Quantum Error Correction Codes

1 Introduction

The problem of quantum error correction is to construct a quantum code (QEC) that can protect a quantum state from decoherence. The basic idea is to encode the state in a larger Hilbert space, so that errors on the encoded state have a larger chance of being detected and corrected.

Quantum error correction codes (QEC) are a natural way to protect a quantum state. The idea is to encode the state in a larger Hilbert space, so that errors on the encoded state have a larger chance of being detected and corrected.

Keywords: Code, Hilbert space, Quantum error correction.

Abstract: I present two general methods to construct quantum codes. One is based on the use of stabilizer codes, which are the quantum analogs of classical linear codes. The other is based on the construction of quantum error correction codes from classical codes.
a (possibly infinite) sequence of classical binary numbers \((a_1, a_2, \ldots, a_m, \ldots)\), the encoding \((b_1, c_1, b_2, c_2, \ldots, b_m, c_m, \ldots)\) with

\[
k_t = a_t + a_{t-2} \text{ mod } 2, \quad c_t = a_t + a_{t-1} + a_{t-2} \text{ mod } 2
\]

for all \(a_t\) and \(a_0 = a_{-2} = 0\) is an example of classical convolutional code that can correct up to one error for every four consecutive bits (see, for example, chap. 4 in Ref. [13] and Lemma 8 in Section 3 for details).

In classical error correction, good convolutional codes often outperform their corresponding block codes in the sense that they have higher encoding efficiencies [13, 20]. Thus, it is instructive to find quantum convolutional codes (QCCs) and to analyze their performance. Here, I report two ways to construct QCCs. And from one of these methods, I construct a QCC of rate 1/4 that can correct one quantum error for every eight consecutive quantum registers (see Ref. [11] for more details).

2 Constructing Quantum Convolutional Codes From Quantum Block Codes

In this Section, I report a general scheme to construct QCCs from quantum block codes (QBCs). But before doing so, let me first introduce some basic notations. Suppose each quantum register has \(N\) orthogonal eigenstates, where \(N\) is an integer greater than one. Then, the basis of a general quantum state making up of a collection of possibly infinite quantum registers can be chosen as \(\{|k\rangle \equiv \{|k_1, k_2, \ldots, k_m, \ldots\rangle\}\), where \(k_m \in \mathbb{Z}_N\) for all \(m \in \mathbb{Z}\) with \(N \geq 2\). Moreover, I abuse the notation by defining \(k_m = 0\) for all \(m \leq 0\). Finally, all additions and multiplications in all state kets below are modulo \(N\).

**Definition 1.** Let \(\langle x \rangle \equiv \sum_{k_1, k_2, \ldots, k_m} a_{k_1, k_2, \ldots, k_m} |k_1, k_2, \ldots, k_m, \ldots\rangle \equiv \sum_{\{k\}} a_k |k\rangle\) be a quantum state. Any quantum error can be regarded as an error operator \(\mathcal{E}\) acting on this state. In particular, there is a **spin flip error** occurring at quantum register \(m\) (with respect to the basis \(\{|k\rangle\}\) if and only if \(\mathcal{E}|x\rangle = \sum_{\{k\}} a_k |k_1, k_2, \ldots, k_{m-1}, k_m, k_{m+1}, \ldots\rangle\), where \(k_m(k_m, \mathcal{E})\) is a \(\mathbb{Z}_N\)-function of \(k_m\) and \(\mathcal{E}\). Moreover, a spin flip error is said to be **additive** provided that \(k_m(k_m, \mathcal{E}) = k_m + \alpha \text{ mod } N\) for some \(\alpha(\mathcal{E})\).

Similarly, there is a **phase shift error** occurring at quantum register \(m\) (with respect to the basis \(\{|k\rangle\}\) if and only if \(\mathcal{F}|x\rangle = \sum_{\{k\}} a_k f(k_m, \mathcal{E}) |k\rangle\) for some complex-valued function \(f(k_m, \mathcal{E})\) with \(|f|^2 = 1\). Spin flip and phase shift errors occurring at more than one quantum register are defined in a similar way.

