CM-P00056929

Archives

RESONANCES IN THE SCATTERING OF PSEUDOSCALAR MESONS

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*) Supported in part by the U.S. Atomic Energy Commission.

**) Supported in part by the U.S. Air Force Office of Scientific Research and Development Command.

A B S T R A C T

The bootstrap mechanism of Zachariasen and Zemach³⁾ is used to study the resonant scattering of pseudoscalar mesons under the assumption of mass degeneracy and the existence of the vector resonances. The general bootstrap conditions on the mass and the coupling constants are formulated and Capps' result⁶⁾ on SU_3 symmetry is rederived from our more general equations. The question of the possible existence of further resonances is reduced to a simple eigenvalue problem. The forces due to the vector exchange are found to be repulsive in the 10 , $\overline{10}$, and 27 SU_3 channels but attractive in the 1 and 8 , and when calculated with the self-consistent parameters are strong enough to bind scalar resonances in the 1 and 8 channels. These resonances may contribute to the interaction forces. Taking into account the scalar octet exchange in addition to the vector exchange we have bootstrapped the two octets together in a double self-consistency calculation. A self-consistent solution is obtained at $m_v \sim 1000$ MeV, $m_s \sim 500$ MeV, and a small value for the ratio of the scalar coupling strength to the vector coupling strength (5%). The singlet scalar meson has not been bootstrapped for the sake of simplicity, but we have some reasons to believe that its effect should be small. We have tentatively assigned the weakly coupled $K'(725)$ to the $I = 1/2$ members of the scalar octet, the $K_1 K_1$ threshold anomaly to the $I = 0$ member of the octet, and the ABC "particle" to the scalar unitary singlet. Spin 2^+ "particles" are also predicted using the self-consistent parameters in the 1 and 8 channels at higher energies, the singlet being here tentatively identified as f_0 while the octet occurs at presumably too high energies to be seen. Because of the crudeness of the model, no quantitative significance of the results is claimed, but the model is expected to be qualitatively correct and to give the right orders of magnitude for the various parameters.

INTRODUCTION

The ideas of unitary symmetry ^{1),2)} and the bootstrap mechanism ^{3),4)} have had some success in the classification of particles and in the determination of their parameters in strong interactions. We shall here investigate in a crude model the scattering of pseudoscalar mesons within the framework of these ideas. The object is to see, in the light of present knowledge, whether we would expect further resonances to occur and if so to which supermultiplets we would assign these resonances. The crude model here employed could presumably be applied to other systems (e.g., pseudoscalar meson scattering from baryons) but we shall deal here only with the mutual scattering of pseudoscalar mesons since it is the simplest case one can handle. In addition to its particularly simple kinematics, this problem has the advantage of a fairly high inelastic threshold, which gives more confidence in dealing with the problem in isolation from related processes like vector meson production or baryon pair production.

We shall employ throughout this paper the bootstrap procedure of Zachariasen and Zemach ³⁾ without discussing its validity or justification. We believe that other variants of the bootstrap procedure will give qualitatively similar results.

Since the conservation of charge and strangeness holds even in electrodynamics, it is reasonable to regard it as having an origin outside the realm of strong interaction. Taking the viewpoint that the present bootstrap mechanism, if valid at all, should apply only to strong interactions ⁵⁾, we shall consider the conservation of charge and strangeness as given a priori in this paper, and not to be expected as consequences.

The position with the conservation of isotopic spin, however, is not quite so clear, but it will also be assumed as given a priori in this paper for lack of anything better.

Under these assumptions we consider the mutual scattering of the eight pseudoscalar meson (π, K, η). The best established resonances in the interaction of these mesons are the eight vector mesons (ρ, K^*, ω or ϕ), and it is reasonable to assume as a beginning that the exchange of these eight vector mesons constitutes the main contribution to the interaction forces between the pseudoscalar mesons. The bootstrap condition then requires that the same eight vector mesons should occur as poles in the direct channel. In this crude model, we neglect the mass differences between the pseudoscalar mesons (π, K, η) and between the vector mesons (ρ, K^*, ω or ϕ). It is then found that the bootstrap condition, in addition to determining the mass of the vector meson and its coupling strength, requires that the interaction should satisfy SU_3 symmetry⁶⁾. In other words, the coupling constants of a vector meson to two pseudoscalar mesons are proportional to the structure constants of the SU_3 group.

The scattering amplitude calculated with forces due to the exchange of only the vector mesons may have further poles in the direct channel in addition to those corresponding to the vector mesons themselves, and these poles must, for consistency, be regarded also as resonances in the scattering of pseudoscalar mesons. The problem of determining the position of these further poles, if any, is found to reduce to a simple eigenvalue problem. Because of the SU_3 symmetry implied by the bootstrap condition on the vector mesons, these poles group automatically into supermultiplets. In the decomposition $8 \times 8 = 1 + 8 + 8' + 10 + \overline{10} + 27$, it is found that only in the channels 1, 8, and $8'$ poles occur, the forces in the other three channels being either zero or repulsive and can thus give neither bound states nor resonances. From the symmetry properties it is seen that the representations 1 and 8 correspond to even spin particles (scalar mesons) and $8'$ to odd spin particles (vector mesons). It would appear, therefore, that this crude model predicts an octet and a singlet of scalar mesons in addition to the original octet of vector mesons. The masses of these new mesons are given by the positions of the poles and their coupling strengths by the residues at the poles. The actual values for these are found numerically in section 6.

If these new scalar particles do in fact exist, they could also contribute to the forces of interaction between the pseudoscalar mesons. The values for the masses and coupling constants calculated in section 6 are thus not self-consistent since we had only taken the forces due to exchange of vector mesons alone. For complete self-consistency, one should exchange in addition both the singlet and the octet of scalar mesons, and bootstrap all these particles to find their masses and coupling constants. In section 7, we shall formulate the double bootstrap problem of both the vector octet and the scalar octet. We have not bootstrapped the scalar singlet, mainly because its inclusion would make the numerical work too cumbersome for such a crude model, but also because we have some reasons to believe that the coupling is small, (see sections 7 and 9).

The double-bootstrap problem yields a mass value for the vector mesons at ~ 1000 MeV, and for the scalar octet at ~ 500 MeV. The self-consistent coupling strength of the scalar is much weaker than that of the vector octet, ($\rho = g^2/4\pi^2 = 0.043$). A quantitative comparison of the results with experiments is perhaps not too meaningful because of the crudeness of the model, but it is interesting that the vector meson mass is of the right order and that the scalar mass is not far from the $K'(725)$. For a tentative assignment of the scalar octet, the reader is referred to section 9. The small value of ρ is consistent with our assumption at the beginning on the predominance of the vector mesons and also affords an explanation why the scalar mesons, if they exist, were not discovered earlier. We note that the $K'(725)$ has a much narrower width ($\gtrsim 10$ MeV) compared with the vector mesons. Poles are found also in the $\ell = 2$ states at higher energies, for a discussion of which the reader is again referred to section 9.

