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SINGULAR POTENTIALS WITH SHORT RANGE^{*)}

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ABSTRACT

Previous works have shown the properties and the possibility of useful approximate methods for the Jost functions, and the radial wave functions in the case of Yukawa generalized potentials, less or equally singular than the centrifugal term. Here we extend some of these results to a larger class of short-range potentials which, near the origin, can be represented as a finite superposition of inverse powers of the radial variable; namely, the most singular term is repulsive with an arbitrary inverse-power behaviour. For simplicity, the degrees of the other terms are obtained from one of the leading terms by the decreasing of unities. The key (as for the inverse power-square case) is to find a singular factor removing the singular part near the origin of the outgoing (ingoing) waves, and keeping unaltered the asymptotic character of the outgoing (ingoing) waves. After this, the analysis follows, as closely as possible, the usual development. By investigating the asymptotic behaviour of the Laplace transform, the existence of the Jost function is proved. The definition of the Jost function as a series, whose p -th term comes from the contribution of the iteration of order p of the inverse Laplace transform, is divergent. A method is presented for evaluating the Jost function by means of convergent sequences, which allow the calculation of approximations which are always convergent without any arbitrary cut-off or hard-core procedures. All the above formalism is given for the $\ell = 0$ case.

I. INTRODUCTION

Our aim in this paper is to determine the Jost functions and the S matrix for a class of short-range potentials, singular and repulsive near the origin, of the form

$$U = V_1 + V_2 \quad (1.1)$$

$$V_1 = e^{-\mu r} \left[\frac{G^2}{r^{2n}} - \sum_{2n-1}^{N_0} \frac{A_p}{r^p} \right] \quad (1.2)$$

where $G^2 > 0$ and n arbitrary > 1 . For integer n , $N_0 = 2$. For non-integer n , N_0 is the smallest number deduced from $2n$ by subtraction of successive integers, such that $2 < N_0 < 3$.

$$V_2 = \int_{\mu}^{\infty} e^{-\alpha r} C(\alpha) d\alpha \quad (|C(\alpha)| < \alpha^{1-\eta}, \eta > 0) \quad (1.3)$$

We want to develop a general formalism for this problem without introducing at any stage any cut-off procedure.

As is well known, when the potential is more singular than $1/r^2$, the origin is an irregular singular point. Accordingly, the partial-wave Schrödinger equation is not of the Fuch's type and the corresponding solutions do not have a power-like behaviour at the origin, but rather essential singularities.

If $G^2 < 0$, it is well known also that it is not so clear in which way one can distinguish between the "regular" and the "irregular" solutions¹⁾. In the following, we restrict ourselves to the case $G^2 > 0$, so that we may unambiguously distinguish between the "regular" and the "irregular" solutions.

The Schrödinger equation for singular potentials of short range has been investigated previously²⁾ in order to obtain analytic properties of the Jost functions with respect to the energy and to the angular momentum.

Recently, several explorations have been made in the domain of unrenormalizable field theory³⁾. In order to legitimate certain cut-off procedures, new interest has been devoted to the study of repulsive singular potentials, because some of the difficulties of convergence encountered in the two fields seem to be similar⁴⁾. However, in this scheme the main problem was not the determination of the Jost function independently of the S matrix.

In order to solve the problem of the scattering, and in particular of the bound states, it is very useful to work with Jost functions which are finite at any stage. First one introduces the solutions of the Schrödinger equation, which asymptotically are ingoing (outgoing) waves. The Jost functions can then be considered as the contribution of the singular part of these solutions when $r \rightarrow 0$.

Now we consider the case $\ell = 0$. In this case, if one considers potentials less singular than $1/r^2$, the "irregular" solution at the origin is a constant and it is easy to extract the contribution of the singular part. On the contrary, in the case of potentials more or equally singular than $1/r^2$, the singularities of the potential determine the behaviour near the origin of both the "regular" and the "irregular" solutions. Therefore the potential cannot be considered as a perturbation near the origin.

In order to be able to reduce to the very comfortable case found in the problem of potentials less singular than $1/r^2$, we shall make a change of function by factorizing a singular factor. This factor (k independent) will be subjected to two conditions. Firstly, it must incorporate the most divergent part of the solution near the origin; secondly it must preserve the character of the solutions of being asymptotically ingoing (outgoing) waves. After this procedure is accomplished, we are reduced to having "new" functions which, while still being asymptotically ingoing (outgoing) waves, go to a constant for $r \rightarrow 0$.

