CONICAL SPACE-TIMES: A DISTRIBUTION THEORY

APPROACH

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Abstract

We consider the problem of calculating the Gaussian curvature of a conical 2-dimensional space by using concepts and techniques of distribution theory. We apply the results obtained to calculate the Riemannian curvature of the 4-dimensional conical space-time. We show that the method can be extended for calculating the curvature of a special class of more general space-times with conical singularity.

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I. INTRODUCTION

Although their great popularity in recent years may be mainly attributed to their close connection with cosmic strings [1], conical space-times made their appearance in the physics literature by the end of the fifties through the work of Marder [2], whose interest was focused basically on topological aspects of locally isometric Riemannian spaces. A follow-up of Marder’s seminal findings was to come some years later with an article by Sokolov and Starobinskii [3] who, combining the celebrated Gauss-Bonet theorem of differential geometry and Einstein field equations, established a link between conical singularity and gravity. Just few years later conical space-times were rediscovered by Vilenkin [1] motivated by the investigation of gravitational effects of topological structures such as cosmic strings predicted by gauge theories [4]. Vilenkin’s solution was found using the linear approximation of General Relativity. Then, Hiscock [5] and Gott [6], followed later by Linet [7], worked out the exact solution by matching interior matter generated and exterior vacuum geometries, an approach which, in fact, had been developed with more generality by Israel in his attempts to characterize line sources in General Relativity [8].

In this paper we are concerned with the problem of how to evaluate the Riemann curvature tensor of a conical 4-dimensional space-time whose metric is known, thereby obtaining (via Einstein field equations) the energy-momentum tensor of the matter source. As in Sokolov’s article, the specific form of the metric we consider reduces the problem to the calculation of the gaussian curvature or, equivalently, the curvature scalar of 2-dimensional spaces with conical singularity. However, rather than resorting to non-local concepts in order to circumvent singularities we make use of distribution theory to extend the concept of curvature. We show that at least for a number of cases this extension allows one to define curvature at points of the manifold where there is no tangent space. In the particular case of a conical space-time this approach reproduces in a very simple way the results obtained in the previous works mentioned already.
In the course of implementing the ideas above we became aware of other works which attack the problem of conical singularities [9]. In particular, it should be mentioned the recent paper of Clark, Vickers and Wilson who apply Colombeau’s generalized functions theory to calculate the distributional curvature of cosmic strings [10]. On the other hand, topological defects and space-times with discontinuity in the derivatives of the metric tensor have been examined by Lichnerowicz, Israel, Taub and Letelier, among others [11–14]. A quite general formulation of a mathematical framework to treat concentrated sources in General Relativity using distribution theory has been put forward by Geroch and Traschen [15]. However, these approaches differ from ours either from a mathematical standpoint or in the degree of generality pursued by the authors.

II. PRELIMINARY CONCEPTS AND DEFINITIONS

In this section we briefly review some elementary concepts of distribution theory which will be used in extending the definition of curvature. For a clear and systematic treatment of distribution theory in Euclidian space the reader is referred to ref. [16].

To start with let us introduce some definitions. Consider a 2-dimensional manifold $S$ and a local coordinate system $(u, v)$. A $C^\infty$-scalar function $\varphi = \varphi(u, v)$ with compact support defined on $S$ is called a test-function. A continuous linear functional $F^x$, or a distribution, is a rule which associates a test-function $\varphi$ with a real number $(F^x, \varphi)$ such that the following conditions are satisfied:

i) $(F^x, a_1\varphi_1 + a_2\varphi_2) = a_1(F^x, \varphi_1) + a_2(F^x, \varphi_2)$, where $a_1$ and $a_2$ are real numbers (linearity condition);

ii) If the sequence of test-functions $\varphi_1, \varphi_2, ..., \varphi_n$ tends uniformly to zero, then the sequence of real numbers $(F^x, \varphi_1), (F^x, \varphi_2), ..., (F^x, \varphi_n)$ approaches zero as well (continuity condition).
A scalar function $F = F(u,v)$ defined on $S$ is said to be locally integrable if

$$\int_{U} F(u,v)\varphi(u,v)\sqrt{g}dudv < \infty$$

for any test-function $\varphi$ and an arbitrary compact domain $U \subset S$. Then, it is easy to see that any locally integrable function $F$ defines a distribution $F^*$ by the formula

$$(F^*, \varphi) = \int_{U} F\varphi\sqrt{g}dudv,$$  \hspace{1cm} (1)

where $\sqrt{g}$ denotes the determinant of the metric tensor $g_{ij}$ defined on $S$. At this point let us just note that due to $F$ and $\varphi$ being scalar functions the definition above is not coordinate-dependent. Any functional of the form (1) is called a regular distribution. If a given distribution is not regular, i.e., if it cannot be put in the form (1), it is called a singular distribution. The product of a given distribution $F^*$ by a scalar function $\alpha(x) \in C^\infty$ is the functional $(\alpha F^*)$ defined by