4.

2. APPROXIMATE N FUNCTION

We shall first write the ℓ th partial wave amplitude in N/D form and follow Zachariasen and Zemach³⁾ in approximating the N function by the Born approximation. Since we assume the exchange of the eight vector mesons to be the main contribution to the interaction forces, we shall take at first only those diagrams with vector mesons exchanged. The full amplitude of the reaction $(ab \rightarrow cd)$ is then approximated by

Fig. 1

Fig. 2

$$T = 4 \sum_r \gamma_{ac}^r \gamma_{bd}^r \frac{s-u}{m^2-t} + 4 \sum_r \gamma_{ad}^r \gamma_{bc}^r \frac{s-t}{m^2-u} \quad (2.1)$$

where, as usual

$$\begin{aligned} s-u &= s + 2q^2(1+z), \\ s-t &= s + 2q^2(1-z), \\ m^2-t &= 2q^2(1-z) + m^2, \\ m^2-u &= 2q^2(1+z) + m^2. \end{aligned} \quad (2.2)$$

with $z = \cos \theta_{ac}$, and q = centre of mass momentum. The coupling constants γ_{ab}^r may, without loss of generality, be chosen real by taking real combinations of the interacting fields, and antisymmetric in the indices a and b by using the vector nature of the exchanged particles. (Let ϕ_a, ϕ_b be pseudoscalar and A^r vector fields, then expression (2.1) comes from the interaction

$$\mathcal{L}_I = \sum_{a,b,r} \gamma_{ab}^r (\phi_a \partial_\mu \phi_b - \phi_b \partial_\mu \phi_a) A^{r\mu}.$$

Projecting out the contribution to the ℓ th partial wave in the direct channel, one has

$$e^{i\delta_\ell} \sin \delta_\ell = \frac{1}{16\pi} \frac{q}{\sqrt{s}} \int_{-1}^1 T(s, \theta) P_\ell(\cos \theta) d\cos \theta \quad (2.3)$$

It can readily be seen that the contributions of the two terms in T differ only by a sign $(-1)^\ell$, so that the Born approximation to the N function may be written as

$$N_{\ell;ab,cd} = -V_{\ell;ab,cd} F_\ell(s) \quad (2.4)$$

where

$$V_{\ell;ab,cd} = \sum_r [\gamma_{ac}^r \gamma_{bd}^r + (-1)^\ell \gamma_{ad}^r \gamma_{bc}^r] \quad (2.5)$$

and

$$F_\ell(s) = -\frac{1}{8\pi} \int_{-1}^1 dz P_\ell(z) \left[1 + \frac{z + (s + m^2)/2q^2}{z - 1 - m^2/2q^2} \right] \quad (2.6)$$

Note that $F_\ell(s)$ is independent of the particle indices.

The functions $F_\ell(s)$ are best expressed in terms of the Legendre functions of the second kind. Thus on introducing the dimensionless quantities

$$x = s/4\mu^2, \quad M = m^2/4\mu^2, \quad (2.7)$$

we have

$$\begin{aligned} 4\pi F_0(x) &= -1 + \frac{2}{x-1} (M+2x-1) Q_0\left(\frac{2M+x-1}{x-1}\right) \\ 4\pi F_\ell(x) &= \frac{2}{x-1} (M+2x-1) Q_\ell\left(\frac{2M+x-1}{x-1}\right), \quad \ell > 0. \end{aligned} \quad (2.8)$$

(The -1 term in F_0 would be absent in a purely dispersion theoretical evaluation of the Born approximation. Its omission would not affect the calculations significantly.)

Writing the once subtracted dispersion relation for $D_\ell(x)$ and using the unitary condition in the usual way, one has

$$D_{\ell;ab,cd} = \delta_{ab,cd} - V_{\ell;ab,cd} d_\ell(x) \quad (2.9)$$

where

$$\begin{aligned} \delta_{\ell;ab,cd} &= \frac{1}{2} [\delta_{ac} \delta_{bd} + (-1)^\ell \delta_{ad} \delta_{bc}] \\ d_\ell(x) &= \frac{x - x_t}{\pi} \int_1^\infty dx' \sqrt{\frac{x'-1}{x'}} \frac{F_\ell(x')}{(x' - x_t)(x' - x - i\varepsilon)}, \end{aligned} \quad (2.10)$$

or in matrix notation ⁷⁾

$$N_\ell(x) = -F_\ell(x) V_\ell,$$

$$D_\ell(x) = 1 - \alpha_\ell(x) V_\ell,$$

(2.11)

$$T_\ell(x) = F_\ell(x) V_\ell [1 - \alpha_\ell(x) V_\ell]^{-1}.$$

Using the property of $Q_\ell(z)$ that for $z > 1$, $Q_\ell(z) > 0$, one readily sees that $F_\ell(x) > 0$ for $x > 0$. In the integral $\alpha_\ell(x)$, if we choose the subtraction point x_t to be the threshold of the left-hand cut ³⁾, then for $x < 0$, $\alpha_\ell(x) > 0$. For $x > 0$, i.e., above the physical threshold, the integral $\alpha_\ell(x)$ is complex and we have to take the principal value. The sign of $\alpha_\ell(x)$ is then not so readily seen but in the nearby region not far above threshold, one would still expect $\alpha_\ell(x)$ to be positive. Now for bound states or resonances to occur, $D_\ell(x)$ must vanish. The sign of $\alpha_\ell(x)$ then implies that only in those channels where $V > 0$ will there be bound states or resonances. For those channels where $V < 0$, the forces will be "repulsive" and no bound state or resonance will occur. This is a useful criterion for the discussion following. The question of whether bound states and resonances do occur, and if so at what energies, can of course only be answered by actually finding the zeros of $D_\ell(x)$ (section 6).

8.

3. THE BOOTSTRAP CONDITION

The matrix $V_{\ell; ab, cd}$ is real and symmetric and has thus a complete set of orthonormal eigenvectors with real eigenvalues. Let $\Psi_{\ell}^1, \Psi_{\ell}^2, \dots$ be such a set of eigenvectors with values $\bar{\lambda}_{\ell}^1, \bar{\lambda}_{\ell}^2, \dots$. The orthogonal matrix

$$O_{\ell} = (\Psi_{\ell}^1, \Psi_{\ell}^2, \dots) \quad (3.1)$$

then simultaneously diagonalizes V, N, D and T , so that

$$\begin{aligned} V_{\ell} &= O_{\ell} \bar{\Lambda} O_{\ell}^{-1}, \\ N_{\ell} &= -F_{\ell}(x) O_{\ell} \bar{\Lambda}_{\ell} O_{\ell}^{-1}, \\ D_{\ell} &= O_{\ell} [1 - \alpha_{\ell}(x) \bar{\Lambda}_{\ell}] O_{\ell}^{-1}, \\ T_{\ell} &= O_{\ell} F_{\ell}(x) \bar{\Lambda}_{\ell} [I - \alpha_{\ell}(x) \bar{\Lambda}_{\ell}]^{-1} O_{\ell}^{-1}, \end{aligned} \quad (3.2)$$

where $\bar{\Lambda}_{\ell}$ is a diagonal matrix,

$$\bar{\Lambda}_{\ell} = \text{diag.} (\bar{\lambda}_{\ell}^1, \bar{\lambda}_{\ell}^2, \dots).$$

