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DETERMINATION OF THE S MATRIX FOR A SUPERPOSITION OF
EXPONENTIAL POTENTIALS SINGULAR LIKE r^{-2} AT THE ORIGIN

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ABSTRACT

Previous works have shown the properties and the possibility of useful approximate methods for Jost functions and radial wave functions in the case of Yukawa generalized potentials. Here we extend some of these results for the same family of short range potentials but with a behaviour near the origin like $-G/r^2$, $[G < (\ell + 1/2)^2]$. The key is to find a singular factor such that the analysis follows as closely as possible the usual development. A method is presented for calculating the Jost function by use of Laplace transform giving approximations always convergent, without any cut-off or hard-core procedure.

I. INTRODUCTION

The aim of this paper is to determine the Jost function and the S matrix for potentials like superposition of exponentials singular as $-Gr^{-2}$ at the origin. Recently many people were interested in the $-Gr^{-2}$ potential and the corresponding singularities in the angular momentum variable. Here we do not consider analytical properties ^{*)} although the method given can be used to study properties with respect to the energy.

Potentials like $-Gr^{-2}(r \rightarrow 0)$ are very particular and in fact can be considered as "transition potentials". Consider first only the mathematical point of view. For the uncoupled radial differential equations in the neighbourhood of the origin, suppose that the potentials are $\underset{r \rightarrow 0}{\sim} -Gr^{-\epsilon}$ (G real).

1a) $\epsilon \leq 2$. The origin is a regular singular point, with solutions of Fuch's type.

2a) $\epsilon > 2$. The origin is an irregular singular point for which power behaviour of the solutions near $r=0$ is not possible.

But we can also consider another classification :

1b) $\epsilon < 2$. We can define a regular solution and an irregular one independently of the potential (ex $U_R \sim r^{\ell+1}$ $U_{IR} \sim r^{-\ell}$). The potential appears as a perturbation.

2b) $\epsilon \geq 2$. The singularities of the potential near the origin determine the behaviour ^{**)} of the solution at $r=0$. For $\epsilon = 2$, the potential enters into the Fuchsian conditions at the origin.

And consequently another choice is this :

1c) It is possible to define unambiguously regular and irregular solutions $\epsilon < 2$.

2c) There are particular values of G such that it is difficult to say what is the irregular one (for instance, $G > (\ell - \frac{1}{2})^2$ $\epsilon=2$, $G > 0$ for $\epsilon > 2$). It can be noted also that with respect to the ℓ analytic properties $\epsilon=2$ is a very particular case.

*) For analytic properties see, for instance, N. Limić, Nuovo Cimento 26, 581 (1962).

***) Note that in the coupled case, because the tensor potential $V_T \simeq C^T r^{-\epsilon_T}$, even for $0 \leq \epsilon_T < 2$ we are in case 2b). H. Cornille, Nuovo Cimento 23, 526 (1962).

The r^{-2} potential and the difficulties arising from the fact that the roots of indicial equation become complex ($G > (\ell + 1/2)^2$) have been studied by many authors ^{1),2),3)}. Here we want to study rather potentials of short range like superpositions of exponentials. The appearance of the Mandelstam representation and many other interesting properties have pointed out the role of Yukawian potentials in providing non-relativistic models of scattering amplitudes. With these potentials, Martin ⁴⁾ was the first who showed the great advantage of using the Laplace transform to obtain the Jost function. Usually one defines ingoing or outgoing waves and the physical solution as a linear combination of the two. The regularity condition at the origin which extracts the singular part of the ingoing or outgoing waves gives the Jost function. Here we are concerned for $\epsilon=2$ only for $0 < G < (\ell + 1/2)^2$ (attractive) or $G < 0$ (repulsive potential) with the possibility of defining unambiguously the regular and the irregular parts of the solutions at $r=0$.

Fortunately this restriction for $\epsilon=2$ and the corresponding one ($G < 0$) for more singular potentials is not too serious in physical applications. As has been pointed out by Predazzi and Regge ⁵⁾ in studying singular potentials, the most striking feature of nuclear potentials near the origin is that they are strongly repulsive.

Another reason for studying the $\epsilon=2$ case is the great analogy with the ladder approximation of the Bethe-Salpeter equation in particular cases, as has been discussed recently by Bastai et al. ⁶⁾.

For $\epsilon=2$, the singularity of the potential determines the behaviour of the singular part at the origin of the outgoing as well as the ingoing wave. The method used is to obtain a factor, independent of the energy which is the same for the two waves. On the one hand, this factor is such that the degree of singularity of the singular parts of the two waves becomes independent of the potential, just as in the familiar case for obtaining the Jost function. On the other hand, this factor gives also the possibility of obtaining the Jost function by approximations that are always convergent and have the same interesting iterative properties of the Laplace transform as in the usual case ⁴⁾ $\epsilon < 2$. We consider in Section II only s waves where we explain the method, and in Section III we extend briefly the results to $\ell \neq 0$.

II. CASE OF $\ell = 0$

We consider the Schrödinger equation for $\ell = 0$:

$$\left\{ \frac{d^2}{dr^2} + k^2 \right\} u = Uu \quad (1)$$

where $U(r) = -Ge^{-\mu r}/r^2 + V(r)$. We suppose that $V(r) \xrightarrow{r \rightarrow \infty} 0$ as fast as a decreasing exponential and $V(r)$ is less singular as Cte/r^2 when r goes to 0. Here we have a regular singular point of Fuch's type at $r=0$, with roots of the indicial equation

$$s(s-1) + G = 0 \quad s_{\pm} = \frac{1 \pm \sqrt{1-4G}}{2} \quad s_- = \lambda \quad (2)$$

For $G > 1/4$ [and more generally $G > (\ell + 1/2)^2$], the $s_{\pm}$ are complex numbers with equal real parts and it is not possible to distinguish what is the regular solution and what is the irregular one. Also the solutions cannot be distinguished by the quadratically integrable condition near the origin. These particular features of the Schrödinger equation for potentials $\xrightarrow{r \rightarrow 0} -G/r^2$ [and a similar situation for more singular potentials $\simeq -G/r^n$, $r > 2$, $G > 0$] have been studied by many authors ^{1), 2), 3)} and we do not consider here these cases. Thus we take always $G < 1/4$ ($G \neq 0$).

