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"AMBIGUITY" OF HARMONIC OSCILLATOR COMMUTATION RELATIONSDavid G. Boulware ^{*)} and S. Deser ⁺⁾

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A B S T R A C T

The possibility, first noticed by Wigner, that the form of the Hamiltonian together with the Heisenberg equations do not imply canonical commutation relations for the harmonic oscillator variables, is re-examined here. It is shown that this freedom of commutation relations just expresses the known possibility of quantizing the system by Bose or Fermi (and para-Fermi) statistics, and of altering the zero-point energy.

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I. INTRODUCTION

Some time ago, Wigner ¹⁾ discussed an apparent ambiguity in the commutation relations for the simple harmonic oscillator. He showed that for a particular choice of quantum Hamiltonian ²⁾, say

$$H = \frac{1}{2} [a^+, a]_+ \quad (1)$$

and the requirement that the Heisenberg and Lagrange equations agree,

$$[H, a] = -a, \quad [H, a^+] = a^+ \quad (2)$$

one could not uniquely obtain the canonical commutation relations $[a, a^+] = 1$, but rather the one-parameter family,

$$[a, a^+] = 1 + (2E_0 - 1)e^{i\pi(H - E_0)} \quad (3)$$

Here E_0 is a positive parameter which gives the zero-point energy of the system (the latter need no longer be $\frac{1}{2}$, owing to the non-normal form of the relations (3)). The operator $\exp [i\pi(H - E_0)]$ is Hermitian, since the spectrum of $(H - E_0)$ runs over the integers, by virtue of (2) and the completeness of this set a, a^+ . When $E_0 = \frac{1}{2}$, one correctly obtains the normal form, of course.

Wigner's analysis of (1) has recently been extended by O'Raiffeartaigh and Ryan ³⁾, who classify all the representations of a and a^+ corresponding to the general case

$$\begin{aligned} H &= \mu a a^+ + (1-\mu) a^+ a & 0 \leq \mu \leq 1 \\ &= a^+ a + \mu [a, a^+] \end{aligned} \quad (4)$$

according to the value of the parameter μ . From the results of these authors, we show, by means of explicit transformations, that the entire

freedom of the $[a, a^+]$ relations allowed by requirement (2) for the general case (4) consists in having Bose, Fermi or para-Fermi statistics ⁴⁾.

II. CLASSIFICATION OF CASES

We begin with case (A), $1 \geq \mu \geq \frac{1}{2}$, E_0 arbitrary but positive. In this case the representation of a, a^+ is ∞ dimensional, with (to within an arbitrary over-all phase factor)

$$a_{r,r+1} = \left\{ (E_0 - \mu) \left[1 - \left(-\frac{1-\mu}{\mu} \right)^{r+1} + r+1 \right] \right\}^{1/2} \quad (5)$$

all other matrix elements vanishing. The standard Dirac representation is, of course,

$$a_{r,r+1} = \sqrt{r+1} = a_{r+1,r}^* \quad (6)$$

From the form (5), keeping to the energy representation, it is easy to obtain an operator U such that $A \equiv Ua$ has the relation

$$[A, A^+] = 1 \quad (7)$$

rather than the non-canonical relation for $[a, a^+]$ implied by (5). The operator U performing this (non-canonical) point transformation is just

$$U = \bar{H}^{-1/2} \left[(E_0 - \mu) \left(1 - \left(-\frac{1-\mu}{\mu} \right)^{\bar{H}+1} \right) \right]^{-1/2}, \quad \bar{H} \equiv H - E_0 + 1 \quad (8)$$

The important point is that the Hamiltonian becomes ⁵⁾, in terms of A, A^+ , just

$$H = A^+ A + E_0 \quad (9)$$

namely the normal simple harmonic oscillator, with the arbitrary (positive) zero-point energy ⁶⁾, E_0 . Wigner's example, $\mu = \frac{1}{2}$ is a particular case, and we see that the sole effect of the "abnormal" relations (3) is to shift the ground state energy of the Hamiltonian from its normal value of $\frac{1}{2}$.