Now $T_{\ell}(x)$ has a pole at $x = \bar{M}$ whenever

$$1 - \alpha_{\ell}(\bar{M}) \bar{\lambda}_{\ell}^i = 0.$$

Since $\alpha_\ell(x) > 0$ in the region of interest, this can only happen for positive $\bar{\lambda}_\ell^i$. The mass of the "particle" with spin ℓ corresponding to this pole of T_ℓ is then given by

$$\alpha_\ell(M_\ell^i) = 1 / \bar{\lambda}_\ell^i. \quad (3.3)$$

Moreover the coupling constants of this "particle" to the pseudoscalar mesons are given by the residue at the pole. Consider first the case when the pole is non-degenerate. Then

$$\gamma_{\ell,ab}^i \gamma_{\ell,ab}^i = g_\ell^2(M_\ell^i) \psi_{\ell,ab}^i \psi_{\ell,ab}^i \quad (3.4)$$

where $g_\ell^2(M_\ell^i)$ is a proportionality constant independent of the indices of the external pseudoscalar mesons. This means the coupling constants $\gamma_{\ell,ab}^i$ may be taken as the components of the eigenvectors Ψ_ℓ^i , apart from an over-all normalization factor.

In the case when the eigenvalue $\bar{\lambda}_\ell^i$ is n -fold degenerate, we have an n -tuple of "particles" of spin ℓ all with the same mass given by (3.3). The coupling constants are then given by

$$\sum_{r=1}^n \gamma_{\ell,ab}^r \gamma_{\ell,ab}^r = g_\ell^2(M_\ell^i) \sum_{r=1}^n \psi_{\ell,ab}^r \psi_{\ell,ab}^r \quad (3.5)$$

It is obvious that an orthonormal set of eigenvectors ψ_{ab}^r may be chosen such that

$$\gamma_{\ell,ab}^r = g_\ell(M_\ell^i) \psi_{ab}^r. \quad (3.6)$$

In particular, the vector mesons which are exchanged to give the forces of interaction must themselves occur as bound states (or resonances) in the direct channel, so that we must have an eight-fold degenerate eigenvalue $\bar{\lambda}$ ($\ell=1$) with

$$\alpha_1(M) = 1/\bar{\lambda}, \quad (3.7)$$

where M is the squared mass of the vector meson exchanged, in units of $4\mu^2$, ($M = m^2/4\mu^2$). The coupling constants γ_{ab}^r will then satisfy

$$\sum_{r,c,d} (\gamma_{ac}^r \gamma_{bd}^r - \gamma_{ad}^r \gamma_{bc}^r) \gamma_{cd}^s = \bar{\lambda} \gamma_{ab}^s \quad (3.8a)$$

$$\sum_{ab} \gamma_{ab}^r \gamma_{ab}^s = g_1^2(M) \delta^{rs}. \quad (3.8b)$$

The factor $g_1^2(M)$ for $\ell=1$ is found to be

$$g_1^2(M) = 12\pi F_1(M)/(M-1)(d\alpha_1/dx)_{x=M}. \quad (3.9)$$

It is convenient to introduce a normalized set of coupling constants, g_{ab}^r

$$\gamma_{ab}^r = g_1(M) g_{ab}^r \quad (3.10)$$

so that ⁸⁾

$$\sum_{r,c,d} (g_{ac}^r g_{bd}^r - g_{ad}^r g_{bc}^r) g_{cd}^s = \lambda g_{ab}^s \quad (3.11a)$$

$$\sum_{ab} g_{ab}^r g_{ab}^s = \delta^{rs} \quad (3.11b)$$

with $\bar{\lambda} = g_1^2(M) \lambda$.

The general case for other bound states or resonances which may occur in the direct channel is similar. The mass condition reads

$$\alpha_\ell^i(M_\ell^i) = 1 / \bar{\lambda}_\ell^i. \quad (3.12)$$

Introducing $f_{\ell,ab}^i$ and λ_ℓ^i with

$$\gamma_{\ell,ab}^i = g_\ell(M_\ell^i) f_{\ell,ab}^i \quad (3.13)$$

and

$$\bar{\lambda}_\ell^i = g_\ell^2(M_\ell^i) \lambda_\ell^i, \quad (3.14)$$

the conditions on the coupling constants then read

$$\sum_{r,c,d} [g_{ac}^r g_{bd}^r + (-1)^\ell g_{ad}^r g_{bc}^r] f_{\ell,cd}^i = \lambda_\ell^i f_{\ell,ab}^i \quad (3.15a')$$

$$\sum_{ab} f_{\ell,ab}^i f_{\ell,ab}^j = \delta^{ij}, \quad (3.15b)$$

or using the fact that $f_{\ell,ab}^i = (-1)^\ell f_{\ell,ba}^i$, (3.15a') becomes

$$2 \sum_{r,c,d} g_{ac}^r g_{bd}^r f_{\ell,cd}^i = \lambda_\ell^i f_{\ell,ab}^i \quad (3.15a)$$

4. SU₃ SYMMETRY FROM BOOTSTRAP

We now solve the equations (3.11a) and (3.11b) or equivalently

$$2 \sum_{r,c,d} g_{ac}^r g_{bd}^r g_{cd}^s = \lambda g_{ab}^s \quad (4.1a)$$

$$\sum_{ab} g_{ab}^r g_{ab}^s = \delta^{rs} \quad (4.1b)$$

for g_{ab}^r under the assumption of charge conjugation invariance and conservation of charge, strangeness and isotopic spin. It is most convenient to factorize g_{ab}^r into an isospin Clebsch-Gordan coefficient and an isoscalar factor, and substitute these expressions for g_{ab}^r into equations (4.1) giving relations between the isoscalar factors. The calculation is straightforward and yields for the isoscalar factors the following equations

$$\begin{aligned} \frac{3}{2} g_{KK}^{P^2} + \frac{1}{2} g_{KK}^{W^2} &= \lambda \\ g_{\pi\pi}^{P^2} + \frac{2\sqrt{2}}{3} g_{\pi\pi}^{K^{*2}} g_{KK}^P / g_{\pi\pi}^P &= \lambda \\ g_{K\eta}^{K^{*2}} + g_{K\pi}^{K^{*2}} &= \lambda \\ -\frac{1}{2} g_{KK}^{P^2} + \frac{1}{2} g_{KK}^{W^2} + \frac{2\sqrt{2}}{3} g_{K\pi}^{K^{*2}} g_{\pi\pi}^P / g_{KK}^P &= \lambda \\ \sqrt{2} g_{KK}^P g_{\pi\pi}^P - \frac{1}{3} g_{K\pi}^{K^{*2}} + g_{K\eta}^{K^{*2}} &= \lambda \end{aligned} \quad (4.2a)$$