The advantage of this procedure has been illustrated previously in the $1/r^2$ case⁵⁾.

We recall that to construct explicitly the Jost functions, previous works⁶⁾ have shown that for Yukawian potentials, very useful methods can be developed through the use of Laplace transform yielding to very peculiar and convenient iterative properties.

In the case of potentials more or equally singular than $1/r^2$, the singular factor must be chosen in such a way as to still give the same peculiar iterative properties. This has already been done for the $1/r^2$ case⁵⁾.

In a short paper published recently⁷⁾, we have presented some preliminary results of the study of potentials, such as the family (1.1) for peculiar cases $n = 3/2$ and $n = 2$. The complete proofs for these cases will be given here, as well as the general case (1.1). In the present paper we shall always limit ourselves to the $\ell = 0$ case.

In section II the general formalism will be developed in the radial co-ordinate variable. For the given family of potentials (1.1), we give the method of constructing explicitly the above-mentioned singular factor, thus we are able to write the S matrix as the ratio of two quantities, both of them being finite. Furthermore, the "regular" solution is written in terms of the "new" functions (asymptotically ingoing or outgoing waves, finite at the origin) in a form similar to the usual one; the only change being a factor whose r dependence is the singular factor. The bound state problem then takes a form similar to the usual one.

In section III we study the properties of the "new" functions by means of Laplace transform and we show that the iterative properties previously recalled⁶⁾ are still valid.

The presence of essential singularities at $r = 0$, reflects into essential singularities at infinity in the conjugate variable through the Laplace transform. One has then to be very careful in order to investigate the asymptotic behaviour of the inverse Laplace transform.

The Jost function is defined as the limit of the Laplace transform when $r \rightarrow 0$ (r being real). The proof of the existence of this limit is given in section IV, by using the so-called "saddle point" or "col" method.

Due to the fact that in the present case the Jost function cannot be represented by an infinite sum of terms corresponding to the iteration of the inverse Laplace transform, we must consider other methods for explicitly constructing the Jost function.

In section V it is proved that the Jost functions can be defined as the limits of convergent sequences, taking great advantage of the iterative properties of the solutions, in order to obtain approximate results. By connecting two limiting procedures it is shown that, approximating the Jost function by a term of the sequence of sufficiently high order, we can also prove that what one neglects becomes smaller and smaller.

It should be mentioned that the present approach can, in principle, be extended to more general classes of singular interactions whose dominant term at the origin is repulsive.

Here, we do not consider analytical properties although the method given could be used for this aim. For instance, it could be very interesting to study the convergence with respect to the different couplings of the potential. Another possible extension is the study of the $\ell \neq 0$ case and of the Bethe-Salpeter problem in the ladder approximation.

II. DEFINITION OF THE S MATRIX AND OF THE JOST FUNCTIONS

We start from the S wave Schrödinger equation:

$$\left(\frac{d^2}{dr^2} + k^2 \right) \Phi = U \Phi \quad (2.1)$$

where $U = V_1 + V_2$. V_1 contains the most singular part of the potential chosen of the form (1.2)

$$V_1 = e^{-\mu r} \left[\frac{G^2}{r^{2n}} - \sum_{2n-1}^{N_0} \frac{A_p}{r^p} \right] \quad (2.2)$$

while V_2 is, as usual, less singular at the origin than r^{-2} and is exponentially decreasing at large distances. As is clear from (2.2), Eq. (2.1) is not of the Fuchs' type near the origin and the corresponding solutions have not a power-like behaviour. It is well known^{s)} that in the neighbourhood of singular irregular points (like here is the origin) the solutions have essential singularities. It is also well known¹⁾ that for the case $G^2 < 0$ it is not so clear how to distinguish between the "regular" solution and the "irregular" one. This will, however, appear clear when dealing with the explicit behaviour of the solutions of (2.1) near the origin. In the following we shall limit ourselves to the case $G^2 > 0$.

We consider the solutions $R(\pm k, r)$ of (2.1) such that

$$R(k, r) = Z(A_p, \mu, r) F(k, r) \underset{r \rightarrow \infty}{\sim} e^{-ikr}. \quad (2.3)$$

As in Ref. 5, Z is a singular k independent factor such that the following conditions hold

$$a) \quad Z(r) \underset{r \rightarrow \infty}{\rightarrow} 1 \quad \text{or} \quad F(k, r) \underset{r \rightarrow \infty}{\rightarrow} e^{-ikr}$$

b) for $r \rightarrow 0$, $Z(r)$ must behave as the singular part of

$$R(k, r) \quad \text{or} \quad F(k, r) \underset{r \rightarrow 0}{\rightarrow} \text{const.}$$

6.