With the above notations and definition in mind, a QBC and a QCC can be defined as follows:
Definition 2. The linear map sending
\[ |k\rangle \mapsto \sum_{i_1, i_2, \ldots, i_m} a_{i_1, i_2, \ldots, i_m} |i_1, i_2, \ldots, i_m\rangle \equiv \sum_{i} a_{i}^{k} |i\rangle \equiv |k_{\text{encode}}\rangle, \tag{3} \]
where \( a_{i}^{k} \in \mathcal{A} \), and \( k_i \in \mathbb{Z}_N \) for all \( i = 1, 2, \ldots, N \), is said to be a quantum block code (QBC) that can correct errors in the set \( E \) if and only if Eq. (1) is satisfied for all \( \mathcal{A}, \mathcal{B} \in E \). Since Eq. (3) encodes every \( n \) quantum registers to \( m \) registers, the rate of this code is, therefore, defined as \( n/m \). In addition, one can encode the quantum state \( \bigotimes_{i} |k^{(i)}\rangle \) using the above QBC as \( \bigotimes_{i} |k^{(i)}_{\text{encode}}\rangle \).

On the other hand, if the encoding scheme expressed in Eq. (3) depends on current as well as past quantum states (that is, the coefficients \( a_{i}^{k} \) in Eq. (3) depend on more than one \( k^{(j)} \)), then it is called a quantum convolutional code (QCC). The rate of this convolutional code equals \( n/m \) because it asymptotically encodes every \( n \) quantum registers as \( m \) registers.

With the above definitions in mind, one can construct a family of QCCs from a QBC as follows:

**Theorem 3.** Given a QBC in Eq. (3) and a quantum state \( |k\rangle \equiv \bigotimes_{i=1}^{+\infty} |k_{i}\rangle \) making up of possibly infinitely many quantum registers, then the mapping
\[ |k\rangle \equiv \bigotimes_{i=1}^{+\infty} |k_{i}\rangle \longmapsto |k_{\text{encode}}\rangle \equiv \bigotimes_{i=1}^{+\infty} \sum_{j_{i}} a_{j_{i}}^{(\sum_{p} \mu_{p} k_{p})} |j_{i}\rangle \] \( \tag{4} \)
forms a QCC of rate \( n/m \) provided that the matrix \( \mu_{ij} \) is invertible. This QCC handles errors in the set \( \mathcal{E} \equiv \mathcal{E}_{1} \oplus \mathcal{E}_{2} \oplus \mathcal{E}_{3} \oplus \cdots \) and \( \mathcal{E}' \equiv \mathcal{E}'_{1} \oplus \mathcal{E}'_{2} \oplus \mathcal{E}'_{3} \oplus \cdots \) on the encoded quantum registers by computing \( \langle k_{\text{encode}}' | \mathcal{E}' \rangle \).

**Proof.** Let me consider the effects of error \( \mathcal{E}' \equiv \mathcal{E}_{1} \oplus \mathcal{E}_{2} \oplus \mathcal{E}_{3} \oplus \cdots \) and \( \mathcal{E}' \equiv \mathcal{E}'_{1} \oplus \mathcal{E}'_{2} \oplus \mathcal{E}'_{3} \oplus \cdots \) on the encoded quantum registers by computing \( \langle k_{\text{encode}}' | \mathcal{E}' \rangle \).