and

$$\begin{aligned}
g_{\pi\pi}^{\rho^2} + g_{KK}^{\rho^2} &= 1 \\
g_{KK}^{\omega^2} &= 1 \\
g_{K\pi}^{K^{*2}} + g_{K\eta}^{K^{*2}} &= 1
\end{aligned} \tag{4.2b}$$

from (4.1a) and (4.1b) respectively. These equations can be solved readily giving ⁹⁾

$$\begin{aligned}
g_{\pi\pi}^{\rho^2} : g_{KK}^{\rho^2} : g_{KK}^{\omega^2} : g_{\pi K}^{K^{*2}} : g_{K\eta}^{K^{*2}} &= \frac{2}{3} : \frac{1}{3} : 1 : \frac{1}{2} : \frac{1}{2} \\
g_{\pi\pi}^{\rho} g_{KK}^{\rho} &> 0
\end{aligned} \tag{4.3}$$

These relations between the isoscalar factors are exactly those required by SU_3 symmetry, and the constants g_{ab}^r , which are products of these isoscalar factors with the corresponding isospin Clebsch-Gordan coefficients, must then be precisely the Clebsch-Gordan coefficients in the SU_3 group in the coupling $8 \times 8 \rightarrow 8$.

There are two isometries in the coupling of $D^8(1, 1) \times D^8(1, 1)$ to $D^8(1, 1)$. Here since the g_{ab}^r are antisymmetric ($g_{ab}^r = -g_{ba}^r$), the coupling must correspond to the isometry where the coupling constants are just the structure constants of the group.

The usual basis for the 8 dimensional regular (adjoint) representation in particle physics is ¹⁰⁾

$$\begin{aligned}
|\pi^+\rangle, |K^+\rangle, |K^0\rangle, |\pi^-\rangle, |K^-\rangle, |\bar{K}^0\rangle, |\pi^0\rangle, |\eta\rangle; \\
|p^+\rangle, |K^{*+}\rangle, |K^{*0}\rangle, |p^-\rangle, |K^{*-}\rangle, |\bar{K}^{*0}\rangle, |p^0\rangle, |\omega\rangle.
\end{aligned} \tag{4.4}$$

corresponding to the eight generators (in that order)

$$E_{-\alpha}, E_{-\beta}, E_{-\gamma}, -E_{\alpha}, -E_{\beta}, -E_{\gamma}, -H_1, -H_2 \quad (4.5)$$

as basis of the Lie algebra. Here we are using the usual notation of the theory of semi-simple Lie groups, where H_i , $i = 1, 2$ are the two commuting operators and E_{ξ} , $\xi = \pm \alpha, \pm \beta, \pm \gamma$ are the raising and lowering operators corresponding to the roots α, β, γ as indicated in the root diagram (Fig. 3).

Fig. 3

In this paper we use the real fields, i.e., we use as the basis of the regular representation the following particle states

$$\begin{aligned} &|\pi_1\rangle, |K_1^c\rangle, |K_1^o\rangle, |\pi_2\rangle, |K_2^c\rangle, |K_2^o\rangle, |\pi_3\rangle, |\eta\rangle; \\ &|\rho_1\rangle, |K_1^{c*}\rangle, |K_1^{o*}\rangle, |\rho_2\rangle, |K_2^{c*}\rangle, |K_2^{o*}\rangle, |\rho_3\rangle, |\omega\rangle; \end{aligned} \quad (4.6)$$

corresponding to the eight generators

$$-E_{y\alpha}, -E_{y\beta}, -E_{y\gamma}, -E_{x\alpha}, -E_{x\beta}, -E_{x\gamma}, -H_1, -H_2 \quad (4.7)$$

as the basis of the Lie algebra, where

$$\begin{aligned}\pi_1 &= \frac{1}{\sqrt{2}} (\pi^+ + \pi^-), \quad \pi_2 = \frac{1}{\sqrt{2}i} (\pi^+ - \pi^-), \quad \pi_3 = \pi^0; \\ K_1^c &= \frac{1}{\sqrt{2}} (K^+ + K^-), \quad K_2^c = \frac{1}{\sqrt{2}i} (K^+ - K^-); \\ K_1^o &= \frac{1}{\sqrt{2}} (K^0 + \bar{K}^0), \quad K_2^o = \frac{1}{\sqrt{2}i} (K^0 - \bar{K}^0);\end{aligned}\tag{4.8}$$

(similarly for the vector particles), and

$$\begin{aligned}E_{x\xi} &= \frac{1}{\sqrt{2}} (E_{+\xi} + E_{-\xi}), \quad E_{y\xi} = \frac{1}{\sqrt{2}i} (E_{+\xi} - E_{-\xi}); \\ \xi &= \alpha, \beta, \gamma.\end{aligned}\tag{4.9}$$

The operators (4.7) are in fact just constant multiples of Gell-Mann's F 's¹⁾. We shall denote them by $iG_r, r = 1, \dots, 8$ since they are differently normalized.

It can readily be shown by direct computations, or otherwise, that the structure constants, C_{st}^r , defined by

$$[iG_s, iG_t] = -i C_{st}^r G_r\tag{4.10}$$

are antisymmetric¹¹⁾ in all three indices, thus

$$C_{st}^r = -C_{ts}^r = -C_{rt}^s,\tag{4.11}$$

and that the metric defined by

$$g_{\alpha\beta} = \sum_{r,t} C_{\alpha t}^r C_{\beta r}^t\tag{4.12}$$

is Euclidean, namely

$$g_{\alpha\beta} = \delta_{\alpha\beta} \quad (4.13)$$

Moreover, as is well-known, the structure constants C_{st}^r may be taken as the elements of the matrices representing the generators G_r in the regular representation, thus

$$g_{ab}^r = C_{ab}^r = (G_r)_{ab} \quad (4.14)$$

These remarks will be useful in solving the eigenvalue problems of W in section 5.

From the well-known properties, it is readily checked that the C_{ab}^r are in fact solutions of the equations (4.1a) and (4.1b) with $\lambda = 1$. Equation (4.1a) is the Jacobi Identity and (4.1b) the metric condition (4.13). In fact these equations are satisfied by the structure constants of any semi-simple Lie group corresponding to the generators (4.7) as basis of the Lie algebra. The assumption of 8 particles, and the additional conditions implied by the conservation laws then restrict the structure constants of SU_3 to be the unique solution.