We put formally

$$Z(r) = e^{\int_r^\infty \psi(r') dr'}$$

Equation (2.1) becomes

$$\left(\frac{d^2}{dr^2} + 2\psi \frac{d}{dr} + k^2 \right) F = WF \quad (W = U - \psi' - \psi^2). \quad (2.4)$$

We choose

$$\psi = e^{-Ar} \left(\sum_{q=n}^{n_0} \frac{a_q}{r^q} + \frac{a_1}{r} \right) \quad (2.5)$$

where A is a positive constant which at this stage is still arbitrary. For integer n , $n_0 = 2$. For non-integer n , n_0 is the smallest number deduced from n by subtraction of successive integers, such that $1 < n_0 < 2$.

We notice that because of (2.5), condition a) is automatically satisfied. At this stage it is also useful to take a particular choice of A , for instance $A = \frac{1}{2}$ but in fact other choices for A are possible.

To satisfy condition b) we make use of the arbitrariness of the parameters of (2.5) in order to remove the most singular part in W , in such a way that

$$W = U - \psi' - \psi^2 \sim 0(r^{-m}) \quad r \rightarrow 0$$

where $m = n$ for integer n and $m = n + n_0 - 1$ for non-integer n .

We write

$$W(r) = \left[e^{-\mu r} \left\{ \frac{G^2}{r^{2n}} - \sum_{p=2n-1}^{m+1} \frac{A_p}{r^p} - \frac{a_n^2}{r^{2n}} - \sum_{p=n-1}^{n_0} \frac{2a_n a_p + B_p}{r^{n+p}} - \frac{2a_n a_1 + B_1}{r^{n+1}} \right\} + e^{-\frac{\mu r}{2}} \frac{na_n}{r^{n+1}} \right] + 0(r^{-m}) \quad (2.6)$$

where B_p are given by linear combinations of products of two a_q 's ($q \neq p, n$). For instance

$$\left\{ \begin{array}{ll} B_{n-1} & = 0 \\ B_{n-2} & = a_{n-1}^2 \\ B_{n-3} & = 2a_{n-1} a_{n-2} \\ \cdot & \\ \cdot & \\ \cdot & \\ B_1 & = \begin{cases} 2a_2 a_{n-1} + 2a_3 a_{n-2} + \dots & (\text{integer } n) \\ 0 & (\text{non-integer } n) \end{cases} \end{array} \right.$$

We now choose the parameters in such a way that the bracket at the r.h.s. of (2.6) be $O(r^{-m})$.

$$\left\{ \begin{array}{ll} a_n & = \epsilon G \quad (\epsilon = \pm 1, G > 0) \\ a_{n-1} & = -\epsilon \frac{A_{2n-1}}{2G} \\ a_{n-2} & = -\epsilon \frac{1}{2G} \left(A_{2n-2} + \frac{A_{2n-1}^2}{4G^2} \right) \\ a_{n-p} & = \dots \dots \dots \\ a_1 & = \frac{n}{2} - \epsilon \frac{B_1}{2G} \end{array} \right. \quad (2.7)$$

B_p being independent of ϵ , all the a_p ($p \neq 1$) are proportional to ϵ .

We now consider $Z_\epsilon(r)$ near the origin

$$Z_\epsilon(r) \underset{r \rightarrow 0}{\sim} Y_\epsilon(r) X_\epsilon(A_p, \mu)$$

where $X_\epsilon(A_p, \mu)$ is a constant (see for instance Table 1) and

$$Y_\epsilon(r) \sim \exp \left[-\epsilon \left(\sum_{n=1}^{n_0-1} \frac{b_n}{r^n} + b_1 \ln r \right) + \frac{n}{2} \ln r \right] \quad (2.8)$$

with

$$b_{n-1} = \frac{G}{n-1}, \quad b_{n-2} = -\frac{1}{2} \left(\frac{A_{2n-1}}{G} + \mu G \right) \frac{1}{n-2}, \quad \dots$$

b_p and b_1 are independent of ϵ and $b_1 = 0$ for non-integer n .

