a and b belong to multiplets (i.e., irreducible representations) of the group. Of course, the theorem is only expected to apply in nature to groups which leave the basic interactions invariant, because only for such groups can we expect that the exchange contributions belonging to different irreducible representations will split at high energies into a dominant one and various negligible ones.

The next section introduces the notation and gives the crossing relations for a general group. Section 3 establishes the main result concerning exchange amplitudes between particles a and b . These amplitudes refer to exchange in the channel $a\bar{a} \rightarrow b\bar{b}$ for which we shall use the conventional name of t channel. The last section is devoted to the mathematically analogous problem of exchange contributions in the u channel, i.e., the channel $a\bar{b} \rightarrow \bar{a}b$. While this question seems to be physically of academic character, because there is no experimental situation where one single exchange contribution appears to dominate in the u channel, it can be studied as a mathematical exercise. It is of much greater complexity than t channel exchange, and we have only been able to establish a dominance theorem for the case of isospin. In this case, the only amplitude which can dominate turns out to be the one of maximum isospin.

2. FORMULATION OF THE CROSSING RELATIONS

Consider four unitary representation spaces for a general group $\mathcal{G}$ which we designate by A, B, C, D . In each space we select an orthonormal basis and represent the vectors of these bases by $|Aa\rangle, |Bb\rangle, |Cc\rangle, |Dd\rangle$. In general we shall drop the capital letters in these symbols writing simply $|a\rangle$ for $|Aa\rangle$. The conjugate spaces to A, B, C, D will be designated $\bar{A}, \bar{B}, \bar{C}, \bar{D}$, and the basis of the conjugate representation $\bar{A}$, associated with the basis $|a\rangle$ in A , will be designated by $|\bar{A}\bar{a}\rangle \equiv |\bar{a}\rangle$.

Let M_s be a linear mapping of the product space $A \times B$ on the $C \times D$ (characterized by the basis vectors $|ab\rangle \equiv |Aa, Bb\rangle$ and $|cd\rangle \equiv |Cc, Dd\rangle$ respectively), specified by its matrix elements $\langle cd|M_s|ab\rangle$.

Associated with the previous mapping we can define other possible mappings between different products of our four representations A, B, C, D . The correspondence between the different mappings is defined through their matrix elements as, for instance

$$\langle cd|M_s|ab\rangle \equiv \langle \bar{b}d|M_t|\bar{a}c\rangle \equiv \langle c\bar{b}|M_u|a\bar{d}\rangle \quad (1)$$

If we consider M_s to be the scattering matrix for the reaction

(called s channel in the following), the mapping M_t and M_u will then be related to the analytical continuation of the S matrix for the reactions $A(a) + \bar{C}(\bar{c}) \rightarrow \bar{B}(\bar{b}) + D(d)$ and $A(a) + \bar{D}(\bar{d}) \rightarrow C(c) + \bar{B}(\bar{b})$ respectively (called t and u channel in the following) to unphysical values of the energies and momentum transfers. We wish to stress that the matrices M_t and M_u are defined whether or not such analytical continuation is indeed possible.

The spaces A, B, C, D , may be either reducible or irreducible representations for $\mathcal{J}$, but the product spaces we have introduced are (with trivial exceptions) certainly reducible. We can then decompose the space $A \times B$ (and $C \times D$) into irreducible subspaces under $\mathcal{J}$. We shall use the symbol X to represent both such an irreducible subspace and the representation induced in it. In each such subspace we may introduce an orthonormal basis $|X(A \times B)\rangle$ ($|X(C \times D)\rangle$) and the totality of these are then a new basis for the whole space $A \times B$. The transformation coefficient $\langle X(A \times B) | AaBb \rangle$, will often be abbreviated as $\langle X | ab \rangle$ ^{*)}. The decomposition of a matrix element of M_S with respect to the irreducible representations of $\mathcal{J}$ is given by

$$\langle cd | M_S | ab \rangle = \sum_{XxX'x'} \langle cd | X'x' \rangle \langle X'x' | M_S | Xx \rangle \langle Xx | ab \rangle \quad (2)$$

By the use of the orthogonality properties of the transformation coefficients, Eq. (2) can be inverted to give

$$\langle X'x' | M_S | Xx \rangle = \sum_{abcd} \langle X'x' | cd \rangle \langle cd | M_S | ab \rangle \langle ab | Xx \rangle$$

Analogous reduction of the product representations can be done for the channels t and u ; we shall use the symbols Y and Z to represent the respective irreducible subspaces.