5. THE EIGENVALUES OF W

Once we have determined that the g_{ab}^r are the structure constants of the SU_3 group, the eigenvalue problem (3.17) is not difficult to solve. Taking f_{ab}^i to be the Clebsch-Gordan coefficients coupling 8×8 to any irreducible representation, the equation (3.17) is satisfied with λ^i as the Racah coefficient (or crossing matrix element), similar to the familiar case of the coupling of three angular momenta. The crossing matrix elements can be calculated in several different ways¹²⁾. We shall present yet another but particularly simple and illuminating method for this special problem, employing the Casimir Operator.

We may write

$$W = 2 \sum_r G_1^r \times G_2^r \quad (5.1)$$

where $\times$ denotes direct product and $G_1^r = (g_{ac}^r)$, $G_2^r = (g_{bd}^r)$. Equivalently we have

$$W = \sum_r (G_1^r + G_2^r)^2 - \sum_r G_1^{r^2} - \sum_r G_2^{r^2}. \quad (5.2)$$

Now for the particular basis (4.7) of the Lie algebra, the metric is Euclidean, $g_{\alpha\beta} = \delta_{\alpha\beta}$, so that the Casimir operator simplifies

$$F = - \sum_{rs} g_{rs} G^r G^s = - \sum_r G^{r^2} \quad (5.3)$$

Thus

$$W = -F + F_1 + F_2, \quad (5.4)$$

where F is the Casimir operator in the product space 1×2 .

F commutes with all G^r and is by Schur's lemma a scalar in any irreducible representation. Denote the value of F in the A dimensional representation by F^A . Then using the well-known complete reduction

$$8 \times 8 = 1 + 8 + 8' + 10 + \overline{10} + 27 \quad (5.5)$$

for representations of SU_3 , one has immediately the result that the eigenvalues of W are degenerate as follows

$$1 + 8 + 8 + 10 + 10 + 27 = 64 \quad (5.6)$$

with the values of λ^A given by

$$\lambda^A = -F^A + F_1^8 + F_2^8 \quad (5.7)$$

In the notation of (4.7), (4.5) we have

$$\begin{aligned} F &= \sum_i H_i^2 + \sum_{\xi} (E_{x\xi}^2 + E_{y\xi}^2) \\ &= \sum_i H_i^2 + \sum_{\xi} (E_{+\xi} E_{-\xi} + E_{-\xi} E_{+\xi}). \end{aligned} \quad (5.8)$$

Let $|m^A\rangle$ be the eigenvector corresponding to the highest weight m^A characterizing the irreducible representation A , then

$$\begin{aligned} E_{+\xi} |m^A\rangle &= 0 \\ F |m^A\rangle &= \left\{ \sum_i m_i^A{}^2 + \sum_{\xi, i} \xi^i m_i^A \right\} |m^A\rangle. \end{aligned} \quad (5.9)$$

Since F is a scalar in representation A

$$F^A = \sum_i m_i^A z + \sum_{\xi, i} \xi^i m_i^A. \quad (5.10)$$

Reading up the roots ξ^i of SU_3 and the highest weights m^A of the various relevant representations from tables ¹⁰⁾, we have

$$F^1 = 0, F^8 = 1, F^{10} = F^{\overline{10}} = 2, F^{27} = 8/3, \quad (5.11)$$

which gives

$$\lambda^1 = 2, \lambda^8 = 1, \lambda^{10} = \lambda^{\overline{10}} = 0, \lambda^{27} = -2/3. \quad (5.12)$$

Now $\lambda > 0$ ($\lambda < 0$) corresponds to attractive (repulsive) force in that channel. Thus we expect in this model no new particle in the $10, \overline{10}$ and 27 channels, while in the $1, 8, 8'$ channels the attractive force may give rise to bound states or resonances.

The coupling constants $f_{\ell, ab}^i$ satisfy

$$f_{\ell, ab}^i = (-1)^\ell f_{\ell, ba}^i \quad (5.13)$$

In the decomposition (5.5) of the product representation, it is well-known that the basis tensors of $1, 8, 27$ are symmetric and $8', 10, \overline{10}$ are antisymmetric, so that even spin "particles" are expected in channels 1 and 8 while odd spin "particles" are expected in channel $8'$. If we restrict the discussion tentatively

only to the two lowest angular momentum states, $\ell = 0, 1$, then in addition to the vector mesons in $8'$, we would expect in this crude model a singlet and an octet of scalar mesons. One has still to see of course whether the attraction in the 1 and 8 channels is strong enough to bind scalar "particles". This is done by solving (3.7), (3.12) numerically for the zeros of D in section 6.

6. NUMERICAL RESULTS OF THE VECTOR BOOTSTRAP

The bootstrap equations for the mass and coupling constant of the vector mesons may be rewritten,

$$\lambda g^2 \alpha_1(M) = 1, \quad (6.1)$$

$$g^2 = 12\pi F_1(M)/(M-1)\alpha'_1(M) \quad (6.2)$$

where $\lambda = 1$, (section 4 or 5). Since $F_1(x)$, $\alpha_1(x)$ and $\alpha'_1(x)$ are all known functions, the equations (6.1), (6.2) can be solved numerically. This gives $M = m^2/4\mu^2 = 1.62$, and $g^2/4\pi = 2.60$ ⁶⁾.

The masses of the scalar mesons if they exist, are then given by

$$\lambda' g^2 \alpha_0(M') = 1, \quad (6.3)$$

where $\lambda' = 2$ for the singlet and $\lambda' = 1$ for the octet of scalar mesons. This equation has also been solved and gives solution at $M' = m'^2/4\mu^2 = 0.5$ for the octet and $M' = 0.05$ for the singlet. This means that, in this model at least, the attraction in the 1 and 8 scalar channels due to the vector exchange force is certainly strong enough to bind new "particles".

The significance of these results in relation to experimental facts will be discussed in a later section. We only point out here that the values given here for the masses and coupling constants are not yet completely self-consistent even within the framework of our present scheme, since we have taken into account only the forces due to the exchange of vector mesons. For complete self-consistency we should bootstrap the scalar mesons as well as the vector mesons and make sure that the inclusion of scalar exchange forces will not alter our conclusion. This will be done in sections 7 and 8.

7. DOUBLE BOOTSTRAP PROBLEM INCLUDING THE EXCHANGE OF THE SCALAR OCTET

Since any bound state or resonance formed in the direct channel may also be exchanged to give rise to interaction forces between the pseudoscalar mesons, complete self-consistency of the bootstrap problem would require that all such "particles" be bootstrapped together with the vector mesons. Thus ideally, both the scalar octet and singlet and any higher spin "particles" which may be found in the calculation of section 6 should all be bootstrapped together. The complexity of the actual solution of the bootstrap problem, however, increases rapidly with the number of particles bootstrapped. To keep the problem within bounds in accord with the simplicity of the present model, we shall bootstrap here only the scalar octet in addition to the vector octet. The neglect of the forces due to the exchange of higher spin particles is in accord with the spirit of the present scheme, since the centrifugal barrier will keep higher spin "particles", if they exist, at a higher energy, and their effect is thus "negligible". We have no such "natural" reason for neglecting the forces due to the exchange of the scalar singlet. However, the crossing matrix element for the singlet exchange is positive and small in all SU_3 channels ($1/4$ cf. 1 for the octet exchange) and its effect would be an additional weak attraction in all the channels. Thus, unless the coupling constant of the singlet is unusually large, it will only help to bind the bound particles tighter without altering the over-all picture. We shall check the validity of the assumption after the solution of the bootstrap problem by calculating the singlet coupling strength using the self-consistent parameters for the octets.