a) Definition of the S matrix

We consider $R(\pm k, r)$ solutions of Eq. (1) such that

$$R(k, r) = F(k, A, r) Z(A, \lambda, r) \xrightarrow{r \rightarrow \infty} e^{-ikr} \quad (3)$$

where A is some positive number and Z a factor

$$\left\{ \begin{aligned} Z(A, \lambda, r) &= e^{\lambda \int_{\infty}^r \frac{e^{-Ar'}}{r'} dr'} \xrightarrow{r \rightarrow \infty} 1 \\ &\xrightarrow{r \rightarrow 0} X(A, \lambda) r^{\lambda} \end{aligned} \right. \quad (4)$$

$$X(A, \lambda) = \exp \left(\lambda \left(\int_{\infty}^1 \frac{e^{-Ax}}{x} dx + \int_1^0 \frac{e^{-Ax}-1}{x} dx \right) \right) > 0$$

We define a regular solution $\varphi(k, r)$ of (1) by $\varphi(k, r) \underset{r \rightarrow 0}{\sim} r^{s_+}$ and the Jost function $f(k)$;

$$f(k) = \lim_{r \rightarrow 0} \left(R(k, r) / r^\lambda \right) = F(k, A, 0) X(A, \lambda) = \frac{1}{\sqrt{1-4G}} W \left(R(k, r), \varphi(k, r) \right) \quad (5)$$

where $W(R, \varphi)$ is the Wronskian. If we write φ as a linear combination of $R(\pm k, r)$ we obtain with Eqs. (4) and (5) :

$$\varphi(k, r) = \frac{\sqrt{1-4G}}{2ik} Z(A, \lambda, r) X(A, \lambda) \left[F(-k, A, r) F(k, A, 0) - F(k, A, r) F(-k, A, 0) \right] \quad (6)$$

and the S matrix

$$S(k) = \frac{f(k)}{f(-k)} = \frac{F(k, A, 0)}{F(-k, A, 0)}$$

with $F(k, A, 0)$ finite. $X(A, \lambda)$ being independent of k , the solutions of $f(-k) = 0$ are those $F(-k, A, 0) = 0$ depending only on k . This procedure introduces no aberrant poles of the S matrix. A bound state for example is a relation between k , G and the parameters of V , independently of A . The introduction of this parameter A is not equivalent to a cut-off or hard-core procedure. $F(k, A, r)$ satisfies the differential equation

$$F'' + k^2 F = \frac{\lambda e^{-Ar}}{r} (-2F' + AF) + \frac{F}{r^2} \left[\lambda (e^{-Ar} - e^{-\mu r}) + \lambda^2 (e^{-\mu r} - e^{-2Ar}) \right] + VF \quad (7)$$

with the boundary condition $F(k, A, r) \underset{r \rightarrow \infty}{\sim} e^{-ikr}$.

Now for $r \rightarrow 0$ the roots of the indicial equation of (7) are 0 and $s_+ - s_- = \sqrt{1-4G} > 0$. Usually the Jost function is determined by starting with a solution which is asymptotically purely outgoing (or ingoing) wave. Then it is necessary to extract its contribution to the irregular solution at $r=0$; for example here $\lim_{r \rightarrow 0} R(k, r) / r^\lambda$ (λ being not integer). Now why have we made the substitution from R to F ? There are two reasons.

Firstly, the change $R \rightarrow ZF$ still gives for F the feature of an asymptotically outgoing (or ingoing) wave. On the other hand, the factor $Z(A, \lambda, r)$, (independent of k) is such that the irregular solution of (7) at $r=0$ goes to a constant when $r \rightarrow 0$. Then it is not necessary to use a limiting process when $r \rightarrow 0$ to extract the Jost function $F(k, A, 0)$. Secondly, while the left-hand side of (7) gives the possibility of starting asymptotically with a purely outgoing or ingoing wave, the right-hand side appears for such solutions as a perturbation with an exponential decreasing factor. Then it is obvious that if we suppose for V a superposition of Yukawa potentials ^{*)} we can still obtain for the Laplace transform of the solution the particular iterative properties first obtained by Martin in the case where $-Ge^{-\mu r}/r^2$ is not present.

b) Determination of the Jost function $f(k)$ for Yukawa generalized potentials

Now we suppose

$$V(r) = \int_{\mu}^{\infty} C(\alpha) e^{-\alpha r} d\alpha \quad |C(\alpha)| < C^{te} \quad (8)$$

At this stage of our study, it is useful to take a particular choice of A . For example $A = n\mu/2$. We arbitrarily choose $A = \mu$, but in fact other choices of A are also possible. We put in (7)

$$F(k, \mu, r) = \int_0^{\infty} e^{-i(k-i\alpha)r} \rho(k, \alpha) d\alpha$$

and obtain for the Laplace transform ρ :

$$\left\{ \begin{aligned} \rho(k, \alpha) &= \delta(\alpha) + \frac{1}{\alpha(\alpha+2ik)} \int_0^{\alpha-\mu} [2\lambda(\beta+ik) + D(\alpha-\beta)] \rho(k, \beta) d\beta \\ D(\alpha) &= \vartheta(\alpha-\mu) \left(\lambda\mu + C(\alpha) \right) + \lambda^2 \left[(\alpha-\mu)\vartheta(\alpha-\mu) - (\alpha-2\mu)\vartheta(\alpha-2\mu) \right]; |D(\alpha)| < C^{te} \end{aligned} \right. \quad (9)$$

^{*)} Even for $\epsilon < 2$, appropriate factors can be useful in view of iterative properties of the Laplace transform. For instance when a Coulombic potential is present with Yukawa generalized potentials (A. Martin, private communication, unpublished work).

We can remark that from (7), for the choice $A = \mu/2$, we obtain for the corresponding ρ in the kernel of the Volterra equation, only λ and not λ^2 . For any λ the Volterra equation has a solution. From (9) we have the following properties.

- 1) If we put $\rho = \delta + \tau$ where $\tau(k, \alpha) = \sum_{n=1}^{\infty} \tau_n(k, \alpha)$, then

$$\tau_n(k, \alpha) \equiv 0 \quad \text{for } \alpha < n\mu$$

$$\tau_n(k, \alpha) = \frac{1}{\alpha(\alpha + 2ik)} \int_{(n-1)\mu}^{\alpha-\mu} \left(2\lambda(\beta + ik) + D(\alpha - \beta) \right) \tau_{n-1}(k, \beta) d\beta$$

we have the same iterative properties as in the usual case ⁴⁾, the approximation of order n gives exactly the solution for $\alpha < n\mu$.