*) It is perhaps well to remark that in some cases (though not in the case of the isospin group) the reduction of a product representation may contain the same irreducible representation more than once. In this case, the decomposition into invariant subspaces is not unique and a particular choice may be dictated by other than group-theoretical considerations or may be quite arbitrary. In such an instance it will be necessary to label an invariant subspace not only by the symbol X but by a further symbol ξ which distinguishes the equivalent invariant subspaces. In this case we shall write for the transformation coefficients $\langle X(\xi) | X(A \times B) | AaBb \rangle$, for example. In general we shall omit this further symbol except where it plays a role in the argument.

The substitution of these results into (1) give what may be called the generalized crossing relations between s, t and u channels :

$$\begin{aligned}
 \langle X'x' | M_s | Xx \rangle &= \\
 &= \sum_{abcd} \sum_{YyY'y'} \langle X'x' | cd \rangle \langle \bar{b}d | Y'y' \rangle \langle Y'y' | M_t | Yy \rangle \langle Yy | a\bar{c} \rangle \langle ab | Xx \rangle = \\
 &= \sum_{abcd} \sum_{ZzZ'z'} \langle X'x' | cd \rangle \langle c\bar{b} | Z'z' \rangle \langle Z'z' | M_u | Zz \rangle \langle Zz | a\bar{d} \rangle \langle ab | Xx \rangle
 \end{aligned} \tag{3}$$

We have so far not assumed that M_s is invariant under the group $\mathcal{J}$. If it is, one can easily prove that M_t and M_u are also invariant under $\mathcal{J}$. In this case, one can establish from Schur's lemma that

$$\langle X'(\xi')x' | M_s | X(\xi)x \rangle = S_X(\xi', \xi) \delta_{X'X} \delta_{x'x}$$

$$\langle Y'(\eta')y' | M_t | Y(\eta)y \rangle = T_Y(\eta', \eta) \delta_{Y'Y} \delta_{y'y}$$

$$\langle Z'(\rho')z' | M_u | Z(\rho)z \rangle = U_Z(\rho', \rho) \delta_{Z'Z} \delta_{z'z}$$

Using these results in (3) gives us the ordinary crossing relation between s, t and u channels

$$\begin{aligned}
 \text{a) } S_X(\xi', \xi) &= \sum_{Y \eta', \eta} P_{XY}(\xi', \xi | \eta', \eta) T_Y(\eta', \eta) = \\
 &= \sum_{Z \rho', \rho} Q_{XZ}(\xi', \xi | \rho', \rho) U_Z(\rho', \rho)
 \end{aligned} \tag{4}$$

6.

$$\begin{aligned}
 \text{b) } P_{XY}(\xi' \xi | \eta' \eta) &= \\
 &= \sum_{abcd} \sum_y \langle X(\xi')_x | cd \rangle \langle \bar{b}d | Y(\eta')_y \rangle \langle Y(\eta)_y | a\bar{c} \rangle \langle ab | X(\xi)_x \rangle
 \end{aligned}
 \tag{4}$$

$$\begin{aligned}
 \text{c) } Q_{XZ}(\xi' \xi | \rho' \rho) &= \\
 &= \sum_{abcd} \sum_z \langle X(\xi')_x | cd \rangle \langle c\bar{b} | Z(\rho')_z \rangle \langle Z(\rho)_z | a\bar{d} \rangle \langle ab | X(\xi)_x \rangle
 \end{aligned}$$

The coefficients P_{XY} and Q_{XZ} are recoupling coefficients which in the isospin group case are simply related to Racah coefficients.

3. DOMINANCE THEOREM FOR THE t CHANNEL

As mentioned in the introduction, the basis for dominance theorems is the optical theorem which requires that the anti-Hermitian part of the T matrix for forward elastic scattering in the (physical) s channel be positive definite (i.e., have positive diagonal elements in all representations) if any reactions take place at all. We shall now identify the matrix M_s introduced in the earlier sections with the anti-Hermitian part of T and restrict ourselves to the case where $A = C$ and $B = D$.