Following the procedure of section 2, we first compute the Born approximation to the full amplitude due to the exchange of the scalar octet for the reaction $(ab \rightarrow cd)$. This gives

$$T = (-4)(4\mu^2) \sum_r \beta_{ac}^r \beta_{bd}^r [1/(t - m'^2)] \\ + (-4)(4\mu^2) \sum_r \beta_{ad}^r \beta_{bc}^r [1/(u - m'^2)], \quad (7.1)$$

The constants β_{ab}^r are symmetric in a and b and have the same physical dimension as the vector coupling constants γ_{ab}^r . The actual coupling constants in the interaction Lagrangian are $2\mu\beta_{ab}^r$. The Zachariasen-Zemach approximation to the N function is then given by

$$\begin{aligned} N_\ell(x) &= -\frac{\sqrt{s}}{2q} \exp(i\delta_\ell) \sin \delta_\ell \\ &= -U_{\ell;ab,cd} G_\ell(x), \end{aligned} \quad (7.2)$$

where

$$U_{\ell;ac,bd} = \sum_r [\beta_{ac}^r \beta_{bd}^r + (-1)^\ell \beta_{ad}^r \beta_{bc}^r] \quad (7.3)$$

$$G_\ell(x) = \frac{1}{4\pi} \cdot \frac{1}{x-1} Q_\ell\left(\frac{2M'+x-1}{x-1}\right). \quad (7.4)$$

Thus the total N function due to both scalar and vector octet exchange is

$$N_{\ell;ab,cd} = -V_{\ell;ab,cd} F_\ell(x) - U_{\ell;ab,cd} G_\ell(x). \quad (7.5)$$

The dispersion relation for D_ℓ and the unitarity condition then yields

$$\begin{aligned} D_{\ell;ab,cd}(x) &= \delta_{ab,cd} - V_{\ell;ab,cd} \alpha_\ell(x) \\ &\quad - U_{\ell;ab,cd} \beta_\ell(x) \end{aligned} \quad (7.6)$$

where

$$\beta_\ell(x) = \frac{x-x_t}{\pi} \int_1^\infty dx' \sqrt{\frac{x'-1}{x'}} \frac{G_\ell(x')}{(x'-x_t)(x'-x-i\varepsilon)} \quad (7.7)$$

and $\alpha_\ell(x)$ is defined as in (2.10).

This gives in matrix notation

$$N_\ell(x) = -V_\ell F_\ell(x) - U_\ell G_\ell(x) \quad (7.8)$$

$$D_\ell(x) = 1 - \alpha_\ell(x) V_\ell - \beta_\ell(x) U_\ell \quad (7.9)$$

$$T_\ell(x) = [V_\ell F_\ell(x) + U_\ell G_\ell(x)] [1 - \alpha_\ell(x) V_\ell - \beta_\ell(x) U_\ell]^{-1} \quad (7.10)$$

Now from the original bootstrap of the vector mesons, (sections 4 and 5), the coupling constants γ_{ab}^r are known to be proportional to the structure constants. Moreover, since β_{ab}^r are the solutions of the eigenvalue Eq. (5.1), they are Clebsch-Gordan coefficients coupling $8 \times 8 \rightarrow 8$ (symmetric). It can be seen that the matrices V_ℓ and U_ℓ are simultaneously diagonalized by vectors whose components are again Clebsch-Gordan coefficients and whose eigenvalues are Racah coefficients or crossing matrix elements. Using the same notation as in section 3, we have

$$N_\ell(x) = O[-\bar{\Lambda} F_\ell(x) - \bar{K} G_\ell(x)] O^{-1} \quad (7.11)$$

$$D_\ell(x) = O[1 - \alpha_\ell(x) \bar{\Lambda} - \beta_\ell(x) \bar{K}] O^{-1} \quad (7.12)$$

$$T_\ell(x) = O[\bar{\Lambda} F_\ell(x) + \bar{K} G_\ell(x)] [1 - \alpha_\ell(x) \bar{\Lambda} - \beta_\ell(x) \bar{K}]^{-1} O^{-1} \quad (7.13)$$

where $\bar{K}$ is a diagonal matrix

$$\bar{K} = \text{diag.} (\bar{K}_1, \dots)$$

The vector mesons occur as poles of $T_1(x)$ at $x = M$, i.e.,

$$1 = \alpha_1(M) \bar{\lambda}^{8'} + \beta_1(M) \bar{\kappa}^{8'} \quad (7.14)$$

and the scalar mesons as poles of $T_0(x)$ at $x = M'$, i.e.,

$$1 = \alpha_0(M') \bar{\lambda}^8 + \beta_0(M') \bar{\kappa}^8. \quad (7.15)$$

The residue of $T_{1;ab,cd}$ at $x = M$ is

$$\left(\sum_{r=1}^8 g_{ab}^r g_{cd}^r \right) [\bar{\lambda}^{8'} F_1(M) + \bar{\kappa}^{8'} G_1(M)] / [-\bar{\lambda}^{8'} \alpha_1'(M) - \bar{\kappa}^{8'} \beta_1'(M)] \quad (7.16)$$

and that of $T_{0;ab,cd}$ at $x = M'$

$$\left(\sum_{r=1}^8 f_{ab}^r f_{cd}^r \right) [\bar{\lambda}^8 F_0(M') + \bar{\kappa}^8 G_0(M')] / [-\bar{\lambda}^8 \alpha_0'(M') - \bar{\kappa}^8 \beta_0'(M')] \quad (7.17)$$

where g_{ab}^r and f_{ab}^r are normalized as in (3.12b) and (3.17b). These residues are, however, connected with the coupling constants. The residue of the graph (Fig. 4) for $\ell = 1$ is

$$- \frac{M-1}{12\pi} \sum_r \gamma_{ab}^r \gamma_{cd}^r$$

and that of the graph (Fig. 5) for $\ell = 0$

$$- \frac{1}{4\pi} \sum_r \beta_{ab}^r \beta_{cd}^r$$

Fig. 4

Fig. 5

Equating these residues to (7.16) and (7.17) respectively gives the equations

$$g^2 = \frac{12\pi}{M-1} \frac{[g^2 \lambda^{8'} F_1(M) + \mathcal{F}^2 \kappa^{8'} G_1(M)]}{[g^2 \lambda^{8'} \alpha_1'(M) + \mathcal{F}^2 \kappa^{8'} \beta_1'(M)]}, \quad (7.18)$$

$$\mathcal{F}^2 = 4\pi \frac{[g^2 \lambda^8 F_0(M') + \mathcal{F}^2 \kappa^8 G_0(M')]}{[g^2 \lambda^8 \alpha_0'(M') + \mathcal{F}^2 \kappa^8 \beta_0'(M')]}, \quad (7.19)$$

for the coupling strengths. The equations for the masses may be rewritten as

$$1 = g^2 \lambda^{8'} \alpha_1(M) + \mathcal{F}^2 \kappa^{8'} \beta_1(M), \quad (7.20)$$

$$1 = g^2 \lambda^8 \alpha_0(M') + \mathcal{F}^2 \kappa^8 \beta_0(M'). \quad (7.21)$$