- 2) The Jost function is $X(\mu, \lambda)F(k, \mu, 0)$ where

$$F(k, \mu, 0) = 1 + \sum_{n=1}^{\infty} \int_{n\mu}^{\infty} \tau_n(\alpha, k) d\alpha = 1 + \int_{\mu}^{\infty} \tau(\alpha, k) d\alpha. \quad (10)$$

We suppose now k outside the cut $[i\mu/2, i\infty]$ which is apparent from (9); we suppose also $|k|$ finite and $\lambda \neq 0$ or $G \neq 0$, then

$$|\tau_n(\alpha, k)|_{\alpha > \alpha_0} < C_n^{\text{te}} \frac{(L\alpha)^{n-1}}{\alpha^2}$$

so that for all finite n , each iteration τ_n gives a convergent contribution for the Jost function (10).

- 3) It is shown in Appendix A that for $|2\lambda| < 1$

$$|\tau(\alpha, k)| \leq \frac{A}{\alpha^{1+\eta}}$$

where η is small and positive, and A is finite. Thus the integral (10) giving the Jost function is convergent.

- 4) For $2\lambda < 1$ or $G < 1/4$ which includes now the repulsive core, one has [see Appendix A]

$$y(\alpha, k) = 1 + \int_{\mu}^{\alpha} \tau(\beta, k) d\beta$$

goes to a constant when $\alpha \rightarrow \infty$.

Thus it is not necessary in order to obtain the Jost function for these kinds of potentials to take any arbitrary cut-off or hard-core procedure.

c) More general factors

We want to show now briefly that $Z(A, \lambda, r)$ defined by (4) is not the only factor which gives the properties just discussed at the end of a). More generally, we consider in (3)

$$Z(A, \lambda, r) = \exp \left(\int_{\infty}^r \psi(\lambda, A, r') dr' \right)$$

with

$$\psi(\lambda, A, r) = \int_A^{\infty} e^{-\alpha r} E(\alpha) d\alpha.$$

We substitute this in (1) with the condition that

$$\psi' + \psi^2 - U(r) = 0 \quad (11a)$$

we then find from (1) that $F(k, A, r)$ satisfies

$$\left\{ \frac{d^2}{dr^2} + 2\psi \frac{d}{dr} + k^2 \right\} F = 0 \quad (12a)$$

We consider for U the form :

$$U(r) = -G \int_{\mu}^{\infty} e^{-\alpha r} \alpha M(\alpha) d\alpha, \quad |M(\alpha)| < 1 \quad \text{and} \quad \lim_{\alpha \rightarrow \infty} M(\alpha) = 1.$$

Suppose now we can obtain a solution of (11a) such that $\psi \xrightarrow[r \rightarrow 0]{\sim} \frac{\lambda}{r}$ and $\psi \xrightarrow[r \rightarrow \infty]{} 0$ as an exponential decreasing.

Then $f(k) = F(k, A, 0)X(A, \lambda)$ where

$$X(A, \lambda) = \exp \left[\int_{\infty}^1 \frac{\psi(\lambda, A, r)}{\lambda} dr + \int_1^0 \left(\frac{\psi(\lambda, A, r)}{\lambda} - \frac{1}{r} \right) dr \right]$$

and the S matrix is still given by $F(k, A, 0)/F(-k, A, 0)$.

We take now the Laplace transform of (11a) and make the choice of $A = \mu$ as previously.

$$E(\alpha) = GM(\alpha) + \frac{1}{\alpha} \int_{\mu}^{\alpha-\mu} E(\beta)E(\alpha-\beta) d\beta \quad (11b)$$

All that is necessary is to show that we can obtain a solution of (11b) such that $E(\alpha) \xrightarrow[\alpha \rightarrow \infty]{} \lambda$ [or $\psi \xrightarrow[r \rightarrow 0]{\sim} \frac{\lambda}{r}$]. We do not want to study here the general case and restrict ourselves for simplicity in the proof to $0 < G < 1/4$. Iterating (11b) with the following "ansatz" $E(\alpha) = \sum_{n_0}^{\infty} G^{n_0} F_{n_0}(\alpha)$ where $n_0 = 1$, we obtain

$$F_1(\alpha) = M(\alpha)\vartheta(\alpha-\mu) \quad \text{and} \quad |F_1| \leq 1$$

$$F_2(\alpha) = \frac{1}{\alpha} \left(\int_{\mu}^{\alpha-\mu} M_1(\beta)M_1(\alpha-\beta) d\beta \right) \vartheta(\alpha-2\mu) \quad \text{and} \quad |F_2| \leq 1$$

$$F_n(\alpha) = \frac{1}{\alpha} \left(\sum_{p=1}^{n-1} \int_{p\mu}^{\alpha-\mu} F_p(\beta)F_{n-p}(\alpha-\beta) d\beta \right) \vartheta(\alpha-n\mu) \quad \text{and} \quad |F_n| \leq a_n$$

We first observe that the value of $E(\alpha)$ for $\alpha < (n+1)\mu$ is still entirely determined by the sum of the first n terms of the series.

Secondly, we can easily obtain the value of a_n . Consider $\lambda = G + \lambda^2$ and suppose $\lambda = \sum_{n=0}^{\infty} G^n b_n$. Then $b_1 = 1$ and $b_n = \sum_{p=1}^{n-1} b_p b_{n-p}$ and we find $a_n = b_n$, $\lambda = 1/2 - 1/2(1-4G)^{1/2}$ and $|E(\alpha)| \leq_{\alpha \rightarrow \infty} \lambda$.

Without going into details, note that because $G = \lambda - \lambda^2$ we can also consider for (11b) a solution $E(\alpha) = \sum_{n=0}^{\infty} \lambda^n F_n(\alpha)$ where now

$$F_1 = M(\alpha) \xrightarrow{\alpha \rightarrow \infty} 1,$$

$$F_2 = -M(\alpha) + \frac{1}{\alpha} \int_{\mu}^{\alpha-\mu} M(\beta) M(\alpha-\beta) d\beta \xrightarrow{\alpha \rightarrow \infty} 0$$

and

$$F_n = \left(\frac{1}{\alpha} \sum_{p=1}^{n-1} \int_{\mu}^{\alpha-\mu} F_p F_{n-p} d\beta \right) \vartheta [\alpha - (n-1)\mu]$$

with similar iterative properties as in the preceding case.