\[ \langle k_{\text{encode}}' | \mathcal{E}' \rangle = \prod_{i=1}^{+\infty} \left[ \sum_{j_{i}} a_{j_{i}}^{(\sum_{p} \mu_{p} k_{p})} \langle j_{i} | \mathcal{E}_{1}^{j_{i}} \mathcal{E}_{1}^{j_{i}} | j_{i}\rangle \right] \]
\[ = \prod_{i=1}^{+\infty} \left[ \left( \left( \sum_{p} \mu_{p} k_{p} \right)_{\text{encode}} | \mathcal{E}_{1}^{j_{i}} \mathcal{E}_{1}^{j_{i}} | \left( \sum_{p} \mu_{p} k_{p} \right)_{\text{encode}} \right) \right] \]
\[ = \prod_{i=1}^{+\infty} \left[ \delta_{\sum_{p} \mu_{p} k_{p}, \sum_{p} \mu_{p} k_{p} A_{\mathcal{E}_{1}^{j_{i}}}} \right] \tag{5} \]
for some constants \( A_{\mathcal{E}_{1}^{j_{i}}}, \mathcal{E}_{1}^{j_{i}} \) independent of \( k \). Because \( \mu \) is invertible, it is clear that \( k_{i} = k_{j}^{i} \) for all \( i \in \mathbb{Z}^{+} \) is the unique solution for the systems of
linear equations \( \sum_{p} \mu_{p} k_{p} = \sum_{p} \mu_{p} k'_{p} \). Consequently, \( \langle \mathbf{k}_{\text{encode}} | \mathbf{c}_j | \mathbf{k}_{\text{encode}} \rangle = \delta_{\mathbf{k}, \mathbf{k}'} \), for some constant \( \lambda_{\mathbf{c}, \mathbf{c}'} \), independent of \( \mathbf{k} \) and \( \mathbf{k}' \). Thus, the mapping in Eq. (4) is a QCC. \( \square \)

Now, let me use Theorem 3 to give an example of a QCC.

**Example 1.** Starting from the five qubit perfect code for \( N = 2 \) [6, 10, 18], Theorem 3 implies that the following QCC can correct up to one error in every five consecutive qubits:

\[
|k_1, k_2, \ldots, k_m, \ldots \rangle \mapsto \bigotimes_{i=1}^{+\infty} \left[ \frac{1}{N^{3/2}} \sum_{p,q,r,i=0}^{N-1} (-1)^{(k_{i} + k_{i+1})(p_{i} + q_{i} + r_{i}) + p_{i}r_{i}} |p_{i}, q_{i}, p_{i} + r_{i}, q_{i} + r_{i}, p_{i} + q_{i} + k_{i} + k_{i+1} \rangle \right]
\]

(6)

where \( k_m \in \{0, 1\} \) for all \( m \in \mathbb{Z}^+ \). The rate of this code is 1/5.

Although the QCC in Eq. (3) looks rather complicated, the actual encoding process can be performed readily. Since \( \mu \) is invertible, one can reversibly map \( |k_1, k_2, \ldots, k_m, \ldots \rangle \) to \( |\sum_{p} \mu_{p} k_{p}, \sum_{p} \mu_{p} k_{p}, \ldots, \sum_{p} \mu_{p} k_{p}, \ldots \rangle \) [1, 2, 12]. Then, one obtains the above five qubit QCC by encoding each quantum register using various encoding procedures described in Refs. [3, 5, 10, 18].

3 Constructing Quantum Convolutional Codes From Classical Convolutional Codes

In this Section, I report a general method to construct QCCs from classical convolutional codes. My construction is based on the following two technical lemmas which hold for both QBCs and QCCs:

**Lemma 4.** Suppose the QECC

\[
|k \rangle \mapsto \sum_{|j\rangle} a_j^{(k)} |j \rangle
\]

(7)

corrects (independent) additive spin flip errors in certain quantum registers. Then, the following QECC, which is obtained by discrete Fourier transforming every quantum register in Eq. (7),

\[
|k \rangle \mapsto \sum_{|j, p \rangle} a_j^{(k)} \prod_{i=1}^{+\infty} \left( \frac{1}{\sqrt{N}} \omega_N^{jp_i} \right) |p \rangle
\]

(8)

corrects (independent) phase errors occurring in the same set of quantum registers. The converse is also true.
Proof. Consider two arbitrary but fixed additive spin flip errors \( \mathcal{E} \equiv \bigotimes_{i=1}^{\infty} \mathcal{E}_i \) and \( \mathcal{E}' \equiv \bigotimes_{i=1}^{\infty} \mathcal{E}'_i \) acting on the code in Eq. (7).\(^7\) denote the set of all quantum registers affected by either one of the above spin flip errors and unaffected by both errors as \( A \) and \( U \), respectively. Then Eqs. (1) and (7) imply that

\[
\sum_{(j, j') \in A} \left[ a_j^\dagger a_j^\dagger \prod_{i \in U} \delta_{j, j'} \prod_{j' \in A} \langle j' | \mathcal{E}'_i | j' \rangle \right] = \delta_{\mathbf{k}, \mathbf{k}'} A_{\mathcal{E}, \mathcal{E}'} \tag{9}
\]

for some constant \( A_{\mathcal{E}, \mathcal{E}'} \), independent of \( \mathbf{k} \) and \( \mathbf{k}' \).