Here the λ 's and the κ 's are the crossing matrix elements for the exchange of vector mesons and scalar mesons respectively. $\lambda^{8'}$ and λ^8 were evaluated in section 5 ($\lambda^{8'} = \lambda^8 = 1$). The constants $\kappa^{8'}$ and κ^8

have been evaluated by Neville ¹²⁾, for example,

$$K^{8'} = 1, \quad K^8 = -3/5. \quad (7.22)$$

Before proceeding to the actual solution of these double bootstrap equations, we shall first discuss qualitatively the additional effect of the scalar meson exchange. From the properties of the Q_ℓ 's one sees that the functions $G_\ell(x)$ are positive definite for $x > 1$, i.e., above threshold. The integrals $\beta_\ell(x)$ are thus positive definite for $x_t < x < 1$, and, by the same loose arguments as in section 3, for $x > 1$ near threshold. A positive crossing matrix element will then again correspond to an attractive force in that channel for either the $\ell = 0$ or $\ell = 1$ exchange. The scalar octet exchange thus gives attraction in the $8'$ channel (vector octet) but may give repulsion in the 8 channel. We would expect, therefore, that the solution of the present problem will pull the vector meson mass down compared with the result of section 6.

Next we shall examine whether the inclusion of scalar meson exchange will affect our conclusion in section 5 concerning the other SU_3 channels. The crossing matrix elements for the scalar meson exchange ¹²⁾ are

$$K^1 = 2, \quad K^{10} = -4/5, \quad K^{\bar{10}} = -4/5, \quad K^{27} = 2/5 \quad (7.23)$$

The effective forces are thus attractive in the singlet state, repulsive in the 10 and $\bar{10}$ channels, and "weakly" attractive in the 27 channel. The singlet scalar will thus be expected to be pulled further down by the inclusion of the scalar exchange, while the 10 , $\bar{10}$ channels will not be expected to give any resonance or bound states. The forces in the 27 channel is "weakly" attractive

due to the scalar exchange ($\kappa^{27} = 2/5$) but "strongly" repulsive ($\lambda^{27} = -2/3$) due to the vector exchange. Whether the resultant force is attractive or not depends on the relative coupling strengths of the vector and scalar mesons. Thus unless our original assumption of the dominating role of the vector mesons is very far from the truth, one would not expect any resonance or bound state in the 27 channel. This qualitative conclusion is to be checked after the solution of the double bootstrap equations for the vector and scalar octet parameters.

Finally, one may check the possible occurrence of higher spin "particles" by solving the equation

$$1 = g^2 \lambda^A \alpha_\ell(M_\ell^A) + F^2 \kappa^A \beta_\ell(M_\ell^A) \quad (7.24)$$

with the crossing matrix elements λ^A and κ^A for the channel A, and the self-consistent values g^2 , F^2 , M and M' from the equations (7.18) - (7.21). Because of the centrifugal barrier, the higher spin "particle" will presumably be pushed too far up along the Regge trajectory for the present crude model to have much significance for the determination of their parameters. One exception to this is the $\ell = 2$ singlet. The forces in the singlet are so strongly attractive ($\lambda^1 = \kappa^1 = 2$) due both to the vector and scalar octet exchange that they pull the $\ell = 2$ pole down to the region near threshold in spite of the centrifugal barrier, further than any other higher spin states.

8. NUMERICAL RESULTS OF THE DOUBLE BOOTSTRAP

We have to solve equations (7.18) - (7.21) for the quantities M , M' , $\mathcal{M}^2$ and $\mathcal{F}^2$. The quantities $F_1, F_0, G_1, G_0, \alpha_1, \alpha_0, \beta_1, \beta_0, \alpha'_1, \alpha'_0, \beta'_1, \beta'_0$ are all known functions of M and M' , and $\lambda^8, \lambda^{8'}, \kappa^8, \kappa^{8'}$ are known constants. We can thus calculate all the coefficients in the Eqs. (7.18), (7.19) for suitable ranges of M and M' , and solve these equations for $\mathcal{M}^2$ and $\mathcal{F}^2$ for each value of M and M' . (The actual range used in our calculation is $1.0 < M < 2.0$, and $0 < M' < 1.0$.) These values for $\mathcal{M}^2$ and $\mathcal{F}^2$ are then substituted to evaluate the r.h.s. of Eqs. (7.20) and (7.21) for every value of M and M' , and we look for the values of M and M' where both these quantities pass simultaneously through unity. A self-consistent solution is found at $M = 1.35$, $M' = 0.36$, $\mathcal{M}^2/4\pi = 2.78$, $\rho = \mathcal{F}^2/\mathcal{M}^2 = 0.043$.

The values of $\mathcal{M}^2$, $\mathcal{F}^2$, M and M' obtained are then substituted into (7.24). One obtains solutions for $\ell = 0$, $A = 1$ (scalar singlet) at $M_0^1 = 0.02$; for $\ell = 2$, $A = 1$ (spin 2 singlet) at $M_2^1 = 1.27$; and for $\ell = 2$, $A = 8$ (spin 2 octet) at $M_2^8 = 2.72$. There may be further solutions with higher ℓ at higher energies but they are not of immediate interest.

We note also that the value of ρ ($= 0.043$) obtained is sufficiently small for the attraction due to the scalar exchange in the 27 channel to be negligible compared with the repulsion due to the vector exchange. The conclusion reached qualitatively in section 7, that there should be no "particles" in the 10, $\overline{10}$ and 27 channels, is thus confirmed.

In the calculation we have chosen the subtraction parameter, following Zachariasen and Zemach³⁾, at the left-hand threshold for the vector exchange, $x_t = 1 - M$. This means we have normalized the vector exchange correctly but not the scalar exchange. To test the sensitivity of the result on the subtraction point, we have varied x_t from $1 - M$ to $1 - M'$. The attractive force becomes progressively weaker and the values of the masses are pushed up, the vector meson mass by 25% and the scalar meson mass by 50%. The ratio ρ remains small, 0.04 to 0.1, so that the vector exchange remains dominant. The values quoted above are for $x_t = 1 - M$, i.e., with the dominant force correctly normalized.

