Is There a General Area Theorem for Black Holes?

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Abstract
The general validity of the area law for black holes still seems to be an open problem. We review the (local) formulation and proof of the area law under additional smoothness assumptions and derive some elementary consequences.

Introduction
It seems to be widely accepted as fact that the surface area of a black hole cannot decrease with time. However, the proofs offered in standard text-books like [HE], [MTW] and [W] are basically content with the remark that this law follows from the non-convergence of the generators of the future event horizon. It would indeed follow from this remark and some elementary differential geometric considerations if the horizon were a sufficiently smooth submanifold. But this is known not to be the case in general and the text-book proofs do not indicate how to proceed without further assumptions. In this note we do not offer a general proof either. Rather, we present our understanding of the text-book proofs with special emphasis on the additional assumptions (on differentiability) they need. Basically we will need piecewise $C^2$-smoothness of the black hole surface on the initial Cauchy slice. However, in general the horizon just satisfies the condition of local Lipschitz continuity (denoted by $C^{1-}$) so that better arguments need to be devised. These might involve suitably smooth approximations of the horizon, or the restriction to just topological and measure-theoretic arguments. Presently we are unaware of such a proof in the literature. However, given the widely believed connection of the area law with thermodynamic
properties of black holes on one side, and the widely expressed hope that this connection may be of heuristic value in understanding certain aspects of quantum gravity on the other, it seems well motivated to call for a proof of the area law without additional differentiability assumptions.

**Notation, Facts and Assumptions**

We assume space-time \((M, g)\) to be strongly asymptotically predictable (in the sense of [W]) and globally hyperbolic. (It would be sufficient to restrict to a globally hyperbolic portion, as in Thm. 12.2.6 of [W].) \(I^+\) (scri-plus) denotes future null infinity, \(J^-(I^+)\) its causal past and \(B := M - J^-(I^+)\) the black-hole region. Its boundary, \(\partial B =: H\), is the future-event-horizon. \(H\) is a closed, imbedded, achronal three-dimensional \(C^1\)-submanifold of \(M\) (Proposition 6.3.1 in [HE]). \(H\) is generated by null geodesics without future end points. Past end points occur only where null geodesics, necessarily coming from \(J^-(I^+)\), join onto \(H\). Such points are called “caustics” of \(H\). At a caustic \(H\) is not \(C^1\) and has therefore no (continuous) normal. Once a null geodesic has joined onto \(H\) it will never encounter a caustic again, never leave \(H\) and not intersect any other generator. See Box 34.1 in [MTW] for a lucid discussion and partial proofs of these statements. Hence there are two different processes through which the area of a black hole may increase: First, new generators can join the horizon and, second, the already existing generators can diverge.

Let \(\Sigma\) be a suitably smooth (here \(C^2\)) Cauchy surface, then \(B := B \cap \Sigma\) is called a black-hole region at time \(\Sigma\) and \(\mathcal{H} := H \cap \Sigma = \partial B\) the (future-event-) horizon at time \(\Sigma\). A connected component \(B_i\) of \(B\) is called a black-hole at time \(\Sigma\). Its surface is \(\mathcal{H}_i = \partial B_i\), which is a two-dimensional, imbedded \(C^1\)-submanifold of \(\Sigma\). In general \(\mathcal{H}\) may contain all kinds of caustic sets, like dense ones and/or those of non-zero measure, which are not easily dealt with in full generality. Below we shall avoid this problem by adding the hypothesis of piecewise \(C^2\)-smoothness.

By \(\exp\) we denote the exponential map \(TM \to M\). Recall that \(\exp_p(v) := \gamma(1)\), where \(\gamma\) is the unique geodesic with initial conditions \(\gamma(0) = p \in M\) and \(\dot{\gamma}(0) = v \in T_p(M)\). For each \(p\) it is well defined for \(v\) in some open neighbourhood of \(0 \in T_p(M)\). One has \(\gamma(t) = \exp_p(tv)\). We shall assume the Lorentzian metric \(g\) of
$M$ to be $C^2$, hence the connection (i.e. the Christoffel Symbols) is $C^1$ and therefore the map $\exp : TM \to M$ is also $C^1$. The last assertion is e.g. proven in [L].