$$(\alpha F^*, \varphi) = (F^*, \alpha \varphi);$$  \hspace{1cm} (2)

whereas the derivative of $F^*$ with respect to the coordinate $u$ is the distribution $\left(\frac{\partial F^*}{\partial u}\right)^*$ given by the formula

$$\left(\left(\frac{\partial F^*}{\partial u}\right)^*, \varphi\right) = -\left(F^*, \frac{1}{\sqrt{g}} \frac{\partial (\sqrt{g} \varphi)}{\partial u}\right),$$  \hspace{1cm} (3)

where it must be assumed that the coordinate system is chosen such that $\sqrt{g}$ and $\frac{1}{\sqrt{g}}$ are also $C^\infty$-functions, a condition that in a sense may restrict the class of 2-dimensional manifolds upon which distributions are to be defined. (In fact, considerations concerning differentiability properties of $\sqrt{g}$ and $\frac{1}{\sqrt{g}}$ becomes relevant because we are broadening the ordinary definition of functional derivative in $\mathbb{R}^n$ to include non-euclidian manifolds.)

Now let us consider geometry. One of the basic geometrical concepts when regarding 2-dimensional manifolds is the notion of Gaussian curvature $K \ [17]$. It is a well known result that in two dimensions $K = \frac{R}{2}$ where $R$ is the curvature scalar. On the other hand if we are given the first quadratic form of a 2-dimensional manifold $S$
\[ ds^2 = Edu^2 + Fdudv + Gdv^2, \]  

where \( E, F \) and \( G \) are functions of the local coordinates \((u, v)\), then the curvature scalar \( R \) can be calculated directly by the formula \[ R = \frac{1}{\sqrt{g}} \left( \frac{\partial P}{\partial v} - \frac{\partial Q}{\partial u} \right) \]  

where

\[ P \equiv \frac{1}{\sqrt{g}} \left( \frac{\partial F}{\partial u} - \frac{\partial E}{\partial v} - \frac{1}{2} \frac{F}{E} \frac{\partial E}{\partial u} \right), \]

\[ Q \equiv \frac{1}{\sqrt{g}} \left( \frac{\partial G}{\partial u} - \frac{1}{2} \frac{F}{E} \frac{\partial E}{\partial v} \right), \]

and \( g = EG - \frac{1}{4} F^2 \).

These geometrical definitions are all very well if the 2-dimensional manifold is what we call more properly a differentiable manifold. However, if there are points where the manifold is not smooth, and as a consequence no tangent space can be defined at these points, then the usual concept of curvature is meaningless. In such cases as, for example, the 2-dimensional cone, which is not a regular surface at the vertex, equations like (5) loses its applicability, and if we insist in defining curvature at points where the manifold is not regular we have to devise another definition outside the scope of the usual differential geometry of surfaces. That is where distribution theory comes into play. Suppose that there exists a coordinate system in which \( P \) and \( Q \) are locally integrable functions. Then, we can define the curvature scalar functional by the following:

\[ R^* = \frac{1}{\sqrt{g}} \left( \frac{\partial P^*}{\partial v} - \frac{\partial Q^*}{\partial u} \right), \]

where \( P^* \) an \( Q^* \) are the regular distributions constructed, respectively, from the functions \( P \) and \( Q \) according to the prescription (1). At this point let us note that although the functions \( P \) and \( Q \) themselves are not scalars the combination in which they appear in equation (8) behaves as a scalar. Therefore, applying \( R^* \) to a test-function \( \varphi \) we have
\[(R^2, \varphi) = -\left( P^2, \frac{1}{\sqrt{g}} \frac{\partial \varphi}{\partial v} \right) + \left( Q^2, \frac{1}{\sqrt{g}} \frac{\partial \varphi}{\partial u} \right), \]

whence

\[(R^2, \varphi) = \int_s \left\{ -P \frac{\partial \varphi}{\partial v} + Q \frac{\partial \varphi}{\partial u} \right\} \, du dv. \]

Thus, given a 2-dimensional manifold with the metric tensor (4) the equation (10) written above may be considered as a definition of the curvature scalar regarded now as a functional or distribution. As we shall see in the next section, this extension of the concept of curvature will permit us to define and evaluate \( R \) (or \( K \)) for a conical surface including its vertex.