For additive spin errors, \( \langle j' | \mathcal{E}'_i | j' \rangle = \langle j'_i + a_j | j'_i + a_i \rangle = \delta_{j'_i + a_j, j'_i + a_i} \) for some constants \( a_i, a'_i \in \mathbb{Z}_N \). In other words, \( \langle j' | \mathcal{E}'_i | j' \rangle \) is a binary function of \( j_i - j'_i \) only. Thus, Eq. (9) still holds if I replace \( \langle j' | \mathcal{E}'_i | j' \rangle \) by any complex-valued function \( g \) taking arguments on \( j_i - j'_i \) for all \( i \in A \). That is to say,

\[
\sum_{(j, j') \in A} \left[ a_j^\dagger a_j^\dagger \prod_{i \in U} \delta_{j, j'} \prod_{j' \in A} g(j_i - j'_i) \right] = \delta_{\mathbf{k}, \mathbf{k}'} A_g \tag{10}
\]

for some complex-valued \( A_g \), independent of \( \mathbf{k} \) and \( \mathbf{k}' \). Conversely, it is obvious that if \( a_j^\dagger a_j^\dagger \) satisfies Eq. (10), then Eq. (7) is a QECC that can be constructed by correcting the above additive spin flip errors. In other words, Eq. (10) is a necessary and sufficient condition for the QECC to correct additive spin flip errors.

Now, I consider the actions of two phase shift errors \( \mathcal{F} \) and \( \mathcal{F}' \) acting on the same set of quantum registers as those in \( \mathcal{E} \) and \( \mathcal{E}' \), respectively. Then

\[
\langle \mathbf{k}'_{\text{encode}} | \mathcal{F}'^\dagger \mathcal{F} | \mathbf{k}_{\text{encode}} \rangle = \sum_{(j, j') \in A} \left[ a_j^\dagger a_j^\dagger \prod_{i \in U} \left( \frac{1}{\sqrt{N}} \delta_{j, j'} \right) \prod_{j' \in A} \langle j' | \mathcal{F}'_i | j' \rangle \right] \times \prod_{j' \in A} \left( \frac{1}{\sqrt{N}} \langle j' | \mathcal{F}_i | j' \rangle \right)
\]

\[
= \sum_{(j, j') \in A} \left[ a_j^\dagger a_j^\dagger \prod_{i \in U} \delta_{j, j'} \prod_{j' \in A} \left( \omega_N^{\langle j, j'-\gamma | j' \rangle} \right) \sum_{j'' \in A} \langle j | \mathcal{F}_i | j'' \rangle \right] \times \prod_{j' \in A} \left( \frac{1}{\sqrt{N}} \langle j' | \mathcal{F}_i | j' \rangle \right) \tag{11}
\]

For phase shift errors, \( \langle j' | \mathcal{F}'_i | j' \rangle = \delta_{j'_i, j_i} h \) for some complex-valued function \( h \) of \( h_i : i \in A \) with \( |h|^2 = 1 \). Consequently, Eq. (11) can be further simplified as

\[
\langle \mathbf{k}'_{\text{encode}} | \mathcal{F}'^\dagger \mathcal{F} | \mathbf{k}_{\text{encode}} \rangle
\]
\[
\sum_{\mathbf{z}} \left( \sum_{\mathbf{j} \in \mathbb{Z}} a_{\mathbf{j}}^{(\mathbf{k})} \left( \prod_{i \in U} \delta_{j_i, j'_i} \right) \omega_{N}^{\sum_{i \in A} \epsilon_{i}(j_i - j'_i)} h(p_i : i \in A) \right) = \sum_{\mathbf{z}} \left( \sum_{\mathbf{j} \in \mathbb{Z}} a_{\mathbf{j}}^{(\mathbf{k})} \left( \prod_{i \in U} \delta_{j_i, j'_i} \right) h'(j_i - j'_i : i \in A) \right),
\]

