Reconstructing the global topology of the universe from the cosmic microwave background

Jeffrey R. Weeks
88 State St., Canton NY 13617, USA

Abstract. If the universe is multiply-connected and sufficiently small, then the last scattering surface wraps around the universe and intersects itself. Each circle of intersection appears as two distinct circles on the microwave sky. The present article shows how to use the matched circles to explicitly reconstruct the global topology of space.

1. Introduction

If the universe is multiply-connected and sufficiently small, then the last scattering surface (LSS) wraps around the universe and intersects itself [1]. Each circle of intersection appears as two distinct circles on the microwave sky, even though the two images correspond to the same circle of points in space itself. In their article in this issue, Cornish, Spergel and Starkman [2] show how to find such pairs of matching circles from the high-resolution data to be provided by NASA’s Microwave Anisotropy Probe (MAP) in the year 2001, or by the ESA’s Planck satellite a few years later. The present article shows how to use the matching circles to explicitly reconstruct the global topology of space.

The microwave background is isotropic to 1 part in $10^5$ [3], which implies that the curvature of space is constant to 1 part in $10^4$ [4]. Our methods work equally well in the spherical, Euclidean, and hyperbolic cases. Current evidence suggests space is hyperbolic with $\Omega_0$ approximately 0.3 or 0.4 [5]. If $\Omega_0$ is 0.4, the LSS will have a radius of about 2 and enclose a volume of about 75, in units of the curvature radius. (The curvature radius provides a natural length scale in hyperbolic as well as spherical geometry. In spherical geometry it’s usually called a “radius”, so we will apply that term in the hyperbolic case as well.) Thousands of closed hyperbolic 3-manifolds of volume less than $7$ are known [6, 7]; each would correspond to a universe in which the LSS encloses 10 or more images of each object in space, and in which the topology would be easily detectable. Moreover, the volume of a hyperbolic 3-manifold is a good measure of its complexity, and a least action argument [8, 9] suggests that low-volume universes are more probable than high-volume one. We don’t rely on such arguments, but they give us hope that the cosmic topology will be detectable.

2. Mathematical background

We will consider space as the quotient of the 3-sphere $S^3$, Euclidean 3-space $E^3$, or hyperbolic 3-space $H^3$ by a group of covering transformations. For example, the 3-torus is $E^3$ modulo the group generated by $x \rightarrow x+1, y \rightarrow y+1, z \rightarrow z+1$; a fundamental domain is the cube $0 \leq x, y, z \leq 1$. By choosing different groups of covering transformations, exactly ten topologically distinct Euclidean 3-manifolds may be obtained[10, 11]. Similarly, every spherical manifold may be obtained as the quotient of $S^3$ by a group of covering transformations. Infinitely many spherical 3-manifolds are possible, but they all fall into a few well understood families [11, 12, 13]. Hyperbolic 3-manifolds, obtainable as quotients of $H^3$, offer the greatest variety. Infinitely many are possible, and their structure is far richer than that of the spherical manifolds [14].

The description of a manifold as a quotient of $S^3, E^3, H^3$ by a group of covering transformations may easily be converted to a description as a fundamental domain. Pick an arbitrary point in the manifold, and start a balloon expanding at that point (Figure 1). Eventually the balloon grows so large that it wraps around the space and touches itself. When this happens, let the balloon keep expanding. At the points where it has touched itself, let it press against itself, forming a planar disk of contact just as two real balloons (of equal internal pressure) would form when pressed against one another.
Figure 1. We may “inflate a balloon” to construct a fundamental domain for a closed universe. For clarity the illustration shows multiple images of the balloon in the universal cover, but the construction is best imagined in the space itself, where there is only one balloon, which wraps around the space and presses against itself.

Let the balloon keep expanding until it fills the entire space. When it has filled the space, the balloon will have the shape of a polyhedron, with pairs of faces identified to form the original closed manifold.

3. Reconstructing the cosmic topology

We take as our starting point the following data, all of which may be deduced from the microwave data provided by the MAP or Planck satellites [2].