**III. THE CURVATURE SCALAR OF A CONICAL SURFACE**

In this section we apply the ideas developed previously to treat the problem of calculating the curvature scalar of the cone, the metric of which may be written in the form

\[ds^2 = d\rho^2 + \lambda^2 \rho^2 d\theta^2,\]

with \( 0 \leq \rho < \infty, 0 \leq \rho < 2\pi \) and \( \lambda = \text{const} > 0 \). It is quite known that although (11) leads to a vanishing curvature everywhere except for \( \rho = 0 \), one cannot define a global coordinate system in which the metric tensor components are constants. The non-regular character of the conical space (11) also manifests itself in that near the origin \( g_{22}(\rho) = \lambda^2 \rho^2 \) does not fulfill the regularity conditions: \( \sqrt{g_{22}}(\rho) \sim \rho, \frac{d\sqrt{g_{22}}(\rho)}{d\rho} \sim 1 \) \([7,19]\). Also, regarding the cone as a surface embedded in \( \mathbb{R}^3 \) a simple demonstration that the cone is not a regular surface follows directly from the fact that it does not admit a differentiable parametrization in the neighborhood of the vertex \([17]\).

Naturally, the conical space owes its name to the fact that its geometry may be identified with the geometry of a cone isometrically embedded in the 3-dimensional euclidian space. The metric induced on the one-sheeted cone parametrized by the equation \( z = a\rho \) may also be expressed, using cartesian coordinates, as
\[ ds^2 = \left( 1 + \frac{a^2 x^2}{x^2 + y^2} \right) dx^2 + \left( 1 + \frac{a^2 y^2}{x^2 + y^2} \right) dy^2 + \frac{2a^2 xy}{x^2 + y^2} dxdy \]  

(12)

Before calculating the curvature scalar \( R \) of the conical space as a functional we note that (11) is not written in suitable coordinates as \( \frac{1}{\sqrt{g}} \) is not \( C^\infty \) everywhere. On the other hand, starting from (12) one can check directly that \( P, Q, \sqrt{g} \) and \( \frac{1}{\sqrt{g}} \) all satisfy the conditions afore mentioned. Indeed, from (6), (7) and (12) we have

\[
P = \frac{2a^2}{\sqrt{1 + a^2}} \left[ \frac{y^3}{(x^2 + y^2)(x^2 + y^2 + a^2 x^2)} \right],
\]

(13)

\[
Q = -\frac{2a^2}{\sqrt{1 + a^2}} \left[ \frac{xy^2}{(x^2 + y^2)(x^2 + y^2 + a^2 x^2)} \right],
\]

(14)

and \( \sqrt{g} = \sqrt{1 + a^2} \). At first sight, it might appear that the functions \( P \) and \( Q \) are not locally integrable as they are not bounded. That this is not so one can immediately see by going to the new coordinates defined by \( x = r \cos \xi, \ y = r \sin \xi \) (in fact, the Jacobian of the transformation above regularize the singularity of \( P \) and \( Q \) at \( r = 0 \), thereby leading to finite integrals).

Now, let us consider the integral (10) which yields the curvature scalar as a distribution. We shall calculate this integral by first defining a small disc of radius \( \epsilon \) with center at the vertex of the cone. We remove the disc from \( S \) and call the remaining region \( S_\epsilon \). Then, we have

\[
(R^\epsilon, \varphi) = \lim_{\epsilon \to 0} \int \int_{S_\epsilon} \left( -P \frac{\partial \varphi}{\partial y} + Q \frac{\partial \varphi}{\partial x} \right) dxdy.
\]

(15)

Clearly the legitimacy of the procedure above is guaranteed by the fact that the integrand is a locally integrable function.

Recalling that \( \varphi \) has compact support and applying Green’s theorem to the right-hand side of (15) we obtain

\[
(R^\epsilon, \varphi) = \lim_{\epsilon \to 0} \left[ \int_{S_\epsilon} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \varphi dxdy - \int_{\partial S_\epsilon} (Pdx + Qdy) \varphi \right],
\]

(16)

where \( \partial S_\epsilon \) denotes the boundary of \( S_\epsilon \) and the integration is performed in the anticlockwise sense.
A quick look at eq. (16) will reveal us the presence of a term proportional to the curvature scalar in the integrand of the surface integral (see eq. (5)). As for the line integral let us find out its geometrical meaning. With this purpose let us consider the covariant derivative of the vector \( \hat{e}_u = \frac{1}{\sqrt{E}} \partial_u \) along the curve \( \gamma \) defined parametrically by \( \gamma(s) = (u(s), v(s)) \). We have

\[
\frac{D \hat{e}_u}{Ds} = \frac{d}{ds} \left( \frac{1}{\sqrt{E}} \right) \partial_u + \frac{1}{\sqrt{E}} \Gamma^\nu_{\mu u} \left( \frac{d\gamma}{ds} \right)^\nu \partial_\mu,
\]

the indices \((\mu, \nu)\) running through \(u\) and \(v\). Let us define the vector