for some complex-valued function \(h(p_i : i \in A)\). Summing over all the \(p_i\)s in Eq. (12), I obtain

\[
\langle k'_{\text{encode}} | F^{\dagger} \mathcal{F} | k_{\text{encode}} \rangle = \sum_{\mathbf{z}} \left( \sum_{\mathbf{j} \in \mathbb{Z}} a_{\mathbf{j}}^{(\mathbf{k})} \left( \prod_{i \in U} \delta_{j_i, j'_i} \right) h'(j_i - j'_i : i \in A) \right)
\]

for some complex-valued function \(h'(j_i - j'_i : i \in A)\). Comparing Eqs. (10) and (13), one concludes that \(\langle k'_{\text{encode}} | F^{\dagger} \mathcal{F} | k_{\text{encode}} \rangle = \delta_{\mathbf{k}, \mathbf{k}'}, \Lambda_{\mathcal{F}, \mathcal{F}}\), for some \(\Lambda_{\mathcal{F}, \mathcal{F}}\) independent of both \(\mathbf{k}\) and \(\mathbf{k}'\). Thus, the QEC \\_4 given in Eq. (8) corrects the phase shift errors as promised.

Conversely, from Eq. (13), one concludes that Eq. (8) corrects phase errors if and only if

\[
\sum_{\mathbf{z}} \left( \sum_{\mathbf{j} \in \mathbb{Z}} a_{\mathbf{j}}^{(\mathbf{k})} \left( \prod_{i \in U} \delta_{j_i, j'_i} \right) h'(j_i - j'_i : i \in A) \right) = \delta_{\mathbf{k}, \mathbf{k}'}, \Lambda_{\mathcal{F}, \mathcal{F}}
\]

for any complex-valued function \(h'(j_i - j'_i : i \in A)\). Hence, from Eq. (10), one concludes that Eq. (12) is able to correct additive spin flip errors. \(\square\)

In essence, Lemma 4 tells us that the abilities to correct additive spin flip and phase shift form a dual pair under the discrete Fourier transform of quantum registers. An interesting case occurs when \(N = 2\). Here, additive spin flip is the only possible kind of spin flip error. As a result, the abilities to correct spin flip and phase shift errors in \(N = 2\) form a dual pair under Lemma 4. And this special form of Lemma 4 was proven earlier by various authors (see, for example, Refs. [6, 7, 16]).

**Corollary 5.** If a QECC handles both spin flip and phase shift errors on the same set of quantum registers, then this QECC handles any general quantum errors occurring at the same set of quantum registers.

**Proof.** Combining Eqs. (10) and (12), one knows that Eq. (10) holds for any complex-valued function \(g(j_i, j'_i : i \in A)\). By putting \(\langle j_i | \mathcal{E}_i | j'_i \rangle = g(j_i, j'_i)\) for all \(i \in A\), then one concludes that the above QEC is capable of correcting any general quantum errors as promised. \(\square\)

**Lemma 6.** Suppose QECCs \(C1\) and \(C2\) handle phase shift and spin flip errors, respectively, for the same set of quantum registers. Then, pasting the two codes together by first encodes the quantum state using \(C1\) then further encodes the resultant quantum state using \(C2\), one obtains a QECC which corrects general errors in the same set of quantum registers.
Proof. Clearly $C$ can handle spin flip errors occurring at the specified quantum registers. So from Corollary 5, it remains to show that $C$ corrects phase errors as well. Let the encodings for $C_1$ and $C_2$ be $|k\rangle \lor \sum_j q_j^k |j\rangle$ and $|j\rangle \lor \sum_p b_p^j |p\rangle$, respectively. Then using the same set of notations as in the proof of Lemma 4, one knows that

$$
\langle k'_\text{encode}|\mathcal{F}^\dagger\mathcal{F}|k\text{encode}\rangle = \sum_{j,j',p,p'} \delta_{j,j'} \delta_{p,p} \left( \prod_{i \in U} \delta_{p',p'_i} \right) \left( \prod_{i \in A} g_i(p_i : i \in A) \right)
$$

for some complex-valued functions $g_i(p_i : i \in A)$ for all $i \in A$.

