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A LOWER BOUND FOR LARGE ANGLE
ELASTIC SCATTERING AT HIGH ENERGIES

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A B S T R A C T

Under certain analyticity assumptions we prove that the scattering amplitude around 90° in the centre of mass system cannot decrease faster than $s^{-C\sqrt{s}}$, where s is the square of the centre of mass energy, as s goes to infinity, if one requires that the total cross-section be a slowly varying function of energy.

There are some experimental indications that the elastic cross-section at 90° in the c.m.s. is a fast-decreasing function ¹⁾ of $\sqrt{s}$, the c.m. energy, whereas the total cross-section, related through the optical theorem to the forward scattering amplitude, is at most only decreasing slowly or possibly constant.

Using analyticity properties of the scattering amplitude $f(s, z)$ in the variable $z = \cos \Theta$ (Θ , the c.m. scattering angle) it can be shown that too fast a decrease of the 90° elastic scattering is not compatible with a slowly varying total cross-section.

We assume that the scattering amplitude $f(s, z)$ is analytic and bounded by s^N (N is independent of z and s) in a sub-domain D of the cut z -plane (Fig. 1). The cuts run from $-\infty$ to $-\rho$ and from ρ to $+\infty$. The shape of D will be specified afterwards. The distance of the cuts to the points $-1, +1$ is given by $\rho = 1 + \frac{\alpha}{k^2}$ (α a positive constant, k the c.m. momentum). Suppose further that on a segment of the real axis, with end points $-a, +a$, we are given an upper bound of $|f(s, z)|$ which depends on s and which we may write as

$$e^{-\phi(s)} \geq |f(s, z)| \quad -a \leq z \leq +a$$

where $\phi(s)$ is a positive function of s .

We shall derive from these hypotheses an upper bound for $|f(s, 1)|$, the modulus of the forward scattering amplitude. In order to find an upper bound at a point located between the segment $(-a, +a)$ and the boundary of D it is convenient to transform both these lines into concentric circles.

A first step maps the cut z -plane onto the interior of a circle of radius ρ in the plane of the variable w (Fig. 2) :

$$w = \frac{\rho}{z} [\rho - \sqrt{\rho^2 - z^2}]$$

2.

The points $z = \pm 1$ are transformed into $w = e \equiv \pm [\rho - \sqrt{\rho^2 - 1}] \rho$ and the points $\pm a$ into $w = a' \equiv \pm \frac{\rho}{a} [\rho - \sqrt{\rho^2 - a^2}]$.

In the second step we map the w -plane on the plane of the variable τ in such a way that a cut along the real axis in w from $-a$ to $+a$ is transformed into the unit circle (Fig. 3). This transformation reads

$$\tau = \frac{1}{a'} [w + \sqrt{w^2 - a'^2}]$$

The point $w = \rho$ is mapped in $\tau = R \equiv \frac{1}{a'} [\rho + \sqrt{\rho^2 - a'^2}]$;
the point $w = e$ is mapped in $\tau = E = \frac{1}{a'} [e + \sqrt{e^2 - a'^2}]$.

We consider the three concentric circles around the origin, with radii $1, E, R$. Our assumption on the domain D will now be that the transform of D in the τ -plane contains the circle around the origin with radius R (Fig. 1).

The image of D in the w -plane is an ellipse tangent in $\pm \rho$ with the circle of radius ρ centred at the origin, and semi-major axis $\sqrt{\frac{1 + \sqrt{1 - a^2}}{a}} - 1$

In the z -plane it is a dumb-bell shaped curve, tangent at the cuts in their end points and cutting the imaginary axis in a point given by

$$z = \pm i \frac{2\rho^2 B}{\rho^2 - B^2}$$

where

$$B = \frac{\rho}{a} \frac{(\rho - \sqrt{\rho^2 - a^2})^2}{a + \sqrt{a^2 - (\rho - \sqrt{\rho^2 - a^2})^2}}$$

On the three circles in the τ -plane we apply Hadamard's three-circle theorem ²⁾ which states that if a function is analytic in the crown-shaped region between the circles with radii 1 and R , has upper

M_1 and M_2 respectively on those circles, it has an upper bound M on an intermediate concentric circle with radius E , given by

$$M = M_1 \frac{\log R - \log E}{\log R} M_2 \frac{\log E}{\log R}$$

We can make use of the fact that in our problem the points E and R come infinitesimally near to each other with increasing s , because their images in the z -plane are separated by the distance $\frac{\alpha}{k^2}$. Developing to first order in $\frac{1}{k}$ we find

$$\frac{\log E}{\log R} = 1 - \frac{\sqrt{2\alpha}}{k} C$$

with
$$C = \frac{a}{[a^2 - (1 - \sqrt{1 - a^2})^2]^{1/2}}$$

and consequently, for large s , we have the bound for $z = 1$

$$|f(s, 1)| \leq e^{-\phi(s) \frac{\sqrt{2\alpha}}{k} C} e^{N \log s (1 - \frac{\sqrt{2\alpha}}{k} C)}$$

It is possible to make the argument for angles different from 0 by taking a smaller intermediate circle, whose image in the z -plane cuts the real axis between a and 1. The formula is especially simple if expressed in the variable t , the momentum transfer, with $z = 1 + \frac{t}{2k^2}$:

$$|f(s, t)| \leq e^{-\phi(s) C \frac{\sqrt{2\alpha - t}}{k}} e^{N \log s (1 - C \frac{\sqrt{2\alpha - t}}{k})}$$

where higher terms in $\frac{\sqrt{|t|}}{k}$ have been neglected.