\[
\hat{e}_u^{\perp} = \frac{\hat{e}_u - <\hat{e}_u, \hat{e}_v> \hat{e}_u}{\| \hat{e}_v - <\hat{e}_u, \hat{e}_v> \hat{e}_u \|} = \frac{\sqrt{E}}{\sqrt{g}} \left( \partial_v - \frac{1}{2} \frac{F}{E} \partial_u \right),
\]

where \( \hat{e}_v = \frac{1}{\sqrt{g}} \partial_v \) and the symbols \( \|, <, > \) denote norm and inner product, respectively. It is clear that the pair \( \{ \hat{e}_u, \hat{e}_u^{\perp} \} \) constitute a positive vector basis provided that \( \{ \hat{e}_u, \hat{e}_v \} \) is positive. The projection of the covariant derivative \( \frac{D \hat{e}_u}{Ds} \) onto the orthogonal vector \( \hat{e}_u^{\perp} \) is called the algebraic value [17] of the derivative and is denoted by \( \left[ \frac{D \hat{e}_u}{Ds} \right] \), i.e.,

\[
\left[ \frac{D \hat{e}_u}{Ds} \right] = \left\langle \frac{D \hat{e}_u}{Ds}, \hat{e}_u^{\perp} \right\rangle .
\]

From (17) and (18) it follows that

\[
\left[ \frac{D \hat{e}_u}{Ds} \right] = \frac{\sqrt{g}}{E} \Gamma^u_{\nu u} \left( \frac{d\gamma}{ds} \right)^\lambda .
\]

Since we have

\[
\Gamma^v_{uu} = \frac{1}{g} \left( -\frac{1}{4} F \frac{\partial E}{\partial u} + \frac{1}{2} E \frac{\partial F}{\partial u} - \frac{1}{2} E \frac{\partial E}{\partial v} \right),
\]

and

\[
\Gamma^v_{uv} = \frac{1}{g} \left( -\frac{1}{4} F \frac{\partial E}{\partial v} + \frac{1}{2} E \frac{\partial G}{\partial u} \right),
\]

we are led to the equation

\[
\left[ \frac{D \hat{e}_u}{Ds} \right] = \frac{1}{2} \left( P \frac{du}{ds} + Q \frac{dv}{ds} \right). \]
Therefore, (15) has the form

\[ (R^*, \varphi) = \lim_{\epsilon \to 0} \left[ \int_{S_{\epsilon}} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \varphi dx dy - 2 \int_{\partial S_{\epsilon}} \left[ \frac{D \hat{e}_x}{Ds} \right] \varphi ds \right], \tag{23} \]

where the geometrical meaning of the integrand of the line integral is explicitly displayed.

To get further insight into the concept of the algebraic value of the covariant derivative of a given unit vector \( \hat{\omega} \), let us make use of the following result [17]: if \( \chi \) is the angle between any two unit vectors \( \hat{\omega} \) and \( \hat{z} \), both defined along a certain curve \( \lambda(s) \) and \( \chi \) is taken from \( \hat{z} \) to \( \hat{\omega} \), then we have

\[ \left[ \frac{D \hat{\omega}}{Ds} \right] = \left[ \frac{D \hat{z}}{Ds} \right] + \frac{d\chi}{ds}. \tag{24} \]

Let us suppose that the vector field \( \hat{\omega} \) is constructed by parallel-transporting it along the curve \( \gamma \). In this case \( \frac{d\hat{\omega}}{ds} = 0 \), hence \( \left[ \frac{D \hat{\omega}}{Ds} \right] = \frac{d\chi}{ds} \). Thus, the algebraic value of the covariant derivative of a unit vector \( \hat{\omega} \) may be regarded as a measure of the variation of the angle between \( \hat{\omega} \) and a parallel transported vector \( \hat{z} \).

After all these considerations (23) takes the form

\[ (R^*, \varphi) = \lim_{\epsilon \to 0} \left[ \int_{S_{\epsilon}} R_{\text{reg}} \varphi dx dy + 2 \int_{\partial S_{\epsilon}} \frac{d\chi}{ds} \varphi ds \right], \tag{25} \]

where we have written \( R_{\text{reg}} \) to highlight the fact that \( R \) is calculated in a region where the conical surface is regular, and \( \chi \) denotes the angle from the vector \( \hat{e}_x \) to a unit vector \( \hat{z} \) parallel-transported along \( \partial S_{\epsilon} \).

It is apparent that the first term of the right-hand side of the equation above vanishes since \( R = 0 \) everywhere except at the origin. Thus, we have

\[ (R^*, \varphi) = \lim_{\epsilon \to 0} 2 \int_{\partial S_{\epsilon}} \frac{d\chi}{ds} \varphi ds \]

\[ = 2\varphi(0) \lim_{\epsilon \to 0} \int_{\partial S_{\epsilon}} \frac{d\chi}{ds} ds, \tag{26} \]

the last step being justified by a known theorem concerning real continuous functions (see appendix).