We shall require the following lemma whose proof is sufficiently simple that we omit it : if A is an irreducible representation space for the group then in the decomposition of the representation space $A \times \bar{A}$, the identity representation (which we shall call Y_0) occurs once and only once. The normalized basis vector in the associated (one-dimensional) subspace then has the form

$$|Y_0(A \times \bar{A})\rangle = \alpha_A^{-\frac{1}{2}} \sum_a |a\bar{a}\rangle \quad (5)$$

implying

$$\langle Y_0 | a\bar{a}' \rangle = \alpha_A^{-\frac{1}{2}} \delta_{aa'}$$

The same is true in the space $B \times \bar{B}$ with the normalized basis vector given by

$$|Y_0(\bar{B} \times B)\rangle = \alpha_B^{-\frac{1}{2}} \sum_b |\bar{b}b\rangle \quad \text{and} \quad \langle Y_0 | \bar{b}b' \rangle = \alpha_B^{-\frac{1}{2}} \delta_{bb'} \quad (5')$$

In the above α_A and α_B are the dimensionalities of the spaces A and B , respectively.

We consider first the case where A and B are both irreducible. The positiveness of the matrix M_S requires that the matrix elements $\langle ab|M_S|ab\rangle$ all be positive. Therefore, using Eq. (1) and the decomposition of the product representation in the t channel, we have :

$$\begin{aligned} \langle ab|M_S|ab\rangle &\equiv \langle \bar{b}b|M_t|a\bar{a}\rangle = \\ &= \sum_{Y Y'} \langle \bar{b}b|Y'Y'\rangle \langle Y|a\bar{a}\rangle \langle Y'Y'|M_t|Y\rangle \geq 0 \end{aligned} \quad (6)$$

From Eqs. (5) and (5')

$$\sum_{a,b} \langle \bar{b}b|Y'Y'\rangle \langle Y|a\bar{a}\rangle = \alpha_A^{-\frac{1}{2}} \alpha_B^{-\frac{1}{2}} \delta_{Y,Y_0} \delta_{Y',Y_0} \quad (7)$$

This immediately implies that if either or both Y and Y' are not the identity representation, then there are values of a and values of b for which the expansion coefficients in Eq. (6) are of opposite sign. When Y and Y' are the identity representation, then from Eqs. (5) and (5')

$$\langle \bar{b}b|Y_0\rangle \langle Y_0|a\bar{a}'\rangle = \alpha_A^{-\frac{1}{2}} \alpha_B^{-\frac{1}{2}} \delta_{aa'} \delta_{bb'} \quad (8)$$

From these considerations it follows :

Theorem 1 :

A necessary condition that M_S be a positive definite matrix is that $\langle Y_0|M_t|Y_0\rangle > 0$.

In fact summing $\langle ab|M_S|ab\rangle$ over a and b in Eq. (6) and making use of Eq. (7) we get

$$\sum_{ab} \langle ab|M_S|ab\rangle = \alpha_A^{-\frac{1}{2}} \alpha_B^{-\frac{1}{2}} \langle Y_0|M_t|Y_0\rangle > 0 \quad (9)$$

Theorem 2 :

If there is one amplitude in the t channel $\langle Y'y' | M_t | Yy \rangle$, which is dominant over the others, this amplitude must be the one corresponding to the identity representation, i.e., $\langle Y_o | M_t | Y_o \rangle$.

In fact, if any other amplitude would dominate, Eq. (7) implies that there are different values of a and b for which $\langle \bar{b}b | Y'y' \rangle \langle Yy | a\bar{a} \rangle$ have opposite signs, whence for some values of a and b the inequality (6) would be violated. For the identity representation instead, as a consequence of (8), the positive amplitude $\langle Y_o | M_t | Y_o \rangle$ would satisfy such an inequality for all a and b .

Moreover, in the case of the dominance of the identity representation in the t channel, it is straightforward to prove the following corollary of the preceding theorem

$$\langle ab | M_s | a'b' \rangle = \alpha_A^{-\frac{1}{2}} \alpha_B^{-\frac{1}{2}} \langle Y_o (A \times \bar{A}) | M_t | Y_o (B \times \bar{B}) \rangle \delta_{aa'} \delta_{bb'} \quad (10)$$

which implies that M_s is diagonal and it is a multiple of the identity. This property of M_s is preserved under a unitary transformation of the basis $|ab\rangle$ and $|a'b'\rangle$ of our representation.

We emphasize here the fact that, in proving the preceding theorems and corollary, we have not made use of the invariance of the matrix M_s under the group J . This means that the theorems apply to the group of ordinary spin rotations of the particles, as well as to the isospin group in the presence of electromagnetic interactions (provided that one neglects mass differences between the members of an isotopic multiplet), and to such other physically interesting situations where invariance is not present, but where mass differences between members of a multiplet associated with the group can be neglected.