Since $C_2$ handles spin flips, one demands that whenever $j \neq j'$,

$$
\sum_p b_p^j \bar{b}_p^{j'} = 0 = \sum_p b_p^j \bar{b}_p^{j'} \mathcal{E}|p\rangle|\mathcal{E}|p\rangle,
$$

where $\mathcal{E}$ denotes a possible spin flip error that can be handled by the Q ECC $C_2$. Consequently,

$$
\sum_p b_p^j \bar{b}_p^{j'} = 0,
$$

where the above primed sum is over either (1) all the $p$ that is affected by the error $\mathcal{E}$, or (2) all the $p$ that is unaffected by the error $\mathcal{E}$.

From Eq. (17), it is easy to see that after summing over all $p$s in Eq. (15), one will arrive at

$$
\langle k'_\text{encode}|\mathcal{F}^\dagger\mathcal{F}|k\text{encode}\rangle = \sum_{j,j'} \prod_{i \in A} g_i(j_i : i \in A) \delta_{j,j'} \delta_{j,j'},
$$

for some complex-valued function $g_i(j_i : i \in A)$. As $C_1$ handles phase shift, one concludes that $\langle k'_\text{encode}|\mathcal{F}^\dagger\mathcal{F}|k\text{encode}\rangle = \delta_{k,k} A_{\mathcal{F},\mathcal{F}}$. Hence, the Lemma is proved. \end{proof}

At this point, I would like to remark that the proof of the abilities to correct both spin flip and phase shift implies the ability to correct a general error for $N = 2$ can be found in Refs. [6, 7, 14, 26]. Moreover, one should notice that the ordering of encoding in Lemma 6 is important. Encoding first using a spin flip code followed by a phase shift code does not, in general, result in a general Q ECC. After proving the above two technical lemmas, I report a method to construct Q ECCs from classical codes.
Theorem 7. Suppose $C$ is a classical (block or convolutional) code of rate $r$ that corrects $p$ (classical) errors for every $q$ consecutive registers. Then, $C$ can be extended to a QECC of rate $r^2$ that corrects at least $p$ quantum errors for every $q^2$ consecutive quantum registers.

Proof. Suppose $C$ is a classical (block or convolutional) code. By mapping $m$ to $|m\rangle$ for all $m \in \mathbb{Z}_N$, $C$ can be converted to a quantum code for spin flip errors. Let $C'$ be the QECC obtained by Fourier transforming each quantum register of $C$. Then Lemma 4 implies that $C'$ is a code for phase shift errors. From Lemma 6, pasting codes $C$ and $C'$ together will create a QECC of rate $r^2$. Finally, the fact that $C'$ corrects at least $p$ quantum errors for every $q^2$ consecutive quantum registers follows directly from Corollary 5.

Theorem 7 provides a powerful way to create high rate QECCs from high rate classical codes.

Example 2 (Shor). Starting with the simplest classical majority block code of rate $1/3$, namely, $|k\rangle \rightarrow |k, k, k\rangle$ for $k = 0, 1$, Theorem 7 returns the famous Shor’s single error correcting nine bit code $[23]$ of rate $1/9$:

$$|k\rangle \rightarrow \sum_{p, q, r = 0}^{q^2} (-1)^{p+q+r} |p, p, q, q, r, r, r\rangle. \quad (19)$$

Alternatively, one may start with a high rate classical convolutional code. One of the simplest codes of this kind is the $1/2$-rate code in Eq. (2). Being a non-systematic$^3$ and non-catastrophic$^3$ code (see, for example, chap. 4 in Ref. [13] for details), it serves as an ideal starting point to construct good QCCs. First, let me write down this code in quantum mechanical form:

**Lemma 8.** The rate $1/2$ QCC

$$\bigotimes_{i=1}^{+\infty} |k_i\rangle \rightarrow |k_{\text{encode}}\rangle \equiv \bigotimes_{i=1}^{+\infty} |k_i + k_{i-2}, k_i + k_{i-1} + k_{i-2}\rangle, \quad (20)$$

where $k_i \in \mathbb{Z}_N$ for all $i \in \mathbb{Z}^+$, can correct up to one spin flip error for every four consecutive quantum registers.