We consider now the Laplace transform of (12a)

$$F(k, A, r) = \int_0^{\infty} e^{-i(k-i\alpha)r} \rho(k, \alpha) d\alpha$$

$$\rho(k, \alpha) = \delta(\alpha) + \frac{2}{\alpha(\alpha+2ik)} \int_0^{\alpha-\mu} E(\alpha-\beta)(\beta+ik)\rho(k, \beta) d\beta \quad (12b)$$

and

$$F(k, A, 0) = 1 + \int_{\mu}^{\infty} \rho(k, \alpha) d\alpha.$$

Because of the iterative properties of the $E(\alpha)$ solution of Eq. (11b), one obtains the same iterative properties for ρ [i.e., $\rho_n = 0$ for $\alpha < n\mu$] and then it is sufficient to have determined $E(\alpha)$ only for $\alpha \leq n\mu$ in order to know ρ in the same range $\alpha \leq n\mu$. If we consider the existence of the Jost function, we are only concerned in (12b) with the asymptotic behaviour of $E(\alpha)$, and when we can show that $E(\alpha) \rightarrow \lambda$, we have a similar asymptotic behaviour for $\rho(\alpha)$ as in b) for Eq. (9).

d) Bound states $G < 1/4$

Consider $F(-k, A, 0) = 0$ with $\text{Im } k > 0$. Then Eq. (6) states that $\varphi(k, r) \rightarrow 0$ both for $r \rightarrow 0$ and $r \rightarrow \infty$. If we make as usual the linear combination of (1) for φ and φ^* we find

$$(k^2 - k^{*2}) \int_0^\infty |\varphi|^2 dr = 0.$$

Then necessarily $\text{Re } k = 0$. Thus as usual $f(-k)$, the denominator of S cannot vanish in the upper half complex plane, except perhaps on the imaginary axis. Suppose always $F(-i\kappa, A, 0) = 0$ with $\kappa > 0$ and consider Eq. (7) for the corresponding $F(-i\kappa, A=\mu, r)$. In this case

$$F \underset{r \rightarrow 0}{\sim} \text{cte } r^{\sqrt{1-4G}}$$

and we obtain after some algebra :

$$-\left[\int_0^\infty r(F'^2 + \kappa^2 F^2) dr \right] = \lambda^2 \int_0^\infty F^2 \left[rV + \frac{e^{-\mu r}}{r} (1 - e^{-\mu r}) \right] dr \quad (13)$$

We see that for $V(r) \geq 0$ we have no bound states ^{*)}.

^{*)} This can also be shown by considering the uncertainty principle (A. Stanghellini, private communication).

III. CASE OF $\ell \neq 0$, $G < (\ell + 1/2)^2$

We consider Eq. (1) with the centrifugal term

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} + \frac{Ge^{-\mu r}}{r^2} \right\} u = Vu \quad (14)$$

with roots of the indicial equation $s(s-1) - \ell(\ell+1) + G = 0$

$$\begin{aligned} s_- &= -\ell + \lambda \\ s_+ &= (\ell+1) - \lambda \end{aligned} \quad \lambda = (\ell+1/2) \left(1 - \sqrt{1 - G/(\ell+1/2)^2} \right) \quad (15)$$

Note that for $\ell = 0$, λ is the same as before. We define the solutions of Eq. (14) :

$$\begin{aligned} R_\ell(k, r) &= Z_\ell(A, \lambda, r) \bar{F}_\ell(k, A, r) \underset{r \rightarrow \infty}{\sim} e^{-ikr} \\ \varphi_\ell(k, r) &\underset{r \rightarrow 0}{\sim} r^{s_+} \end{aligned} \quad (15')$$

The aim is to find a factor Z_ℓ such that for the corresponding F_ℓ one has :

- 1) the most singular power behaviour near the origin is $r^{-\ell}$;
- 2) the solution is purely ingoing (or outgoing) when $r \rightarrow \infty$;
- 3) a natural extension of the $\ell = 0$ case;
- 4) the useful iteration properties of the Laplace transform.

For all these reasons we take the same factor (4) as before. λ is now given of course by (15). Then we have :

$$f_\ell(k) = \lim_{r \rightarrow 0} \left(R_\ell(k, r) / r^{\lambda - \ell} \right) = X_\ell(A, \lambda) F_\ell(k, A) = \frac{1}{(2\ell+1-2\lambda)} W \left(R_\ell(k, r), \varphi_\ell(k, r) \right)$$

where

$$F_\ell(k, A) = \lim_{r \rightarrow 0} r^\ell F_\ell(k, A, r) \quad (16)$$

$$\varphi_\ell = \frac{(2\ell+1-2\lambda)}{2ik} Z_\ell(A, \lambda, r) X_\ell(A, \lambda) \left(F_\ell(-k, A, r) F_\ell(k, A) - F_\ell(k, A, r) F_\ell(-k, A) \right) \quad (16')$$

$$S_\ell(k) = f_\ell(k)/f_\ell(-k) = F_\ell(k, A)/F_\ell(-k, A) \quad .$$

$F_\ell(k, A, r)$ satisfies

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} \right\} F_\ell = \frac{\lambda e^{-Ar}}{r} \left[-2 \left(F'_\ell + \frac{\ell e^{(A-\mu)r}}{r} F_\ell \right) + \right. \\ \left. + \frac{F_\ell}{r^2} \left[\lambda (e^{-Ar} - e^{-\mu r}) + \lambda^2 (e^{-\mu r} - e^{-2Ar}) \right] + V F_\ell \right] \quad (17)$$

$$F_\ell \underset{r \rightarrow \infty}{\sim} e^{-ikr}$$

The roots of the indicial Eq. (17) : $x(x-1)+2\lambda(x+\ell)-\ell(\ell+1) = 0$ are $-\ell$ and $\ell+1-2\lambda$ with $-\ell < \ell+1+2\lambda$. Then condition (1) is satisfied and the limit in (16) exists. When $r \rightarrow \infty$ the right-hand side of (17) goes to zero and the asymptotic behaviour is given by the left-hand side. For $\ell = 0$ Eq. (17) reduces to Eq. (7).