With these choices of $\epsilon = \mp 1$, the corresponding $Y_\epsilon(r)$ behave near the origin like the "singular" and "regular" part, respectively, of $R(\pm k, r)$,

$$R(k, r) \underset{r \rightarrow 0}{\sim} C_1 Y_{-1}(r) + C_2 Y_1(r).$$

This can be seen directly from (2.1) by investigating the behaviour of R near the origin with an expansion like

$$R \sim \exp \left[\sum_i \frac{\beta_i}{r^{\gamma_i}} \right] \cdot r^\lambda \quad (\gamma_i > 0).$$

Equivalently one can verify that with the choice $\epsilon = -1$ an analogous expansion near the origin can be given for $F(k, r)$, (see Appendix). In fact we have:

$$F(k, r) \sim F(k, 0) + \frac{Y_{+1}(r)}{Y_{-1}(r)} \text{ const.} \quad (\text{see Table 1}). \quad (2.9)$$

In order to satisfy condition b), in the following we consider $Z(r) = Z_{\epsilon=-1}(r)$ in equation (2.3) linking R and F .

We also define a regular solution $\varphi(k, r)$ of (2.1) by

$$\varphi(k, r) \underset{r \rightarrow 0}{\sim} Y_{\epsilon=+1}(r)$$

and the Jost function $f(k)$

$$\begin{aligned} f(k) &= \lim_{r \rightarrow 0} \left[R(k, r) / Y_{\epsilon=-1}(r) \right] = X_{\epsilon=-1}(A_p, \mu) F(k, 0) = \\ &= \frac{1}{2G} W(R, \varphi) \end{aligned} \quad (2.10)$$

W being the Wronskian.

Writing, as usual, φ as a linear combination of $R(\pm k, r)$ we obtain from (2.3) and (2.10)

$$\varphi(k, r) = \frac{G}{ik} X_{\epsilon=-1}(A_p, \mu) Z_{\epsilon=-1}(r) [F(k, r) F(-k, 0) - F(-k, r) F(k, 0)] \quad (2.11)$$

and the S matrix is defined by

$$S(k) = \frac{F(k, 0)}{F(-k, 0)} = \lim_{r \rightarrow 0} \frac{R(k, r)}{R(-k, r)} \quad (2.12)$$

The last equality in (2.12) coincides with the result found in Ref. 9 for a larger class of singular repulsive potentials.

Being $X_{-1}(A_p, \mu)$ a k independent factor (see for instance Table 1), from (2.10) it is seen that the energy dependent roots of $f(-k) = 0$ are the same for $F(-k, 0) = 0$. In the following one shall refer $F(\pm k, 0)$ as to the Jost functions.

Consider $F(-k, 0) = 0$ with $\text{Im} k > 0$; then (2.11) states that $\varphi(k, r) \rightarrow 0$, both for $r \rightarrow 0$ and $r \rightarrow \infty$. If we make, as usual, the linear combination of (2.1) for φ and φ^* we find

$$(k^2 - k^{*2}) \int_0^\infty |\varphi|^2 dr = 0 \quad .$$

Then, necessarily $\text{Re } k = 0$ which is the usual result for the bound states problem $F(-k, 0) = 0$.

We would like to notice explicitly that the choice (2.5), of the singular factor Z , has another very useful property; namely, after the most singular part of V_1 has been removed, the remaining terms of $(V_1 - \psi' - \psi^2)$ have still an exponential decreasing factor $e^{-(\mu/2)r}$. This fact will be of the utmost importance in the next sections, where we are going to assume that the "regular" part of the interaction $V_2(r)$ is a generalized Yukawa potential. Indeed it will still allow us to obtain, for the Laplace transform of the solution of (2.4), the particular iterative properties previously obtained for the short-range potentials such that $r^2 V(r) < \text{const}^{5,6}$).

$r \rightarrow 0$

III. USE OF THE LAPLACE TRANSFORM FOR YUKAWA GENERALIZED POTENTIALS

For $V_1(r)$ we still consider the family of singular potentials (2.2) and for $V_2(r)$ we assume the form

$$V_2(r) = \int_{\mu}^{\infty} C(\alpha) e^{-\alpha r} d\alpha \quad |C(\alpha)| < M\alpha^{1-\eta} \quad (\eta > 0). \quad (3.1)$$

We put for $F(k, r)$

$$F(k, r) = \int_0^{\infty} e^{-(ik+\alpha)r} \rho(\alpha, k) d\alpha. \quad (3.2)$$