Proof. Here, I give a “quantum version” of the proof. Using notations in the proof of Theorem 3, I consider $\langle k'_{\text{encode}} | \varepsilon' | k_{\text{encode}}\rangle$ again. Clearly, the worst case happens when errors $\varepsilon$ and $\varepsilon'$ occur at different quantum registers. And in this case, Eq. (20) implies that exactly two of the following four equations hold:

$$\begin{cases}
    k_{2i} + k_{3i-2} = k_{2i} + k_{3i-2} \\
    k_{2i} + k_{3i-1} + k_{3i} = k_{2i} + k_{3i-1} + k_{3i-2} \\
    k_{3i+1} + k_{3i-1} = k_{3i+1} + k_{3i-1} \\
    k_{2i+1} + k_{2i} + k_{2i-1} = k_{2i+1} + k_{2i} + k_{2i-1}
\end{cases} \quad (21)$$

That is, both $b_i$ and $c_i$ are not equal to $a_i$.

That is, a finite number of channel errors does not create an infinite number of decoding errors.
for all $i \in \mathbb{Z}^+$. One may regard $k_i$s as unknowns and $k_i'$s as arbitrary but fixed constants. Then, by straightforward computation, one can show that picking any two equations out of Eq. (21) for each $i$ will form an invertible system with the unique solution $k_i = k_i'$ for all $i \in \mathbb{Z}^+$. Therefore, $(k_{\text{encode}}')^\dagger \mathcal{E} | k_{\text{encode}}') = \delta_{k_i k_i'} \delta_{\mathcal{E} \mathcal{E}'}$, and hence this lemma is proved. \hfill \square

**Example 3.** Theorem 7 and Lemma 8 imply that

$$
\bigotimes_{i=1}^{+\infty} k_i \mapsto \bigotimes_{i=1}^{+\infty} \sum_{p_1, q_1, \ldots} \frac{1}{N} \omega_N^{(k_i+k_{i-1})p_i+(k_i+k_{i-1}+k_{i-2})q_i} [p_i + p_{i-1},

p_i + p_{i-1} + q_{i-1}, q_i + q_{i-1}, q_i + q_{i-1} + p_i) ,

(22)
$$

where $k_i \in \mathbb{Z}_N$ for all $i \in \mathbb{Z}^+$, is a rate 1/4 QCC capable of correcting up to one quantum error for every sixteen consecutive quantum registers.

In what follows, I show that this code can in fact correct up to one quantum error per every eight consecutive quantum registers.

**Proof.** Suppose $\mathcal{E}$ and $\mathcal{E}'$ be two quantum errors affecting at most one quantum register per every eight consecutive ones. By considering $(k_{\text{encode}}')^\dagger \mathcal{E}' | k_{\text{encode}}')$, I know that at least six of the following eight equations hold:

$$
\begin{aligned}
p_{2i-1} + p_{2i-2} &= p_{2i-1}^0 + p_{2i-2}^0, \\
p_{2i-1} + p_{2i-2} + q_{2i-2} &= p_{2i-1}^0 + p_{2i-2}^0 + q_{2i-2}^0, \\
q_{2i-1} + q_{2i-2} &= q_{2i-1}^0 + q_{2i-2}^0, \\
p_{2i} + p_{2i-1} &= p_{2i}^0 + p_{2i-1}^0, \\
p_{2i} + p_{2i-1} + q_{2i-1} &= p_{2i}^0 + p_{2i-1}^0 + q_{2i-1}^0, \\
q_{2i} + q_{2i-1} &= q_{2i}^0 + q_{2i-1}^0, \\
q_{2i} + q_{2i-1} + p_{2i} &= q_{2i}^0 + q_{2i-1}^0 + p_{2i}^0.
\end{aligned}