In the case of the spin rotation for instance, theorem 2 states that the only amplitude that can dominate in the t channel is the one which corresponds to singlet states for $\bar{A}\bar{A}$ and $\bar{B}\bar{B}$. It is simple to generalize the above theorems to the case in which the spaces A, B, C, D are reducible. In fact, let us decompose them into irreducible spaces A_j, B_j, C_j, D_j . The first theorem when extended to this case states that the amplitudes

$$\langle Y_0(A_j \times \bar{A}_j) | M_t | Y_0(\bar{B}_i \times B_i) \rangle$$

must be positive as a consequence of the positive definiteness of the M_s matrix (diagonal elements positive in any representation).

For what concerns the dominance theorem (theorem 2) it is clear that if we ask one amplitude to be dominant, this cannot be an amplitude

$$\langle A_i \times \bar{A}_i | M_t | \bar{B}_j \times B_j \rangle$$

for $i \neq i'$ and/or $j \neq j'$, which relates different irreducible representations. Indeed such a non-diagonal amplitude gives a M_s matrix which is directly seen not to be positive definite. We are therefore reduced to look for the dominant amplitude among the diagonal ones $\langle A_i \times \bar{A}_i | M_t | \bar{B}_j \times B_j \rangle$ and, therefore, the situation is reduced to the case discussed previously. Then, the amplitudes that can dominate in the t channel are one or more corresponding to the identity representations in the product spaces of irreducible representations and their respective conjugate representations, i.e., amplitudes of the form

$$\langle Y_0(A_j \times \bar{A}_j) | M_t | Y_0(\bar{B}_i \times B_i) \rangle$$

A straightforward application of this theorem is to charge conjugation. In this case particles which differ from their anti-particles correspond to a superposition of representations with $C = +1$ and $C = -1$. The theorem then says that if an amplitude dominates it must be one of those which have total C equal $+1$ in the t channel.

4. DOMINANCE THEOREM FOR ISOTOPIC SPIN IN THE u CHANNEL

In the paper of Foldy and Peierls ²⁾ the viewing of an interaction as exchange in a "crossed" sense was introduced (the motivation being crossing relations between what are here called the s channel and the u channel) and a "theorem" derived that the dominant amplitude in the u channel must correspond to minimum isotopic spin exchange. This result is unfortunately incorrect ^{*)} and the correct theorem which specifies that the dominant amplitude in the u channel must be the highest isotopic spin is here derived.

In contrast to the t channel theorem, we have not been able to derive a general dominance theorem for the u channel valid for any group, nor are we certain that there exists such a theorem. For the isospin group, however, we can,

*) Since the reason for the incorrect result of Foldy and Peierls for the u channel theorem is a little subtle and has been the source of some confusion, it may be well to detail at what point the argument is incorrect. The F-P derivation is based on Eqs. (4a) and (4b) with the identification $C = B$, $D = A$ so that one still has forward elastic scattering. This is permissible, but it was then further assumed that with this identification the resultant S_X would be positive. In fact, however, one has

$$S_X = \sum_{aba'b'} \langle Xx | Bb' Aa' \rangle \langle Bb' Aa' | M_S | AaBb \rangle \langle AaBb | Xx \rangle$$

Now one knows that $\langle Bb' Aa' | M_S | AaBb \rangle$ is a positive definite matrix from the optical theorem. If it were also true that $\langle Xx | Bb Aa \rangle = \langle AaBb | Xx \rangle^*$, then S_X would be positive. But the standard Clebsch-Gordan coefficients do not have this symmetry property and hence if one employs these to derive the Racah coefficients, as was done by F-P, one cannot assume the S_X to be positive. This invalidates the theorem of F-P and the correct u channel theorem is presented here.

following the procedure of Ref. ²⁾, employ the explicit evaluation of the crossing matrix in terms of Racah coefficients and the properties of these coefficients to derive the desired theorem for isospin.