Inserting (3.1, 2) into (2.4) we obtain for the inverse Laplace transform ρ

$$\begin{aligned} \rho(\alpha, k) = & \delta(\alpha) + \frac{1}{\alpha(\alpha + 2ik)} \int_0^{\alpha - \mu/2} \left\{ (\beta + ik) \cdot \left[-\frac{2G(\alpha - \beta - \mu/2)^{n-1}}{\Gamma(n)} + \right. \right. \\ & + \frac{A}{G} \frac{(\alpha - \beta - \mu/2)^{n-2}}{\Gamma(n-1)} + \frac{1}{G} \left(B + \frac{A^2}{4G^2} \right) \frac{(\alpha - \beta - \mu/2)^{n-3}}{\Gamma(n-2)} + \dots + n \left. \right] + \\ & \left. + \bar{W}(\alpha - \beta) \right\} \rho(\beta) d\beta \\ & (\bar{A}_{2n-1} = A, \bar{A}_{2n-2} = B, \dots) \end{aligned} \quad (3.3)$$

where $\bar{W}(\alpha)$ is the inverse Laplace transform of $W(r)$ (2.6), and $|\bar{W}(\alpha)| < \text{const } \alpha^{m-1}$.

First we observe that (3.3) is an integral equation of the Volterra type such that for any $G \neq 0$, the solution ρ of (3.3) exists. It is clear from (3.3) that we still have the useful iterative properties for $\rho(\alpha, k)$ encountered in the previous cases^{6, 5}). Since the iteration of order q is 0 for $\alpha < q\mu/2$, it is sufficient to consider the approximation

of order $q-1$ for ρ in order to have the exact solution of ρ in the range $\alpha < q\mu/2$. Moreover, for the same approximation of order q , it is also sufficient to know the inverse Laplace transform of the potential for $\alpha < q\mu/2$.

We write

$$\rho(\alpha, k) = \delta(\alpha) + \sum_{q=1}^{\infty} \rho_q(\alpha, k) \vartheta(\alpha - q\mu/2) \quad (3.4)$$

where

$$\rho_q(\alpha, k) = (-G)^q \alpha^{q(n-1)+(m-n-2)} \tau_q(\alpha, k, G) \quad (3.5)$$

with

$$|\tau_q(\alpha, k, G)| < \text{const} \quad \text{for all } G \neq 0, \quad \alpha \rightarrow \infty$$

and where m has been previously defined in section II.

Because of (2.10), (2.12) and (3.2) we have for the Jost function

$$F(k, 0) = 1 + \lim_{r \rightarrow 0} \int_{\mu/2}^{\infty} e^{-\alpha r} \rho(\alpha, k) d\alpha. \quad (3.6)$$

Now our aim is to prove that (3.6) makes sense.

First of all we notice that by inserting (3.4) into (3.6) and taking into account (3.5), each contribution of order q to the Jost function

$$\lim_{r \rightarrow 0} \int_{\mu/2}^{\infty} e^{-\alpha r} \rho_q(\alpha, k) d\alpha$$

is divergent as $r^{-q(n-1)-(m-n-1)}$. In order to prove the existence of the Jost function, as defined by (3.6), it is not sufficient to consider only the convergence of the single terms of (3.4), but we have to investigate the asymptotic behaviour of the whole function $\rho(\alpha, k)$.

It is easily seen that if we try an asymptotic behaviour $\rho(\alpha) \simeq C_k \alpha^C$, from Eq. (3.3) and by using ¹⁰⁾

$$\frac{1}{\Gamma(q)} \int_0^X y^{p-1} (X-y)^{q-1} dy = \frac{\Gamma(p)}{\Gamma(p+q)} X^{p+q-1} \quad (3.7)$$

and, if we compare the highest power at both sides, we find

$$C_k \alpha^{C+2} \sim C'_k \alpha^{C+n+1}.$$

That means for $n > 1$, a power behaviour for ρ when $\alpha \rightarrow \infty$ is not allowed, which is similar to what happened for its Laplace transform $F(k,r)$ when $r \rightarrow 0$. According to the behaviour (2.9) for $F(k,r)$, we try for ρ

$$\rho(\alpha, k) \underset{\alpha \rightarrow \infty}{\simeq} C_k e^{\sum_i D_i \alpha^{E_i}} \alpha^C = C_k \sum_{q_i} \left(\prod_i \frac{D_i^{q_i}}{q_i!} \right) \alpha^{\sum_i q_i E_i + C} \quad (3.8)$$

where $E_1 > E_2 > \dots > 0$.