Consider now an integral equation corresponding to (17) defined by an asymptotic behaviour of ingoing or outgoing waves. Then the left-hand side gives the free term and the right-hand side appears as a perturbation. Furthermore, the behaviour of the more singular part of the solution near the origin is still the same as that of the free term. The treatment of equations similar to Eq. (17) in order to obtain iterative properties [as in I] has been made in the $\ell \neq 0$ case by using the Laplace transform⁴⁾. It is also possible to consider a transformation with respect to the Hankel spherical functions^{7), 8)}

or with solutions of the left-hand side of (17) purely ingoing or outgoing⁹⁾. We give both the two methods, the last being a kind of generalized Laplace transform, especially appropriate for the Schrödinger equation with $\ell \neq 0$. Now we put $A = \mu$ and always $\lambda < (\ell + 1/2)$.

a) Laplace transform. In fact it is not sufficient to take directly the Laplace transform of (17) [see Appendix B] in order to obtain the desired iterative properties. It is necessary to make a more elaborate treatment⁴⁾, we give only the results for potentials like (8). We put

$$F_{\ell}(k, \mu, r) = e^{-ikr} \left[1 + \int_0^{\infty} \rho_k(\alpha) e^{-\alpha r} d\alpha \right] \quad (18)$$

$$\begin{aligned} \rho_k(\alpha) = & \left\{ P'_L \left(1 + \frac{\alpha}{ik} \right) + \frac{G_k(\alpha)}{\alpha(\alpha + 2ik)} + \int_{\mu}^{\alpha} K(\alpha, \beta) G(\beta) d\beta \right\} + \\ & + \int_0^{\alpha - \mu} \rho_k(\gamma) d\gamma \left[\frac{2\lambda\gamma + G_k(\alpha - \gamma)}{\alpha(\alpha + 2ik)} + \int_{\gamma + \mu}^{\alpha} K(\alpha, \beta) \left(2\lambda\gamma + G_k(\beta - \gamma) \right) d\beta \right] \end{aligned} \quad (19)$$

where

$$G_k(\alpha) = C(\alpha) + \vartheta(\alpha - \mu) \left(\lambda\mu + 2i\lambda k - 2\lambda\ell(\alpha - \mu) \right) + \lambda^2 \left[\vartheta(\alpha - \mu)(\alpha - \mu) - \vartheta(\alpha - 2\mu)(\alpha - 2\mu) \right] \quad (19')$$

and

$$K(\alpha, \beta) = P'_L \left(1 + \frac{\alpha}{ik} \right) Q'_L \left(1 + \frac{\beta}{ik} \right) - Q'_L \left(1 + \frac{\alpha}{ik} \right) P'_L \left(1 + \frac{\beta}{ik} \right)$$

P_L and Q_L are respectively Legendre functions of first and second kind. The integral Eq. (19) has the same structure as the s wave Eq. (9) and in fact reduces to it for $\ell = 0$. When $\rho(\alpha)$ is known for $\alpha < n\mu$, one can get ρ for $\alpha < (n+1)\mu$ by inserting the known functions in the right-hand sides.

Still we suppose k outside the cut $[i\mu/2, i\infty]$, and $|k|$ finite and $\neq 0$. If we iterate (19) $\rho(\alpha) = \sum_0^\infty \rho_n(\alpha)$ we obtain $|\rho_n(\alpha)| < C_n^{te} \alpha^{\ell-1}$ for $\alpha > \alpha_0$. Then ⁴⁾ for each n ,

$$\lim_{r \rightarrow 0} r^\ell \int_{n\mu}^\infty e^{-\alpha r} \rho_n(\alpha) d\alpha$$

exists and we can obtain approximate solutions of the Jost function. When we consider the sum $\rho(\alpha)$, we can show [see Appendix B] for $\lambda < \ell + 1/2$ with similar proofs as in the $\ell = 0$ case that $\rho(\alpha) \underset{\alpha \rightarrow \infty}{\sim} C_k \alpha^{\ell-1}$. Thus the limit in (16) exists and also ⁴⁾ $S(k) = \lim_{\alpha \rightarrow \infty} \rho_k(\alpha) / \rho_{-k}(\alpha)$.

b) We want now to study Eq. (17) by transform with respect to the $h_\ell(\rho)$ which is $\rho \times$ the Hankel spherical function :

$$h_\ell(\rho) = (-i)^{\ell+1} e^{i\rho} \sum_{p=0}^{\ell} \left(\frac{i}{2\rho}\right)^p a_{\ell,p} \quad a_{\ell,p} = \frac{(\ell+p)!}{\ell!(\ell-p)!} \quad (20)$$

We put

$$(-i)^{\ell+1} F_\ell(-k, \mu, r) = \int_0^\infty x_{-k}(\alpha) \left(\frac{k+i\alpha}{k}\right)^\ell h_\ell((k+i\alpha)r) d\alpha \underset{r \rightarrow \infty}{\sim} h_\ell(kr) \underset{r \rightarrow \infty}{\sim} (-i)^{\ell+1} e^{ikr} \quad (21)$$

We rewrite Eq. (17) with $A = \mu$ for potentials like (8)

$$\left\{ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} \right\} F_\ell = -2\lambda \frac{e^{-\mu r}}{r} \left(F'_\ell + \frac{\ell}{r} F_\ell \right) + F_\ell \int_\mu^\infty D(\alpha) e^{-\alpha r} d\alpha \quad (22)$$

$$D(\alpha) = C(\alpha) + \lambda\mu\vartheta(\alpha-\mu) + \lambda^2 \left((\alpha-\mu)\vartheta(\alpha-\mu) - (\alpha-2\mu)\vartheta(\alpha-2\mu) \right) ; \quad |D(\alpha)| < C^{te}$$

We insert (21) into (22) and because of the presence of the derivative F'_ℓ we use the relation :

$$\left\{ \frac{d}{d\rho} + \frac{\ell}{\rho} \right\} h_\ell(\rho) = h_{\ell-1}(\rho) \quad .$$

We obtain for $x_{-k}(\alpha)$ an integral equation of the Volterra type

$$x_{-k}(\alpha) = \delta(\alpha) + \frac{1}{\alpha(\alpha-2ik)} \int_0^{\alpha-\mu} x_{-k}(\beta) \left[-2\lambda H_{\ell-1}^\ell(\alpha, \beta, \mu, k) + \int_\mu^{\alpha-\beta} D(\gamma) K_\ell^\ell(\alpha, \beta, \gamma, k) d\gamma \right] d\beta \quad (23)$$

where

$$h_\ell \left((k+i\beta)r \right) e^{-\gamma r} = \int_0^\infty K_\ell^\ell(\alpha, \beta, \gamma, k) \left(\frac{k+i\alpha}{k+i\beta} \right)^\ell h_\ell \left((k+i\alpha)r \right) d\alpha \quad (24)$$

$$h_{\ell-1} \left((k+i\beta)r \right) \frac{e^{-\mu r}}{r} = \int_0^\infty H_{\ell-1}^\ell(\alpha, \beta, \mu, k) \frac{(k+i\alpha)^\ell}{(k+i\beta)^{\ell+1}} h_\ell \left((k+i\alpha)r \right) d\alpha \quad (25)$$

Kernels like that of Eq. (24) have been calculated (7), (8), (9). We give in Appendix C the method to obtain the kernel of (25). It is easy to see that, because $K_\ell^\ell \equiv 0$ for $\alpha < \beta + \gamma$ and $H_{\ell-1}^\ell \equiv 0$ for $\alpha < \beta + \mu$, Eq. (22) gives the same iterative properties as for $\ell = 0$.