(i) The geometry of space (spherical, Euclidean or hyperbolic).

(ii) The radius of the last scattering surface. If the geometry is spherical or hyperbolic, the radius will be reported in radians. In the Euclidean case, it will be normalized to 1.

(iii) A list of matching circles, as described in the Introduction.

From these data we will reconstruct the topology of the universe both as a group of covering transformations of $S^3$, $E^3$, or $H^3$, and as a fundamental domain. Let us temporarily assume we are given perfect data; after we have laid out the basic algorithm we will consider possible imperfections in the data, and explain how to compensate for them.

The main idea is quite simple. Figure 2a shows the LSS, as seen in the universal covering space. If a pair of circles on the LSS represent the same circle in the quotient manifold, then there is a covering transformation $g$ taking one circle to the other. Figure 2b shows the image of the LSS under the action of $g$. It is straightforward to compute a matrix for $g$; in the spherical case the matrix will be in the orthogonal group $O(4)$, in the hyperbolic case it will be in the Lorentz group $O(3, 1)$, and in the Euclidean case it will be in the subgroup of $O^+(4, R)$ fixing the hyperplane $x_0 = 1$. The transformation $g^{-1}$ interchanges the roles of the two circles.

Constructing a fundamental domain is equally easy. Each matched circle is equidistant from two images of the observer (Figure 2b). But each face of the fundamental domain also lies midway between two images of the observer (Figure 1). So, roughly speaking, the planes of the circles and the planes of the fundamental domain’s faces coincide! (Figure 2c) Of course, only the planes of the largest circles correspond to actual faces of the fundamental domain; the planes of the smaller circles...
Figure 2. Constructing the fundamental domain

are too far away. Conversely, if some face of the fundamental domain lies entirely outside the LSS, then its plane does not contain an observable circle; it lies midway between two images of the observer, but the corresponding images of the LSS are too small to intersect. In the extreme case that all faces of the fundamental domain lie outside the LSS, no circles are observable and the topology of space cannot be detected. Assuming for the moment that all faces of the fundamental domain lie at least partially within the LSS, our algorithm for constructing the fundamental domain is the following:

Algorithm 3.0. Constructing a fundamental domain.

(i) Inputs:
- a space $\mathbf{X} \cong \mathbb{R}^3$, $E^3$, or $H^3$
- the radius $R_{\text{LSS}}$ of the LSS in radians (or 1 in the Euclidean case)
- a list $\mathbf{C}$ of matching circles

(ii) Output:
- A polyhedron $\mathbf{D}$ (typically a fundamental domain for the universe – cf. Proposition 3.1 below)

(iii) Algorithm:
- Begin with a ball $\mathbf{B}$ of radius $R_{\text{LSS}}$ in the simply-connected space $\mathbf{X}$.
- For each circle $\mathbf{c} \in \mathbf{C}$, let $P(\mathbf{c})$ be the plane in $\mathbf{X}$ spanned by $\mathbf{c}$, and $H(\mathbf{c})$ be the halfspace bounded by $P(\mathbf{c})$ and containing the center of the ball $\mathbf{B}$.
- Let the polyhedron $\mathbf{D}$ be the intersection of the halfspaces $H(\mathbf{c})$, for all $\mathbf{c} \in \mathbf{C}$.

If the microwave data reveal any circles at all, they will probably reveal a large number of them, and Algorithm 3.0 will compute a valid fundamental domain for the universe. However, if the universe is just barely small enough for the LSS to intersect itself, then we may observe only a few circle pairs, and Algorithm 3.0 may fail to find all the faces of the fundamental domain. Proposition 3.1 provides a sufficient condition for checking that the fundamental domain is correct.

Even if the data don’t provide enough circles initially, it’s easy to deduce where the missing faces must lie: compute matrix generators for the group of covering transformations (cf. above) and obtain the missing group elements as products of those generators. The simplest 3-manifolds all have 2- or 3-generator groups [6], so 2 or 3 pairs of matched circles would suffice.