9. DISCUSSION

Because of the inherent crudeness of the present model, due both to the uncertain approximation of the Zachariasen-Zemach bootstrap mechanism and the rather unphysical degenerate mass assumption, we would not expect the conclusions reached in this paper to have too much quantitative significance in relation to actual experimental facts. The qualitative conclusions, however, may well be correct. The agreement, or otherwise, of our results with experiments may also be regarded as a test for the present bootstrap mechanisms as a sensible model for strong interaction dynamics, since we have here merely pushed the existing bootstrap model to its logical conclusion.

Apart from bootstrap and mass degeneracy, we have assumed the invariance under charge conjugation, and the conservation of charge, strangeness and isospin. We have also assumed the existence of the eight vector mesons and the eight pseudoscalar mesons with the usual assignment of quantum numbers. The additional assumption made at the beginning on the predominant role of the vector mesons has been verified as consistent from the results obtained ($\rho = 0.043$, "small"). We have not bootstrapped the scalar singlet also found in our model. The value found for its coupling constant in our calculation is "small", ($(\ell f_0^1)^2 / \ell_f^2 \sim 0.3$) and since the crossing matrix element of the singlet exchange on the direct channel is only $1/4$ (for all SU_3 channels) as compared with 1 for the vector exchange, its effect should be "small" (of order $0.3 \times 1/4$ of the vector exchange). In any case, the inclusion of the singlet would give rise to further attraction, and would only bind the particles tighter without altering the qualitative picture.

The following consequences have been derived : (i) SU_3 symmetry (first derived by Capps⁶), (ii) absence of bound states or resonances in the 10 , $\overline{10}$ and 27 supermultiplets, (iii) existence of vector "resonances" in $8^1 \sim 1$ GeV, (iv) existence of scalar "particles" in 1 and 8 channels,

(v) likely existence of "particles" of higher spin in the 1, 8 and $8'$ channels at higher energies.

So far no experimental "anomaly" has yet been reported which necessitates the assumption of resonances in the 10 , $\overline{10}$ or 27 supermultiplets, so that (ii) may tentatively be regarded as correct.

The existence of an octet of vector mesons (iii) is well established. The SU_3 singlet vector meson (ϕ or ω) does not appear in our model, since complete mass degeneracy corresponds to exact SU_3 symmetry in which a singlet vector meson cannot be coupled to two pseudoscalar mesons. It is difficult to compare the mass values obtained in the calculation with actual experimental values since one is not sure exactly what to compare. The value $M = 1.35$ corresponds to a mass of the vector meson at 2.34μ , where μ is the (degenerate) pseudoscalar meson mass. Taking μ as the root mean square mass, the mean mass, and the unperturbed mass of the Gell-Mann/Okubo formula respectively for the pseudoscalar octet, one has $\mu = 410, 370, 550$ MeV. For the vector octet, we get from our model the corresponding values $m_{th} = 960, 860$ and 1290 MeV as compared with the experimental values $m_{ex} = 840, 840, 900$ MeV. As seems common in bootstrap problems, the widths compare worse with experiments than the mass values. The value $g^2/4\pi = 2.72$ calculated here is to be compared with 0.9 , the value deduced from the ρ width. In any case the quantitative comparison may not have much meaning, though it is interesting that the quantities do have the right order of magnitude.

The prediction of this model of the existence of a scalar octet and a scalar singlet may be of greater practical interest. The scalar "particles" here appear below threshold, i.e., as bound states, but when the mass degeneracy is removed, they would presumably appear as resonances decaying into the channels with lower thresholds, e.g., $\pi + \pi$. The calculated value for $M' = 0.36$ corresponds to the mass values $m_{th}' = 500, 450, 670$ MeV, respectively for the

three cases quoted above. Since the coupling constants of the scalar "particles" have a different physical dimension from those of the vector mesons, there is no obvious comparison between the two coupling strengths. We have given the coupling strength for the scalar mesons in units of 2μ , which is a reasonable standard scale for these problems. The calculated value for $\rho = F^2/4f^2 = 0.043$ then presumably implies that if these scalar mesons exist, they should have much narrower widths and be much more weakly coupled than the vector mesons. This and the fact that they are scalars would make them difficult to identify experimentally.

No scalar resonance in pseudoscalar meson scattering has so far been definitely identified experimentally. There are, however, several possible candidates in existence which may be tentatively assigned to the scalar 1 and 8 supermultiplets. One may, for example, assign the low-lying ABC particle (existence not well established) to the scalar SU_3 singlet. The resonances $K^*(725)$ with a width of $\lesssim 10$ MeV would fit in comfortably as the $I = 1/2$ members of the scalar octet, especially since its width is in qualitative agreement with our small value of ρ . This leaves the other members of the octet as weakly coupled scalar resonances "to be discovered"! There is an "anomaly" in the $K\bar{K}$ system $(K_1 K_1)^{13)}$ near threshold which may be explained by an $I = 0$ member of a scalar octet. Taking the mass of this $K_1 K_1$ to be 800 MeV, Glashow¹³⁾ obtained a value of 550 MeV for the remaining $I = 1$ members using the Gell-Mann/Okubo formula. Because of the conservation of parity and G parity the $I = 1$ members of a 0^+ octet has, as the lowest strong decay modes, 5π or $\pi + \eta$, both of which lie above the value 550 MeV. Such resonances can thus only decay electromagnetically.

One could assign the f_0 particle to the $\ell = 2$ singlet. The calculated values for the mass $m_2 = 930, 840, 1250$ MeV is to be compared with the experimental value 1250 MeV. The predicted value for the mass of the $\ell = 2$ octet lies much higher, ($m_2^8 = 1350, 1220, 1820$ MeV) and may not be observable at present. The assignment of the newly discovered B resonances¹⁴⁾ at ~ 1250 MeV to the $I = 1$ members of the $\ell = 2^+$ octet is not possible because of charge conjugation. The favourite assignment at present for the B is 1^+ . If this is true then they will not appear in this model in which we have only intermediate states of parity $(-1)^J$.

ACKNOWLEDGEMENT

The work was begun in Princeton, continued in Florida State University, Brookhaven National Laboratory, Notre Dame University, University of Hong Kong and CERN. We would like to thank Dr. R. Oppenheimer for hospitality at the Institute for Advanced Study, Dr. W. Bleakney for his at Princeton University, Dr. J. Lannutti for his at Florida State University (Tallahassee), Dr. G.C. Wick for his at the Brookhaven National Laboratory, Dr. Y.C. Wong for his at the University of Hong Kong, and Dr. L. Van Hove for his hospitality at CERN.

We have benefited from discussions with Drs. G.C. Wick, L. Balazs, L.F. Cook, S.B. Treiman, J.S. Dowker, K. Dietz, J. Prentki, and C. Wilkin, and we are indebted to Dr. K. Dietz for programming the double bootstrap problem on the IBM computer at CERN.

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