If we iterate Eq. (23), $x = \delta + r$, $\tau_{-k}(\alpha) = \sum_1^\infty \tau_n(\alpha, -k) \vartheta(\alpha - n\mu)$, we find

$$|\tau_n(\alpha, -k)| \leq \frac{A_n}{\alpha^2} \quad .$$

Here also the approximations of the Jost function are always convergent. If we can show that $|\tau_{-k}(\alpha)| < \frac{B}{\alpha^{1+\eta}}$ where $\eta > 0$ and B constant > 0 , finite, then one has :

$$F_\ell(-k, \mu) = \lim_{r \rightarrow 0} r^\ell F_\ell(-k, \mu, r) = \left[1 + \int_\mu^\infty \tau_{-k}(\alpha) d\alpha \right] \left(\frac{i}{2k} \right)^\ell \alpha_{\ell, \ell} \quad (26)$$

The proof of convergence of the integral in Eq. (26) is similar to the S case with restrictions on λ ($\lambda < \ell + 1/2$). This is given in Appendix C.

c) Bound states. Results obtained for $\ell = 0$ are valid for $\ell \neq 0$. If $F_\ell(-k, A) = 0$ for $\text{Im} k > 0$, with Eq. (16') for φ_ℓ and with the usual combination of (14) for φ_ℓ and φ_ℓ^* one can show that necessarily $\text{Re} k = 0$. Now consider

$$R_\ell(-ix, r) = Z_\ell(\mu, s_-, r) F_\ell(-ix, \mu, r) \underset{r \rightarrow \infty}{\sim} e^{-xr}$$

where Z_ℓ is the same phase factor as in (15') but with the change $\lambda \rightarrow s_-$. We obtain for F_ℓ Eq. (17) with $A = \mu$ and $\lambda = s_-$. Suppose that R_ℓ is regular near the origin, $R_\ell \underset{r \rightarrow 0}{\sim} C r^{s_+}$, then we find for the corresponding F_ℓ with some algebra :

$$-\int_0^\infty \left[r(F'^2 + x^2 F^2) + \frac{\ell(\ell+1)}{r} F^2 (1 - e^{-\mu r}) \right] dr = \int_0^\infty F^2 \left(s_-^2 \frac{e^{-\mu r}}{r} (1 - e^{-\mu r}) + rV \right) dr$$

Here also it is necessary that $V(r)$ changes sign somewhere in order to have bound states.

IV. CONCLUSIONS

We have seen that for "generalized Yukawa potentials", singular as r^{-2} at the origin, methods for obtaining Jost functions and wave functions by Laplace transforms are still possible. We have not used cut-off or hard-core. Note that for simplicity in the proofs we have taken in Eq. (8) only $|C(\alpha)| < C^{te}$. In fact the method presented here can be easily extended to the case $|C(\alpha)| < C^{te} \alpha^{1-\xi}$ or $V(r) \underset{r \rightarrow 0}{\sim} C^{te}/r^{2-\xi}$.

It is our hope that we can extend some of the methods presented here for the transition potentials r^{-2} , into more singular potentials repulsive with short range and also for the potentials $\underset{r \rightarrow 0}{\sim} r^{-4}$ in the Bethe-Salpeter equation.

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APPENDIX A

(1 = 0)

We want to show that $\int_{\mu}^{\infty} \tau(K, \alpha) d\alpha$ exists where $K \notin [iM_2, i\infty]$, $|K|$ finite and

$$\tau(K, \alpha) = \frac{1}{\alpha(\alpha + 2iK)} \left(N(\alpha) + \int_{\mu}^{\alpha - \mu} (2\lambda\beta + N(\alpha - \beta)) \tau(K, \beta) d\beta \right) \quad (A.1)$$

$N(\alpha) = D(\alpha) + 2\lambda iK \theta(\alpha - \mu)$; $D(\alpha)$ is given by (8) and (9)
 $|N(\alpha)| < N = c^{te}$.

1) Case of $|2\lambda| < 1$

a) Consider firstly $\text{Im}K \leq 0$, then $|\alpha + 2iK| > \alpha$. We define $\eta < 1$, small and positive such that $\left| \frac{2\lambda}{1 - \eta} \right| = \gamma < 1$.

Suppose that we have :

$$|\tau_n(K, \alpha)| < \frac{1}{\alpha^{1+\eta}} \gamma^{n-1} A_n \theta(\alpha - n\mu) \quad (A.2)$$

where A_n is a finite constant positive. Then we find with (A.1)

$$|\tau_{n+1}(K, \alpha)| < \frac{\gamma^n A_n}{\alpha^{1+\eta}} \left[1 + \frac{B - (n\mu)^{1-\eta}}{\alpha^{1-\eta}} \right] \theta(\alpha - (n+1)\mu) \quad (A.3)$$

where $B = \left(\frac{N}{\gamma} \right) \int_{n\mu}^{\infty} \frac{d\beta}{\beta^{1+\eta}}$ is a constant > 0 , decreasing with respect to n increasing. Then $\exists n_0$ such that $A_{n_0+1} \leq A_n$. Furthermore $\tau_1(\alpha)$

is of the kind of (A.2). Then

$$|\tau(K, \alpha)| \leq \sum_1^{\infty} |\tau_n(K, \alpha)| \leq \frac{A_p}{\alpha^{1+\eta}(1-\gamma)} \quad (\text{A.4})$$

where A_p is the greatest of A_n 's for $n = 1, \dots, n_0$.

b) Consider now $\text{Im}K > 0$. Iterating (A.1) we find always for n finite the form $|\tau_n| < \frac{A_n}{\alpha^{1+\eta}} \theta(\alpha - n\mu)$. Now we can choose m_0 and n such that for $\alpha \geq \frac{m_0}{\mu}$ one has $\left| \frac{\alpha}{\alpha + 2iK} \right| \left| \frac{2\lambda}{1-\eta} \right| < Y < 1$ where Y is finite. Then we can apply the previous arguments for $n \geq m_0$ and find for $\sum_{m_0}^{\infty} |\tau_n(\alpha, K)|$ a formula similar to (A.4).