The limit in (26) can be calculated immediately if we note that
\[
\int_{\partial S_s} \frac{d\chi}{ds} ds = \chi(s_f) - \chi(s_i),
\]
where \(s_f\) and \(s_i\) are the final and initial values of the parameter \(s\) on \(\partial S_e\). Evidently, \(s_f\) and \(s_i\) represent the same point on \(S_e\) since \(\partial S_e\) is a circumference, so the final and initial points coincide.

Further, by the definition of the angle \(\chi\), we know that \(\chi(s_f)\) is the angle between \(\hat{z}(s_f)\) and \(\hat{e}_x(s_f)\), i.e., between the vector parallel-transported and the vector \(\hat{e}_x\), both taken at \(s_f\) on \(\partial S_e\). Analogously, \(\chi(s_i)\) measures the angle between \(\hat{z}(s_i)\) and \(\hat{e}_x(s_i)\). Since the endpoints of \(\partial S_e\) coincide and \(\hat{e}_x\) is a vector field, we have \(\hat{e}_x(s_f) = \hat{e}_x(s_i)\). Thus \(\Delta \chi \equiv \chi(s_f) - \chi(s_i)\) is indeed the angle between \(\hat{z}(s_f)\) and \(\hat{z}(s_i)\), i.e., the angular deviation of the vector that has been parallel-transported along \(\partial S_e\). Clearly, \(\Delta \chi\) is a quantity which depends on the global properties of the manifold \(S\). In this way we see that the singular term of \(R^*\) is related to the topology of the manifold, not only to its geometry.

In the cone case, this term \(\Delta \chi\) is equal to the angular deficit \(\Delta \chi = 2\pi(1 - \lambda)\). Then, from (26), we are led to the result:

\[
(R^*, \varphi) = 2\varphi(0)[2\pi(1 - \lambda)] = 4\pi(1 - \lambda)\varphi(0),
\]
which may be expressed in more familiar form in terms of Dirac delta function as

\[
R^* = 4\pi(1 - \lambda)\delta^{(2)}(\rho),
\]
where, by definition,

\[
\int \delta^{(2)}(\rho) \sqrt{g} d\rho d\theta = 1.
\]

**IV. APPLICATION TO THE GENERALIZED CONE**

We can easily generalize the results obtained in the previous section by considering a surface embedded in the 3-dimensional Euclidean surface \(\Sigma\) defined by the equation \(z = \alpha(\rho)\), where \(\alpha\) is an arbitrary \(C^\infty\)-function. The induced metric in \(\Sigma\) is given by the line element
where prime denotes derivative with respect to $\rho$. In cartesian coordinates, the equation above may be written as

$$ds^2 = [1 + \alpha''] d\rho^2 + \rho^2 d\theta^2,$$

(30)

with $\tilde{E} = 1 + \frac{2\gamma}{x^2+y^2}$, $\tilde{F} = \frac{x'}{x^2+y^2}$, and $\tilde{G} = 1 + \frac{2\gamma}{x^2+y^2}$. As we already know, the curvature scalar corresponding to (31) is obtained from (5) with $\sqrt{\tilde{g}} = \sqrt{1 + \alpha''}$ and the functions $\tilde{P}$ and $\tilde{Q}$ given as below:

$$\tilde{P} (x, y) = P[\alpha'] + P,$$

(32)

$$\tilde{Q} (x, y) = Q[\alpha'] + Q,$$

(33)

where

$$P[\alpha'] = \frac{2\alpha'' \gamma^3}{\sqrt{1 + \alpha''} \left[(1 + \alpha'')x^2 + y^2\right](x^2 + y^2)},$$

(34)

$$P = \frac{2\alpha' \alpha'' \gamma^2 x^2 y}{\sqrt{1 + \alpha''} \sqrt{x^2 + y^2}\left[(1 + \alpha'')x^2 + y^2\right]},$$

(35)

$$Q[\alpha'] = -\frac{2\alpha'' \gamma^3}{\sqrt{1 + \alpha''} \left[(1 + \alpha'')x^2 + y^2\right](x^2 + y^2)},$$

(36)

and

$$\tilde{Q} = \frac{2\alpha' \alpha'' \gamma^2 xy^2}{\sqrt{1 + \alpha''} \sqrt{x^2 + y^2}\left[(1 + \alpha'')x^2 + y^2\right]}.$$

(37)