Case (B), $\frac{1}{2} \geq \mu > 0$, $E_0 = \mu$. Here the only representation is the normal infinite dimensional one (6), and no transformation is needed. The Hamiltonian is just $H = a^+ a + \mu$. The same is true for case (C), $\mu = 0 = E_0$, which permits the infinite dimensional form (6) as well. However, case (C) also allows representations of any finite dimension $D \geq 2$, which are clearly inequivalent to the Bose cases, although the matrix elements of a are still given by (6). In particular, the representation $D = 2$ is just the Fermi oscillator

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad [a, a^+]_+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (10)$$

More generally, the representations of higher dimensions, $D = 3, 4, \dots$ correspond to the so-called para-Fermi statistics (the normalization (6) employed here differs from the usual one in which $a_{r,r^+} = 1$ is taken). If one computes the commutator, one finds that

$$[a, a^+] = \begin{pmatrix} 1 & & \\ & \dots & \\ & & -D+1 \end{pmatrix} \quad (11)$$

so that the trace is correctly zero. The "limit" as $D \rightarrow \infty$ is the normal Bose case ⁷⁾; as mentioned earlier, however, the " $D = \infty$ " possibility can be found directly as well. It should be noted that $\mu = 0$ implies $E_0 = 0$, so no arbitrariness in E_0 exists here.

We are left, finally, with case (D), $\frac{1}{2} > \mu > 0$ but

$D \neq \infty$. The analysis of Ref. 4) shows that even and odd D 's must be treated separately ; for all of them E_0 is a fixed positive function $E_0(\mu, D)$, given explicitly in the Appendix. The forms of $a_{r,r+1}$ which emerge are of the type

$$a_{r,r+1} = [f(D, \mu, r) + r + 1]^{1/2} \quad (12)$$

and transformations similar to the U of (8) are easily found (see Appendix). These yield the "normal" Fermi and para-Fermi representations of type (6) for the new operators $A = Ua$. As in the Bose cases, the Hamiltonian then becomes the "normal" one, plus the zero-point energy :

$$H = A^\dagger A + E_0(\mu, D) \quad (13)$$

There is again a one-parameter family of E_0 for every D , as μ ranges over the open interval $(\frac{1}{2}, 0)$, in analogy with the Bose situation (9).

III. CONCLUSIONS

The non-uniqueness of the commutation relations $[a, a^\dagger]$ for the quantum oscillator, given the general form of the Hamiltonian (4) and the correct Heisenberg equations of motion (2), has been examined. We have shown that the only essential non-uniqueness arises, as it should, from the well-known possibility of employing Fermi or para-Fermi statistics instead of Bose quantization. All other apparently possible commutation relations merely shift the zero-point energy. This is brought out by the fact that one may always transform to variables obeying appropriate statistics, the Hamiltonian in terms of the new variables differing from (4) merely by the (c-number) displaced zero-

point energy. Otherwise put, the Heisenberg equations fix the spectrum of the operator $H-E_0$, but not, of course, E_0 itself.

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A P P E N D I X

We list here for reference the forms of U and E_0 for the finite-dimensional case (D).

a) Even dimension N

$$U = \bar{H}^{1/2} \left[\frac{N}{\alpha^{N-1}} (1 - (-\alpha)^{\bar{H}}) + \bar{H} \right]^{-1/2} \quad \alpha \equiv \frac{1-\mu}{\mu}$$

$$E_0 = \mu + N/(\alpha^N - 1)$$

b) Odd dimension M

$$U = \bar{H}^{1/2} \left[-\frac{M}{\alpha^{M+1}} (1 - (-\alpha)^{\bar{H}}) + \bar{H} \right]^{-1/2}$$

$$E_0 = \mu - \frac{M}{\alpha^{M+1}}$$

In case b), μ turns out to be bounded from above ³⁾ by a function $0 < \mu_0(M) < \frac{1}{2}$, which ensures that E_0 is positive.

FOOTNOTES AND REFERENCES

- 1) E.P. Wigner, Phys.Rev. 77, 711 (1950).
- 2) We use units such that $\hbar \omega = 1 = m$, and the notation $[,]_+$ for anti-commutator, $[,]$ for commutator.
- 3) L. O'Raiffeartaigh and C. Ryan, Proc.Roy.Irish Academy 62A, 93 (1963), to which the reader is referred for the detailed results.
- 4) For our one-dimensional system, there is no real difference between para-Bose and ordinary Bose statistics ; the only distinction is a trivial one in the normalization of the creation operators.
- 5) The result (9) is most easily checked by computing $H|n\rangle$ for the forms (4) and (9) and showing they are equal.
- 6) The "classical" value $E_0 = 0$ is not allowed here, since in that case $a_{0,1} = 0$ and a, a^+ no longer form a complete set.
- 7) There is also a curious limit in which the Bose case may be said to approach the Fermi one. If one takes the limit $E_0 \rightarrow \infty$ in case (A), and defines $a' = a(E_0)^{-1/2}$ then one obtains the relations $[a', a'^+]_+ = 1$ and the infinite-dimensional space breaks up into a direct product of case (C), $D = 2$ representations. These limits, in which Bose and Fermi statistics interchange, are related to the "para-clock" considerations of O.W. Greenberg and A. Messiah (to be published).