2) Case of $2\lambda < 1$.

When 2λ is < 0 and $2\lambda < -1$, it is more tedious to show directly from the iterative solution of (A.1) an asymptotic behaviour of $\tau(K, \alpha)$ giving the existence of the Jost function. The difficulty arises because we are dealing with an alternating series with respect to $|2\lambda|$.

We put $\gamma_K(\alpha) = 1 + \int_{\mu}^{\alpha} \tau(K, \beta) d\beta$ and $F(K, A, 0) = \lim_{\alpha \rightarrow \infty} \gamma_K(\alpha)$.

We derive (A.1)

$$[\alpha(\alpha + 2iK)\gamma'(\alpha)]' - 2\lambda(\alpha - \mu)\gamma'(\alpha - \mu) = \frac{d}{d\alpha} \left[N(\alpha) + \int_{\mu}^{\alpha - \mu} N(\alpha - \beta)\gamma'(\beta) d\beta \right] \quad (\text{A.5})$$

$N(\alpha)$ is bounded. Assume for $\gamma_K(\alpha)$ the asymptotic behaviour $C_K \alpha^{-p}$.

Then we find either $p = 0$ or $p = 1 - 2\lambda = \sqrt{1 - 4G} > 0$.

Considering now the free term of A(3), we see that $\gamma_K(\alpha)$ is bounded when $\alpha \rightarrow \infty$.

A P P E N D I X B

$l \neq 0$ case with Laplace transform method [Section II, a)]

If we substitute Eq. (18) directly in Eq. (17) with $A = \mu$, then we obtain an integral equation for ρ

$$\alpha(\alpha + 2ik)\rho_K(\alpha) - l(l+1)\alpha - l(l+1)\int_0^\alpha (\alpha - \beta)\rho_K(\beta) d\beta = -2\lambda l(\alpha - \mu)\theta(\alpha - \mu) + D(\alpha) + \int_0^{\alpha - \mu} \left[2\lambda(\beta - l(\alpha - \beta - \mu)) + D(\alpha - \beta) \right] \rho_K(\beta) d\beta \quad (B.1)$$

where here

$$D(\alpha) = C(\alpha) + \theta(\alpha - \mu)(\lambda\mu + 2ik\lambda) + \lambda^2[\theta(\alpha - \mu)(\alpha - \mu) - \theta(\alpha - 2\mu)(\alpha - 2\mu)].$$

Because of the centrifugal term, (B.1) is not of the required form which gives the desired properties for the iteration ρ_n . A. Martin ⁴⁾ has given the method for obtaining these properties and the resulting integral equation is quoted in Eqs. (19) and (19'). We wish to show that

$$\rho_K(\alpha) \simeq c_K \alpha^{l-1} \quad (l \neq 0) \quad (B.2)$$

Assume for ρ an asymptotic behaviour $\rho_K(\alpha) \simeq c_K \alpha^p$ ($p > -1$) then (B.1) gives for the dominant term :

$$\frac{c_K \alpha^{p+2}}{(p+1)(p+2)} \left[(p+1)(p+2) - l(l+1) + 2\lambda(l-1-p) \right] = 0 \quad (B.3)$$

Thus either $p = l - 1$ or $p = 2\lambda - l - 2$ and the condition for (B.2) is $\lambda < (1 + \frac{1}{2})$ or $G < (1 + \frac{1}{2})^2$ always supposed in this paper.

APPENDIX C

1) Determination of the kernels H_{1-1}^1 , and K_{1-1}^1

We recall that ⁹⁾ the transform

$f(r) = \int_0^\infty K_\ell(\alpha, K) h_1((K + i\alpha)r) d\alpha$ with $\frac{\partial}{\partial r} [r^1 h_1(Kr)] = Kr^1 h_{1-1}(Kr)$ can be brought back to a Laplace transform.

$$\Delta_\ell(f(r)) = \int_0^\infty e^{-\alpha r} K_\ell(\alpha, K) (K + i\alpha)^\ell d\alpha \quad (C.1)$$

where $\Delta_{1-1}(f(r)) = ie^{-iKr} \left(\frac{1}{r} \frac{\partial}{\partial r}\right)^{1-1} (r^1 f)$. Thus for (24) and (25)

$\frac{(K + i\alpha)^{2\ell}}{(K + i\beta)^\ell} K_{1-1}^1$ is the inverse Laplace transform of $\Delta_{1-1}(h_\ell((K + i\beta)r) r^1 e^{-\beta r})$ and $\frac{(K + i\alpha)^{2\ell}}{(K + i\beta)^{\ell+1}} H_{\ell-1}^\ell$ is the one of $\Delta_\ell(h_{\ell-1}((K + i\beta)r) r^{\ell-1} e^{-\beta r})$.

We want now to show how to obtain H_{1-1}^1 . For any $F(r)$ one has ⁹⁾:

$$\left(\frac{1}{r} \frac{\partial}{\partial r}\right)^\ell (F(r)) = \frac{1}{r^{2\ell}} \sum_{p=1}^\ell b_{\ell,p} F^{(p)} \quad b_{\ell,p} = \left(-\frac{1}{2}\right)^{\ell-p} a_{\ell-1, \ell-p} \quad (C.2)$$

We consider (C.2) for $F = r^{1-1} e^{-\mu r} h_1((K + i\beta)r)$ and obtain for $\Delta_{1-1} F$ terms of the kind : $e^{-(\mu + \beta)r/r^n}$; taking the inverse Laplace transform we obtain :

$$\frac{(K+i\alpha)^l}{(K+i\beta)^{l+1}} \left[\frac{1}{l-1} \right] \left[\alpha, \beta, \mu, K \right] = \frac{(-i)^{l-1} \theta(\alpha - \beta - \mu)}{(K+i\alpha)^l} \left[\sum_{n=0}^{2l-2} \frac{(\alpha - \beta - \mu)^n}{n!} \right] \left[\sum_{p=1}^l b_{l,p} \left[\sum_{q=0}^{l-1} C_p^{S-q} (ik - \mu - \beta)^{S-q} \left(\frac{i}{2(K+i\beta)} \right)^{l-1-q} \frac{a_{l-1, l-1-q}(S-p)}{q!} \right] \right] \quad (C.3)$$

where $S = 2l - 1 - n$, $0 \leq S - q \leq p \leq l$, $a_{\nu, \nu'}$ and $b_{\nu, \nu'}$ are equal to 0 if $\nu' > \nu$.