(23)
$$

for all $i \in \mathbb{Z}^+$. Again, I regard $p_i$ and $q_i$ as unknowns; and $p_i'$ and $q_i'$ as arbitrary but fixed constants. Then, it is straightforward to show that choosing any six equations in Eq. (23) for each $i \in \mathbb{Z}^+$ would result in a consistent system having a unique solution of $p_i = p_i'$ and $q_i = q_i'$ for all $i \in \mathbb{Z}^+$. Consequently,

$$
\begin{aligned}
(k_{\text{encode}}')^\dagger \mathcal{E}' | k_{\text{encode}}')

&= \sum_{(p, q)} \left\{ \prod_{i=1}^{+\infty} \omega_N^{\sum_{j=2i-1}^{2i} p_j (k_j+k_{j-1}+k_{j-2}+k_{j-3})+q_j (k_j+k_{j-1}+k_{j-2}+k_{j-3})}

\times \langle f_i | \mathcal{E}' | g_i \rangle \langle g_i | \mathcal{E}' | g_i \rangle \right\}

(24)
$$

for some linearly independent functions $f_i(p, q)$ and $g_i(p, q)$. 

Now, I consider a basis \( \{ h_i(p, q) \} \) for the orthogonal complement of the span of \( \{ f_i, g_i \} \in \mathbb{Z}^+ \). By summing over all \( h_i \)'s while keeping \( f_i \)'s and \( g_i \)'s constant in Eq. (24), one ends up with the constraints that \( k_i = k_i' \) for all \( i \in \mathbb{Z}^+ \). Thus,

\[
\langle k_{\text{encode}}' | \mathcal{E} | k_{\text{encode}} \rangle = \delta_{k, k'} \sum_{\{ p, q \}} \prod_{i=1}^{k} \left( \langle f_i(p, q) | \mathcal{E} | f_i(p, q) \rangle \langle g_i(p, q) | \mathcal{E} | g_i(p, q) \rangle \right). \tag{25}
\]

Hence, Eq. (22) corrects up to one quantum error per every eight consecutive quantum registers. \( \square \)

From the discussion following Example 1, the encoding in Eq. (20) can be done efficiently with the help of reversible computation [1, 2, 12].

4 Outlook

It is instructive to investigate the coding ability of QCCs as compared to that of QBCs. Knill and Laflamme [17] proved that it is impossible to construct a four qubit QBC that corrects one general quantum error. Their result can be extended to the case when \( N > 2 \) [10]. Here, with a slight modification of Knill and Laflamme’s proof, I show that:

**Theorem 9.** It is not possible to construct a QCC which corrects one general quantum error for every four consecutive quantum registers.

**Proof.** Clearly, the QCC must be of rate 1/4. And with a simple permutation of the quantum registers, a general QCC of rate 1/4 can be written as

\[
|k\rangle \mapsto |k_{\text{encode}}\rangle \equiv \sum_{\{w, x, y, z\}} a_{w,x,y,z}^{(k)} |w, x, y, z\rangle. \tag{26}
\]

Without lost of generality, I may assume that quantum errors occurs in any one of the following four set of registers: \( |w\rangle \), \( |x\rangle \), \( |y\rangle \) and \( |z\rangle \).

Then, following Knill and Laflamme [17] by considering the action of errors in the above four sets of registers, one arrives at \( \rho^{(k)} \rho^{(k')} = 0 \) and \( \rho^{(k)} = \rho^{(k')} \) for all \( k \neq k' \), where

\[
\rho^{(k)}_{w,x,y,z} = \sum_{\{y,z\}} a_{w,x,y,z}^{(k)} a_{w,x,y,z}^{(k)} \tag{27}
\]

Hence, the reduced (Hermitian) density matrices \( \rho^{(k)} \) are nilpotent for all \( k \). This is possible only if \( a_{w,x,y,z}^{(k)} = 0 \) for all \( k, w, x, y, z \). This contradicts the assumption that \( |k_{\text{encode}}\rangle \) is a QCC. \( \square \)
It is, however, unclear if QCC can perform better than QBC in other situations. And further investigation along this line is required.

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References


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