For the isospin group, the symbols A, B, C, D, Z represent the total isospin which itself serves to completely label an irreducible representation, while the corresponding small letters, a, b, c, d, z , label the eigenvalues of the z component of isospin. Using the usual choice of phases for the rotation group, the Clebsch-Gordan coefficients, such as $\langle Zz | AaBb \rangle$, are real and it is consistent to take

$$\langle Zz | Aa \bar{B}\bar{b} \rangle = (-1)^{B-b} \langle Zz | AaB-b \rangle$$

Combining these with the usual symmetry relations ⁴⁾ of the Clebsch-Gordan coefficients and the definition of the Racah coefficients ⁴⁾ allows one to rewrite the sum in (4c) as

$$Q_{XZ} = (-1)^{A+B-C-D} (2Z+1) W(ABDC ; XZ) \quad (11)$$

Now going to the case of elastic scattering where $A = C$, $B = D$ the orthogonality relation for the Racah coefficients can be written in the form

$$\sum_X (2X+1) (2Z+1) W(ABBA ; XZ) W(ABBA ; XZ') = \delta_{ZZ'} \quad (12)$$

Furthermore we have the particular result ^{*)}

$$W(ABBA ; X, A+B) = W(ABBA ; A+B, X) > 0 \quad (13)$$

for all X such that $|A - B| \leq X \leq A + B$.

^{*)} See, for example, Eq. (6.11), p. 113, of Ref. ⁴⁾.

Thus setting $Z' = A + B$ in the orthogonality relation, multiplying by U_Z , and summing over all Z yields

$$\begin{aligned} \sum_X (2X+1) W(ABBA ; X, A+B) \sum_Z \rho_{XZ} U_Z &= \\ = \sum_X (2X+1)^{\frac{1}{2}} W(ABBA ; X, A+B) S_X &= U_{A+B} \end{aligned} \quad (14)$$

Thus if $U_{A+B} = 0$, S_X cannot be positive for all X , while on the other hand it follows from (12) and (13), that if all the U_Z vanish except for U_{A+B} and the latter is positive, then the S_X are indeed positive for all X . Summarizing :

Theorem 3 :

If the anti-Hermitian part of the T matrix in the s channel is decomposed into amplitudes corresponding to various isotopic spins in the u channel, then

- a) the amplitude U_{A+B} corresponding to maximum isotopic spin exchange must be non-vanishing and positive ;
- b) it is the only isotopic spin amplitude which can dominate in the u channel.

It should be noted that in contrast to the theorems of the preceding sections, we have explicitly used here the invariance of the T matrix under the isospin group.

It is of interest to speculate on the generalization of this theorem to other groups. It is not at all clear that it can be generalized to arbitrary groups since there is not in general a simple analogue to the "highest" representation of a group occurring in the decomposition of a product representation into irreducible representations. In the case of semi-simple groups, however, such

an analogue exists, namely the representation of highest weight, and in fact, if A and B are irreducible representations of such a group, then the highest weight representation in the decomposition of $A \times \bar{B}$ into irreducible representations occurs only once in the decomposition. Furthermore, since SU_2 is a subgroup of every semi-simple group, consistency with the isospin (SU_2) theorem proved above would suggest the highest weight representation as the only candidate for a dominating irreducible representation in the u channel. We have not been able to prove, however, that if the corresponding amplitude is the only non-vanishing amplitude in the u channel then all amplitudes in the s channel are of one sign. As discussed in the introduction, the existence of dominance theorems in the u channel may be purely academic since there is no present evidence that the dominance of any u channel amplitude is ever realized experimentally in any energy range.

We are happy to acknowledge the benefit of conversations with Dr. D. Speiser with regard to the possibility of generalizing the u channel dominance theorem to other groups.

R E F E R E N C E S

- 1) I.Ia. Pomeranchuk, J. Exptl. Theoret. Phys. (USSR) 30, 423 (1956)
(translation : Soviet Phys. JETP 3, 306 (1956)) ;
L.B. Okun and I.Ia. Pomeranchuk, *ibid.* 30, 424 (1956)
(translation : *ibid.* 3, 307 (1956)).
- 2) L.L. Foldy and R.F. Peierls, Phys. Rev. 130, 1585 (1963).
- 3) D. Amati, S. Fubini and A. Stanghellini, Nuovo Cimento 26, 896 (1962).
In this paper particular realizations of the dominance theorem were explicitly given in connection with Regge poles.
- 4) See, e.g., M.E. Rose, Elementary Theory of Angular Momentum, (John Wiley and Sons, New York, 1957) ; particularly, pp. 38, 110-115. The significance of amplitudes in crossed channels as well as the utility of Racah coefficients to express crossing relations seems to have been first recognized by F.J. Dyson, Phys. Rev. 100, 344 (1955).