Local Formulation of the Area Law

We consider two $C^2$ Cauchy surfaces with $\Sigma'$ to the future of $\Sigma$. The corresponding black-hole regions and surfaces are denoted as above, with a prime distinguishing those on $\Sigma'$. We make the assumption that $H$ is piecewise $C^2$, i.e. each connected component $H_i$ of $H$ is the union of open subsets $H_i^k$ which are $C^2$ submanifolds of $M$ and whose 2-dimensional measure exhaust that of $H_i$: $\mu(H_i - \bigcup_k H_i^k) = 0$, where $\mu$ is the measure on $H$ induced from the metric $g$.

For each point $p \in H_i^k$ there is a unique future- and outward-pointing null direction perpendicular to $H_i^k$, which we generate by some future directed $l(p) \in T_p(M)$. We can choose a $C^1$-field $p \mapsto l(p)$ of such vectors over $H_i^k$. The geodesics $\gamma_p : t \mapsto \gamma_p(t) := \exp_p(tl(p))$ are generators of $H$ without future end point. This implies that each $\gamma_p$ cuts $\Sigma'$ in a unique point $p' \in H'$ at a unique parameter value $t = \tau(p)$. By appropriately choosing the affine parametrisations of $\gamma_p$ as $p$ varies over $H_i^k$ we can arrange the map $\tau$ to be also $C^1$. Hence $p \mapsto m(p) := \tau(p)l(p)$ is a null vector field of class $C^1$ over $H_i^k$. We can now define the map

$$\Phi^k_i : H_i^k \to H', \quad p \mapsto \Phi^k_i(p) := \exp_p(m(p)), \quad (1)$$

which satisfies the following

**Lemma.** $\Phi^k_i$ is (i) $C^1$, (ii) injective, (iii) non-measure-decreasing.

(i) follows from the fact that the functions $m$ and $\exp$ are $C^1$. Injectivity must hold, since otherwise some of the generators of $H$ through $H_i^k$ would cross in the future. By non-measure-decreasing we mean the following: Let $\mu$ and $\mu'$ be the measures on $H$ and $H'$ induced by the space-time metric $g$. Then $\mu[U] \leq \mu'[\Phi(U)]$ for each measurable $U \subset H_i^k$. Assuming the weak energy condition, this is a consequence of the nowhere negative divergence for the future geodesic congruence $p \mapsto \gamma_p$ (Lemma 9.2.2 in [HE]), as we will now show.

**Proof of (iii):** Set $H_i^k := \bigcup_{p,t} \exp_p(tl(p))$, $\forall p \in H_i^k$ and $\forall t \in \mathbb{R}_+$, which is a $C^1$-submanifold of $M$. Let $l$ be the unique (up to a constant scale) future directed
null geodesic (i.e. $\nabla v = 0$) vector field on $H^k_i$ parallel to the generators. Then
$0 \leq \nabla_\mu \mu = \pi^\mu_\nu \nabla_\mu \nu$, where $\pi$ denotes the map given by the $g$-orthogonal projection
$T(M)|_{H^k_i} \to T(H^k_i)$, followed by the quotient map $T(H^k_i) \to T(H^k_i)/\text{span}\{l\}$. Note
that tangent spaces of $C^1$-cross-sections of $H^k_i$ at the point $p$ are naturally identified
with $T_p(H^k_i)/\text{span}\{l(p)\}$. Since $\pi^\mu_\nu l^\nu = 0$, we also have $\pi^\mu_\nu \nabla_\mu k^\nu \geq 0$ for $k = 
lambda l$ and any $C^1$-function $\lambda : H^k_i \to \mathbb{R}_+$. Hence this inequality is valid for any
future pointing $C^1$-vector-field $k$ on $H^k_i$ parallel to the generators. Given that, let
then $t \mapsto \phi_t$ be the flow of $k$ and $A(t) := \mu_t[\phi_t(U)] := \int_{\phi_t(U)} d\mu_t$, then $\dot{A}(t) = \int
_{\phi_t(U)} \pi^i_\nu(t) \nabla_\mu k^\nu(t) d\mu_t \geq 0$, where $\pi(t)$ projects onto $T(\phi_t(H^k_i))/\text{span}\{l\}$, $k(t) = \frac{d}{dt}|_{t'=t} \phi_t$ and $\mu_t$ = measure on $\phi_t(H^k_i)$. Now choose $k$ such that $\phi_{t=1} = \Phi_i^k$. Then
$\mu'[\Phi_i^k(U)] - \mu[U] = \int_0^1 dt \dot{A}(t) \geq 0$.