If we assume for $\phi(s)$ a form

$$\phi(s) = C' s^\alpha (\log s)^\beta$$

we see that for $\alpha < \frac{1}{2}$ the bound will be of the form s^{+N} for large s , because $k = \sqrt{s^2 - m^2}$ where m is a positive constant, and nothing is gained in the forward direction over the initial assumptions. If $\alpha > \frac{1}{2}$ on the other hand, the forward amplitude will decrease faster than any negative power of s .

For $\alpha = \frac{1}{2}$ and $\beta = 1$ we have the critical behaviour

$$|f(s,1)| \leq e^{(N - \sqrt{2\alpha} CC') \log s} = s^{(N - \sqrt{2\alpha} CC')}$$

and it depends on the relative magnitude of the constants N and

$\sqrt{2\alpha} CC'$ whether the forward amplitude will decrease or not. For $\alpha = \frac{1}{2}$ $\beta > 1$ $f(s,1)$ will always decrease rapidly. We find therefore that any decrease of the elastic scattering amplitude around 90° faster than $s^{-C' \sqrt{s}}$ makes the forward amplitude decrease faster than any power of s . With the critical behaviour $|f(s,z)| \approx s^{-C' \sqrt{s}}$ $-a < z < +a$ it is always possible to find a fixed momentum transfer t_0 such that $|f(s,t_0)|$ tends to zero faster than any power of s , for arbitrary C' .

In these considerations only analyticity and boundedness by s^N have been used. One may ask if unitarity might improve our result, i.e., put a stronger bound on the forward scattering amplitude. We know that unitarity in the s channel gives us the Froissart bound ^{3),4)}

$$|f(s,z)| < C s \log^2 s$$

for $-1 \leq z \leq +1$, and also fixed angle bounds ⁵⁾ such as

$$|f(s,z)| < C \frac{(\log s)^{3/2}}{1-z^2}$$

for $-1 < z < +1$.

However, we shall see that this information does not give any help in finding the critical behaviour of the amplitude for large angle scattering. Indeed, one can construct a very simple example ⁵⁾ satisfying the requirements of unitarity, analyticity in z , and boundedness :

$$f(s, 1 + \frac{t}{2k^2}) = i C' s \log s s^{-C[\sqrt{2\alpha-t} - \sqrt{2\alpha}]}$$

f is analytic in a cut plane, bounded by $C'(\log s) s^{1+C\sqrt{2\alpha}}$. The partial wave projections have a modulus less than unity for a convenient choice of C' . Further, one has to check that the partial wave projections are imaginary positive. This is the case because for any $t < \alpha$ $\frac{1}{i} f$ and all its derivatives with respect to t are positive, in particular for $t = -2k^2$, i.e., $z = 0$. This means that the series expansion of f in terms of z has positive coefficients. Now since z^n can be written as a sum of Legendre polynomials with positive coefficients, we get the desired result.

Now the example we have proposed is precisely such that for $\cos \theta \leq a < 1$ the amplitude is less than $s^{-\text{const} \sqrt{s}}$, where the constant can be made arbitrarily large. This means that a large angle behaviour $s^{-\text{const} \sqrt{s}}$ is consistent with a non-decreasing total cross-section and unitarity (in the weak sense) in the s channel.

One could of course notice that if for $-a < \cos \theta < +a$ $|f|$ is less than $s^{-\text{const} \sqrt{s}}$, one will always be able to find a negative transfer $t < -T$ such that $|f|$ is bounded by s^{-N} where N can be made arbitrarily large. This is inconsistent with the Gribov-Pomeranchuk argument according to which the amplitude, for finite transfer, should not decrease faster than s^{-1} ⁶⁾; however, the validity of Gribov's result necessitates a number of assumptions which are not necessarily true ⁷⁾.

We should also say that for any $\phi(s)$ increasing faster than $\log s$ as s goes to infinity the amplitude decreases faster than any power of s at any fixed angle different from 0 or π . This result was indicated to us by M. Zerner (Marseilles) and initiated the present work.

If future very high energy experiments would be in contradiction with the present result (which we doubt very much), i.e., if one had a fixed angle decrease of the amplitude faster than $s^{-C\sqrt{s}}$ together with a practically constant cross-section, this would show that either the analyticity assumptions or the boundedness by s^N or both are wrong. Let us notice that according to the shape of domain D we are using momentum transfers which can be as large as $C|s|$. According to recent investigations by S. Mandelstam⁸⁾ there is some doubt that the boundedness condition still holds for such momentum transfers increasing with energy. However, it is clear that if the bound s^N is replaced by some weaker bound, one could still use our approach, but, of course, the result would be weakened correspondingly.

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R E F E R E N C E S

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that for $\ell = 0$ and $\ell = 1$ it coincides with the actual physical
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FIGURE CAPTIONS

Fig. 1 The cut plane in the z variable and the subdomain D in which analyticity and boundedness by polynomials are required. D has a fixed limit as the energy goes to infinity.

Fig. 2 The circle of the complex w plane onto which the cut z plane has been mapped. The domain D is mapped on an ellipse.

Fig. 3 In the complex τ plane the slit $(-a', +a')$ in w is mapped on the unit circle. The boundary of D is mapped on the outer circle.

Fig.1

Fig. 2

Fig. 3