At this point two comments are in order. The first is that, as we shall see later, the functions $P[\alpha']$ and $Q[\alpha']$ are recognized as the part of the curvature which accounts for the conical singularity. (Note that $P[\alpha']$ and $Q[\alpha']$ reduce to (13) and (14), respectively, in the particular case $\alpha(\rho) = a\rho$.) The second comment concerns the requirements which must be fulfilled by the function $\alpha(\rho)$: for the curvature scalar to be well defined as a functional we
must assure ourselves that i) $\sqrt{g}$ is $C^\infty$ and ii) $\tilde{P}$ and $\tilde{Q}$ are locally integrable. Then, we define the curvature scalar functional by

$$ (R^*, \varphi) = \lim_{\epsilon \to 0} \int_{\partial S_\epsilon} \left( \frac{\partial \tilde{P}}{\partial y} - \frac{\partial \tilde{Q}}{\partial x} \right) \varphi dx dy - \int_{\partial S_\epsilon} (P dx + Q dy) \varphi, \quad (38) $$

where, the region $S_\epsilon$ and its boundary $\partial S_\epsilon$ are defined as before.

Considering the equation above it is readily seen that $\tilde{P}$ and $\tilde{Q}$ do not contribute to the line integral. Indeed, parametrizing $\partial S_\epsilon$ by $(x = \epsilon \cos \theta, y = \epsilon \sin \theta)$ it is straightforward to see that on the boundary $\partial S_\epsilon$

$$ \tilde{P} dx + \tilde{Q} dy = 0 \quad (39) $$

Thus, the line integral which appears in (38) reduces to

$$ \int_{\partial S_\epsilon} (P \alpha' dx + Q \alpha' dy) \varphi, \quad (40) $$

To compute this term we proceed exactly as we did for the cone case. In this way, one can simply show that,

$$ \int_{\partial S_\epsilon} (P \alpha' dx + Q \alpha' dy) = 4\pi (\lambda(\epsilon) - 1), \quad (41) $$

where $\lambda(\epsilon) = \frac{1}{\sqrt{1 + \alpha^2(\epsilon)}}$; whence it follows that

$$ \lim_{\epsilon \to 0} \int_{\partial S_\epsilon} (P(\alpha') dx + Q(\alpha') dy) \varphi = 4\pi (\lambda(0) - 1) \varphi(0). \quad (42) $$

On the other hand, similarly to the cone case, if we assume that $\Sigma$ is regular at $\rho \neq 0$, the surface integral term of the equation (38) may be put in the form

$$ \int_S R_{reg} \varphi dx dy, $$

where

$$ R_{reg} = \frac{2\alpha \alpha''}{(1 + \alpha'^2)^2 \rho}. $$

Therefore, we finally conclude that the curvature scalar defined as a distribution will be given by

12
According to (30) if we have a conical singularity at $\rho = 0$, then by definition, $\alpha'(0) \neq 0$. In this case, we see that the distribution $R^\alpha$ contains a singular part which appears as a delta function.

V. EXTENSION TO 4-DIMENSIONAL SPACE-TIME

In the previous section we have shown by using techniques imported from distribution theory how to define the curvature scalar of a 2-dimensional manifold at points where conical singularities exist. In particular, we have worked out the case of the 2-dimensional cone by calculating explicitly its curvature scalar as a distribution. It turns out, as we shall see, that the same mathematical treatment may be applied to calculate the Riemannian curvature of a 4-dimensional manifold $M$ which also has conical singularities provided that $M$ near the singularity admits a suitable coordinate system in which its line element has the special form

$$ds^2 = (Adt^2 + Bdtdz + Cdz^2) + (Edu^2 + Fdudv + Gdv^2),$$

where the metric components $(A, B, C)$ depend solely on $(t, z)$ whereas $(E, F, G)$ are functions of $(u, v)$ only.

In fact, this was the case considered by Sokolov [3] in which the conical 4-dimensional space-time $M$ may be regarded as the direct product $\mathbb{R}^2 \otimes Q^2$, $Q^2$ standing for the conical surface. Truly, the possibility of decomposing a 4-dimensional space-time $M$ into 2-dimensional submanifolds $M_1$ and $M_2$ greatly simplifies the calculation of the Riemannian curvature of $M$ allowing, as a consequence, the use of distribution theory to treat the Einstein tensor as a functional. To see this, one can easily verify that the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ of $M$ calculated directly from (44) have the peculiar form
\[ \Gamma^\lambda_{\mu\nu} = \begin{cases} 
(1) \Gamma^\lambda_{\mu\nu}, & \text{for } \lambda, \mu, \nu = t \text{ or } z; \\
(2) \Gamma^\lambda_{\mu\nu}, & \text{for } \lambda, \mu, \nu = u \text{ or } v; \\
0, & \text{otherwise}; 
\end{cases} \]

where \( (1) \Gamma^\lambda_{\mu\nu}(t, z) \) and \( (2) \Gamma^\lambda_{\mu\nu}(u, v) \) are the Christoffel symbols of the submanifolds \( M_1 \) and \( M_2 \), calculated from the metrics \( (1) ds^2 = Adt^2 + Bddz + Cdz^2 \) and \( (2) ds^2 = Edu^2 + Fdudv + Gdv^2 \), respectively. Analogously, the components of the Riemann tensor are also decomposed into separate parts, as below:

\[ R^\lambda_{\mu\nu\kappa} = \begin{cases} 
(1) R^\lambda_{\mu\nu\kappa}, & \text{for } \lambda, \mu, \nu, \kappa = t \text{ or } z; \\
(2) R^\lambda_{\mu\nu\kappa}, & \text{for } \lambda, \mu, \nu, \kappa = u \text{ or } v; \\
0, & \text{otherwise}; 
\end{cases} \]

where, as before, \( (1) R^\lambda_{\mu\nu\kappa}(t, z) \) and \( (2) R^\lambda_{\mu\nu\kappa}(u, v) \) refer to the Riemannian curvature tensors of \( M_1 \) and \( M_2 \), respectively.

Further simplification is achieved by noting that due to a distinctive property of two dimensions, the Riemann tensors of the submanifolds may be written as [18]

\[ (i) R^\lambda_{\mu\nu\kappa} = \frac{1}{2} \left( g^\lambda_{\mu\kappa} g_{\nu\rho} - g^\lambda_{\mu\rho} g_{\nu\kappa} \right) (i^i R), \]  

(45)

where the index \( i = 1, 2 \) refers, evidently, to geometric quantities defined on \( M_1 \) and \( M_2 \).

From (45) we calculate the Ricci tensor of \( M \), which takes the form

\[ R^\lambda_{\mu} = \begin{cases} 
\frac{1}{2} (1) R^\lambda_{\delta\mu}, & \text{for } \lambda, \mu = t \text{ or } z; \\
\frac{1}{2} (2) R^\lambda_{\delta\mu}, & \text{for } \lambda, \mu = u \text{ or } v; \\
0, & \text{otherwise}; 
\end{cases} \]

Contracting the indices \( \lambda \) and \( \mu \) we have

\[ R = (1) R + (2) R, \]

which leads to the following expression for the Einstein tensor:

\[ G^\lambda_{\mu} = \begin{cases} 
-\frac{1}{2} (2) R^\lambda_{\delta\mu}, & \text{for } \lambda, \mu = t \text{ or } z; \\
-\frac{1}{2} (1) R^\lambda_{\delta\mu}, & \text{for } \lambda, \mu = u \text{ or } v; \\
0, & \text{otherwise}; 
\end{cases} \]
Accordingly, it is natural to define the functional \((G^\lambda_\mu)^s\) corresponding to the mixed components of the Einstein tensor by

\[
G^\lambda_\mu = \begin{cases} 
-\frac{1}{2} R^s_\mu^\lambda, & \text{for } \lambda, \mu = t \text{ or } z; \\
-\frac{1}{2} R^s_\mu^\lambda, & \text{for } \lambda, \mu = u \text{ or } v; \\
0, & \text{otherwise};
\end{cases}
\]

where the functionals \((1)^R^s\) and \((2)^R^s\) are defined as in equation (10).

In this way we have found that if the metric of a 4-dimensional manifold \(M = M_1 \otimes M_2\) can be written in the special form (44), then the Einstein tensor of \(M\) is directly obtainable from the curvature scalars of the submanifolds \(M_1\) and \(M_2\). If the manifold \(M\) is not regular everywhere, then such non-regularity may manifest itself as a non-regularity of one of (or both) its 2-dimensional submanifolds. In this situation, as we have shown earlier, the problem of calculating the Riemannian curvature of \(M\) is amenable to a distribution theory approach.

To conclude the section it is worth noting that if we make a coordinate transformation of the type \(t' = t'(t, z), z' = z'(t, z), u' = u'(u, v)\) and \(v' = v'(u, v)\), then the separable form of (44) is preserved. In this case, the components \(G^\lambda_\mu\) of the Einstein tensor do not change, i.e., they are invariant. Naturally, in this new coordinates \(G^\lambda_\mu\) may still be defined as a functional provided the following is also preserved by the coordinates transformation: i) the new functions \(P_i\) and \(Q_i\), as defined in (6) and (7), are locally integrable in each submanifold \(M_i(i = 1, 2)\); ii) \(\sqrt{(2)^g_i}\) and \(\frac{1}{\sqrt{(2)^g_i}}\) are \(C^\infty\) functions, where by \(2g_i\) we are denoting the determinant of \(M_i\).