Consequences

From Proposition 9.2.5 of [HE] it is known that black holes cannot bifurcate in the future. Hence all surface elements $\mathcal{H}_i^k$ of the $i$-th black-hole at time $\Sigma$ are mapped
via $\Phi_i^k$ into the surface of a single black-hole at time $\Sigma'$, whose area therefore cannot
be less then the area of $\mathcal{H}_i$. Note that this does not exclude that the number $N'$ of black-holes at time $\Sigma'$ might be bigger than their number $N$ at time $\Sigma$. But
it implies that this can only be achieved by an intermediate formation of $K \text{ new}$
black-holes $B'_1 \cup \cdots \cup B'_K$, where $K \geq N' - N$. That these black-holes are 'new', i.e.
not present at time $\Sigma$, means that all generators of $H$ which intersect $\cup_{i=1}^{K} \mathcal{H}'_i$
must have past endpoints somewhere between $\Sigma$ and $\Sigma'$. Hence we have

**Assertion 1.** Consider two Cauchy surfaces, $\Sigma$ and $\Sigma'$, with $\Sigma'$ to the future of $\Sigma$. Suppose there is a black hole at time $\Sigma'$ whose area is smaller than any black hole
area at time $\Sigma$. Then all null generators of the future event horizon intersecting the
surface of this black hole must have past end points between $\Sigma$ and $\Sigma'$.

There is another interesting consequence of our analysis: Consider a configuration
of two black holes which merge between $\Sigma$ and $\Sigma'$. We assume $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}'$ to be homeomorphic to two-spheres. Suppose $\mathcal{H}_1$ did not contain any caustics, i.e.,
that $\mathcal{H}_1$ was a $C^1$-submanifold of $\Sigma$. Then we can construct a map $\Phi_1 : H \to \mathcal{H}'$
alogous to the construction of $\Phi_i^k$ above, but now defined on all of $\mathcal{H}_1$. The
$C^1$-condition on $\mathcal{H}_1$ now implies that $\Phi_1$ is $C^0$. $\Phi_1$ is also injective for the same
reason as given for $\Phi^k_i$. Since some generators that cut $\mathcal{H}'$ come from $\mathcal{H}_2$, the map $\Phi_1$ cannot be surjective. If $p'$ is a point of $\mathcal{H}'$ not in the image of $\Phi_1$, we have a continuous injective map $S^2 \cong \mathcal{H}_1 \rightarrow \mathcal{H} - \{p'\} \cong R^2$. But this is impossible since such a map does not exist. One way to see this is through a theorem in topology, due to Borsuk and Ulam (proven e.g. in chapter 9 of [A]), which says that any continuous map $S^2 \rightarrow R^2$ identifies some pair of antipodal points. In particular, it cannot be injective. Hence we obtain a contradiction to the assumption that $\mathcal{H}_1$ was $C^1$. The same applies of course to $\mathcal{H}_2$, or any other black hole that is going to merge at some later time. Note that it does not matter how far back in time $\Sigma$ actually is. Thus, under the assumption of spherical topologies for the surfaces of the black holes (which should not be essential), we have shown the following

**Assertion 2.** At no time before merging can the surface of a black hole be without caustics.

**References**