For instance, for $l = 1$, $\mathcal{K}_1^1$ has been given ¹⁰⁾ and H_0^1 is :

$$\frac{(K+i\alpha)}{(K+i\beta)^2} H_0^1 = \frac{\theta(\alpha - \beta - \mu) i (K+i(\beta + \mu))}{(K+i\alpha)}$$

We have the following bounds for the kernels H_{l-1}^1 and $\mathcal{K}_l^l$

$$\left. \begin{aligned} & \left| \int_{\mu}^{\alpha - \beta} \mathcal{D}(\delta) \mathcal{K}_l^l d\delta \right| < \text{const.} \\ & H_{l-1}^l = \frac{-\beta^{(2l+1)}}{(\alpha - iK)^{2l}} + M(\alpha, \beta, \mu, K), \quad |M| < \text{const} \end{aligned} \right\} \quad (C.4)$$

2) Convergence of $1 + \int_{\mu}^{\infty} \tau_{-K}(\alpha) d\alpha$

Iterating once Eq.(23) and putting $x = \beta + \tau$, $\tau(\alpha) = \sum_{n=1}^{\infty} \tau_n(\alpha) \theta(\alpha - n\mu)$, with (C.4) for H_{l-1}^1 , we obtain for τ an integral equation :

$$\tau(\alpha) = \frac{1}{\alpha(\alpha - 2iK)} \left[\mathcal{L}(\alpha, 0, \mu, K) + \int_{\mu}^{\alpha - \mu} \left(\frac{2\lambda\beta^{2l+1}}{(\alpha - iK)^{2l}} + \mathcal{L}(\alpha, \beta, \mu, K) \right) \tau(\beta) d\beta \right] \quad (C.5)$$

where $\mathcal{L} = -2\lambda M + \int_{\mu}^{\alpha - \mu} \mathcal{K}_l^l d\delta$, $\tau_1 = \mathcal{L} \frac{(\alpha, 0, \mu, K)}{\alpha(\alpha - 2iK)}$ and $|\mathcal{L}| < L = \text{constant}$.

a) Case of $2\lambda < 2\ell + 1$

Consider $Y_{-K}(\alpha) = 1 + \int_{\mu}^{\alpha} \tau_{-K}(\beta) d\beta$. We want to show that $Y_{-K}(\alpha) \rightarrow \text{cte}$. From Eq. (C.5) we obtain for Y an integro-differential equation that we differentiate

$$\left[\gamma'(\alpha - 2iK)(\alpha - iK)^{2\ell} \right]^{-2\lambda(\alpha - \mu)^{2\ell+1}} y'(\alpha - \mu) = \frac{d}{d\alpha} \left[(\alpha - iK)^{2\ell} \left(\frac{\gamma}{\alpha} + \int_{\mu}^{\alpha} \frac{\gamma'}{\beta} d\beta \right) \right] \quad (\text{C.6})$$

Assuming that $Y \underset{\alpha \rightarrow \infty}{\sim} C_{-K} \alpha^{-p}$ and considering the homogeneous terms in (C.6) we find either $p = 0$ or $p = 2\ell + 1 - 2\lambda > 0$ if we consider only $\lambda < 1 + \frac{1}{2}$. Consider now the free term in (C.6) then $Y_{-K}(\alpha) \underset{\alpha \rightarrow \infty}{\rightarrow} \text{const.}$

b) Case of $|2\lambda| < 2\ell + 1$

As in the $l = 0$ case we can show the convergence of (26) by considering directly the iteratives of (C.5). Note that (26) gives $f_1(-K)$. For simplicity we consider only $\text{Im}K \gg 0$, $K \neq 0$ and $|K|$ finite such that $|\alpha - 2iK| > \alpha$ and $|\alpha - iK| > \alpha$.

Firstly we choose η_1 small and positive such that $\left| \frac{2\lambda}{2\ell+1} - \eta_1 \right| < 1$. If we iterate (C.5) we find :

$$|\tau_n| < \frac{A_n \theta(\alpha - n\mu)}{\alpha^{1+\eta_1}}, \quad \forall n \text{ finite } A_n \text{ is finite} \quad (\text{C.7})$$

and

$$|\tau_{n+1}| < \frac{A_n}{\alpha^{1+\eta_1}} \left[\left| \frac{2\lambda}{2\ell+1} - \eta_1 \right| + \frac{|L|}{(n\mu)^{1-\eta_1}} \int_{n\mu}^{\alpha} \frac{d\beta}{\beta^{1+\eta_1}} \right] \theta(\alpha - (n+1)\mu) \quad (\text{C.8})$$

Now we choose η_2 small and positive such that $Y = \left| \frac{2\lambda}{2\ell+1-\eta_1} \right| |1+\eta_2| < 1$.

Then for η_2 given, we can find m_0 such that $\forall_n \geq m_0$.

$$L \left| \frac{2\ell+1-\eta_1}{2\lambda} \right| \frac{1}{(n\mu)^{1-\eta_1}} \int_{n\mu}^{\infty} \frac{d\beta}{\beta^{1+\eta_1}} < \eta_2 \quad (C.9)$$

Thus
$$\sum_{m_0}^{\infty} |\tau_n| < \frac{A_{m_0}}{\alpha^{1+\eta_1}(1-\gamma)}$$

In the conclusion we have said that condition (8) : $|C(\alpha)| < \text{Const}$ can be extended to $|C(\alpha)| < c^{te} \alpha^{1-\xi}$, ($\xi > 0$). We wish to show that here. Now $|C| < L \alpha^{1-\xi}$. We impose a supplementary condition for η_1 which is $\eta_1 \leq \xi$. We have always (C.7) and in (C.8) the only difference is a slightly different term

$$\frac{L}{(n\mu)^{\xi-\eta_1}} \int_{n\mu}^{\infty} \frac{d\beta}{\beta^{1+\eta_1}}$$

Then the proof is the same as previously with the corresponding change in (C.9)

$$(n\mu)^{1-\eta_1} \rightarrow (n\mu)^{\xi-\eta_1}$$

The result can be extended, as in Appendix A, for $\text{Im } K < 0$, K outside the cut $-i[\mu_2, \infty]$, $K \neq 0$ and $|K|$ finite.

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