Proposition 3.1: If the polyhedron $\mathbf{D}$ constructed by Algorithm 3.0 lies in the interior of the LSS, then it is a fundamental domain for the universe.
Proof 3.1. Clearly the true fundamental domain must be a subset of \( D \). If the true fundamental domain included a face which \( D \) lacked, then that face would lie in the interior of the LSS. The plane spanned by the face would intersect the LSS in a “matched circle”. Assuming perfect data (no missing circles), we must have already found that face. QED

The circle detection algorithm described in [2] is quite reliable. Nevertheless, we must be prepared for both missing circles and false matches. Missing circles may be reconstructed by multiplying together the group elements corresponding to known circles. False matches may be detected because they won’t fit in with the group structure. That is, the covering transformations corresponding to all valid circles should fit together to form a discrete group: the composition of any two such covering transformations should give another valid transformation. If we find that a few of the group elements are inconsistent with the overall structure of the discrete group, then we may reject them and their corresponding circles as false matches.

The algorithm for constructing fundamental polyhedra has been implemented as part of the computer program SnapPea, but only for the hyperbolic case [6]. The author is extending it to the spherical and Euclidean cases. SnapPea lets the user compute a wide variety of invariants for the resulting manifold, and also check for homeomorphisms with known manifolds.

4. Verifying the observational data

If we do indeed find matching circles in the microwave data, how can we be certain that our results are correct? How do we know that the MAP or Planck satellite didn’t report bad data? How do we know that our computer programs didn’t contain serious bugs? Fortunately, the discreteness of the group of covering transformations provides a reliable check against both observational and computational errors. The composition of any two covering transformations must yield a third, to within a known tolerance. It is effectively impossible for bad data to yield a discrete group by chance. (As mentioned above, some small portion of the circles may need to be rejected as “false matches”, but they are expected to comprise well under 1% of the total data. The remaining 99% of the circle pair should then yield a discrete group.)

If we get more than just a few circle pairs, then we may confirm the data even more dramatically by using the largest circles to “predict” the smaller ones. That is, we may construct generators for the group of covering transformations using the largest circles only, and then take products of the generators to predict the sizes and locations of all remaining circles. This method is, of course, equivalent to checking the discreteness of the group, but most people find it more convincing to see that a small subset of the data predicts the remainder of the data set.

5. Sharpening \( R_{\text{LSS}} \) and \( \Omega_0 \)

In the spherical and hyperbolic cases (but not the Euclidean case) we may use the geometry to sharpen the reported value for the radius \( R_{\text{LSS}} \) of the last scattering surface (in spherical or hyperbolic radians). The sharpened value for \( R_{\text{LSS}} \) may then be used to sharpen the values of \( \Omega_0 \) and other cosmological parameters.

To sharpen the value of \( R_{\text{LSS}} \), consider the fundamental domain computed by Algorithm 3.0. When its faces are identified in pairs to form the closed manifold, its edges come together in groups of three. In theory, the sum of the dihedral angles of the three edges in each group should be exactly \( 2\pi \). If in practice we find that the angle sums are all, say, slightly greater than \( 2\pi \), this implies that our value of \( R_{\text{LSS}} \) is too low (in the hyperbolic case) or too high (in the spherical case). We should replace the old value of \( R_{\text{LSS}} \) with a new value which makes the average angle sum as close to \( 2\pi \) as possible.

The sharpened value of \( R_{\text{LSS}} \) may be used to sharpen \( \Omega_0 \) as well. The exact relationship between \( R_{\text{LSS}} \) and \( \Omega_0 \) depends on the redshift of the LSS, and takes into account the effects of both matter and radiation. (For rough estimates, it may be approximated to within a few percent by \( R_{\text{LSS}} \equiv \arccos((2 - \Omega_0)/\Omega_0) \) in the spherical case, or \( R_{\text{LSS}} \equiv \mathrm{arcosh}(2 - \Omega_0)/\Omega_0 \) in the hyperbolic case[15].) The sharpened value of \( \Omega_0 \) may in turn be used to sharpen the values of other cosmological parameters.

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References