VI. COSMIC STRINGS
It is widely known that the space-time generated by a static cosmic string is described by the metric

\[ ds^2 = -dt^2 + dz^2 + d\rho^2 + a^2\rho^2 d\theta^2, \]

where \(-\infty < t, z < \infty, 0 < \rho < \infty, 0 \leq \theta < 2\pi\). The metric of this space-time, which possesses a 2-dimensional submanifold with a conical singularity, clearly has the form (44). As is also evident from (44), one of the 2-dimensional manifolds is readily identified with the plane \(\mathbb{R}^2\) while the other is the cone, the curvature scalar of which was calculated in section III. Thus, from the definition of \((G^\lambda_\mu)^*\) and taking into account (28) we end up with

\[ G^t_t = G^z_z = -2\pi(1 - a)\delta^{(2)}(\rho), \]

while all the other components vanish.

If we assume the validity of the Einstein equations \(G^\lambda_\mu = 8\pi G T^\lambda_\mu\) \((G\) is the gravitational constant and units are chosen in which \(c = 1\)), then we determine the energy-momentum tensor \(T^\lambda_\mu\) of the material source that generates the gravitational field described by the metric (46):

\[ T^t_t = T^z_z = \frac{(1 - a)}{4G}\delta^{(2)}(\rho), \]

with all the other components equal to zero. From this result we conclude that the matter source is concentrated on the axis \(\rho = 0\) with a linear mass density \(\mu = \frac{(1 - a)}{4G}\). Such configuration of matter has exactly the same structure as a vacuum string [1], a topological defect predicted by gauge theories with spontaneous symmetry breaking. Actually, it was Sokolov and Starobinskii [3] who first showed the connection between the space-time (46) and the material source described by the energy-momentum tensor (48). Curiously enough the same solution was rediscovery by Vilenkin [1] (also by Hiscock [5] and Gott [6] later) to a certain extent by following the inverse path, i.e., starting from (48), solving the Einstein equations and, then, arriving at (44).
VII. FINAL REMARKS

By applying some concepts and definitions of distribution theory we have developed a formalism which may be applied to calculate, or more precisely, to define the Riemannian curvature of a certain class of space-times wherein conical singularities appear. It is known that the treatment of general space-times using distribution theory has revealed to be rather problematic. One of the main difficulties lies on the non-linearity of the Riemann tensor with respect to the affine connections. An attempt to formulate General Relativity theory in terms of distribution theory combining both mathematical rigour and high degree of generality has been undertaken by Geroch and Traschen [15], who have shown that in arbitrary space-time a product of connections do not make sense as a distribution unless the connections satisfy some specific mathematical properties, e.g., they must be locally square-integrable. From to this last condition as well as the peculiar way how the Riemann tensor is written in terms of the connections Geroch and Traschen go further to demonstrate that singular distributions cannot have support on a submanifold of less then three dimensions. This does not seem to be a desirable situation as a number of space-times, such as the conical space-times, are left out of consideration. In this article we have tried to overcome this difficulty. We have shown that if we renounce the idea of generality we can work out a prescription to treat a class of space-times with conical singularities using distribution theory concepts. This is due to the fact that for the particular class of space-times that we have considered one can express the Riemann tensor in a form which avoids terms containing products of distributions.

The extension of our method to more general space-times than those taken into account in (44) in order to include, for example, general static and axially symmetric space-times is currently under investigation.

Finally, if the underlying theory of gravity is not General Relativity the problem of how to obtain the energy-momentum tensor may get a bit more complicated. This is the case, for example, of the so-called scalar-tensor theories of which Brans-Dicke theory is a par-
ticular case [20]. Space-times with conical singularity have been found in the context of Brans-Dicke theory of gravity [21]. Here an extra difficulty arises due to the fact that the energy-momentum tensor is not determined by the geometry solely but depends also upon the scalar field.

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Appendix

The theorem mentioned in section III states the following:

Let us be given two real functions $g(s)$ and $h(s)$ which are continuous in the interval $L: a \leq s \leq b$. Assuming that $h(s)$ does not change its sign in $L$, then there exists a number $c$ lying on $L$ such that

$$\int_a^b g(s)h(s)ds = g(c) \int_a^b h(s)ds.$$ 

Thus, choosing $g(s)$ and $h(s)$ as $\varphi(s)$ and $\frac{dx}{ds}$, respectively, and taking into account that

$$\frac{dX}{ds} = -\left[\frac{d^2\varphi}{ds^2}\right] = -\frac{1}{\varepsilon} \left[1 - \frac{1}{\lambda^2(1+a^2\cos^2(\frac{s}{\lambda})})\right] \geq 0,$$

we have

$$\int_{\partial S_c} \frac{dx}{ds} \varphi(s)ds = \varphi(c) \int_{\partial S_c} \frac{dx}{ds} ds,$$

where $c \in [0, 2\pi\lambda\varepsilon]$. It may happen that the number $c$ depends on $\partial S_c$, i.e., $c = c(\varepsilon)$; nevertheless due to the continuity of the function $\varphi$ we must have

$$\lim_{\varepsilon \to 0} \varphi(c) = \varphi(0).$$

18
REFERENCES