Now we insert (3.8) into (3.3) and we consider only the highest contribution of ρ , with respect to α . We can note that for integer n it is easy to introduce the exponential form of the Ansatz (3.8) chosen for ρ in order to deduce the parameter E_1 because in this case it is straightforward to derivate or integrate both sides of (3.3). However, for non-integer n the r.h.s. of (3.3) appears as the non-integer primitive of $\alpha \rho$ and ρ , and it is more convenient to use the second form $\sum_{q_i}$ in (3.8). Our aim then is finally to obtain the asymptotic behaviour of non-integer primitives of exponentials. With the use of (3.7) we find

$$\begin{aligned}
& \sum_{q_i} \left(\prod_i \frac{D_i^{q_i}}{q_i!} \right) \alpha^{\sum_i q_i E_i + C + 2} \left[-1 + (-2G) \alpha^{n-1} \frac{\Gamma(\sum_i q_i E_i + C + 2)}{\Gamma(\sum_i q_i E_i + C + 2 + n)} \right. \\
& + \left(\frac{A}{G} + \mu G \right) \alpha^{n-2} \frac{\Gamma(\sum_i q_i E_i + C + 2)}{\Gamma(\sum_i q_i E_i + C + 1 + n)} + \left\{ \frac{1}{G} \left(B + \frac{A^2}{4G^2} \right) \frac{A\mu}{2G} - \frac{G\mu^2}{4} \right\} \alpha^{n-1} \frac{\Gamma(\sum_i q_i E_i + C + 2)}{\Gamma(\sum_i q_i E_i + C + n)} \\
& \left. + \dots + \frac{n}{\sum_i q_i E_i + C + 2} \right] \sim 0 . \quad (3.9)
\end{aligned}$$

(3.9) must be considered as an asymptotic form for α and q_i going to infinity. However, because $E_1 > E_2 \dots$, we can suppose that $E_1 q_1$ is much larger than $E_2 q_2$ and so on. In fact, it is possible to obtain a correspondence between q_i and $D_i \alpha^{E_i}$ when both go to infinity, and with the use of the Stirling¹⁰⁾ formula

$$\frac{\Gamma(X+a)}{\Gamma(X+b)} \underset{X \rightarrow \infty}{\sim} X^{a-b} \left[1 + (a-b)(a+b+1) X^{-2} + O(X^{-2}) \right] \quad (3.10)$$

we find

$$\sum_{q_i} \frac{D_i^{q_i} \alpha^{q_i E_i}}{\Gamma(q_i + 1)} q_i^z \simeq \sum_{q_i} \frac{D_i^{q_i} \alpha^{q_i E_i}}{\Gamma(q_i + 1)} D_i^z \alpha^{z E_i} \left(1 + z \frac{(z-1)}{2q_i} \right) \quad (3.11)$$

(z arbitrary) .

With the correspondence (3.11) it is now possible to compare the different terms of the bracket in (3.9), in such a way that this bracket is identically zero. To be more precise, the bracket in (3.9) can be considered as an expansion in decreasing powers of α with coefficients independent of q_i and α . We note that the highest contribution of each of its terms is larger than the one coming from the next term. In fact, the first term and the highest contribution of the second one in the bracket of (3.9) are constants. The highest contribution of the third term is of the

order $\alpha^{-1/n}$, while for the fourth we would have $\alpha^{-2/n}$ and so on until the last, whose highest contribution is of the order $\alpha^{-1+1/n}$.

So far we have neglected terms in the bracket (3.9), of the order

$$\alpha^{n-2} \frac{(-2Gik) \Gamma(\sum_i q_i E_i + C + 1)}{\Gamma(\sum_i q_i E_i + C + 1 + n)} \quad (3.9a)$$

$$\text{const } \alpha^{m-2} \frac{\Gamma(\sum_i q_i E_i + C + 1)}{\Gamma(\sum_i q_i E_i + C + 1 + n)} \quad (3.9b)$$

where the second one is the leading term coming from $\bar{W}(\alpha)$. However, these terms (3.9a, 9b) can be neglected because their highest contributions behave as α^{-1} and $\alpha^{-1-(1-m/n)}$ respectively which are always dominated by the highest contribution of the last term in the bracket (3.9), i.e., by $\alpha^{-1+1/n}$. The requirement that the bracket (3.9) be identically zero implies therefore that all the coefficients of the decreasing powers of α be zero.

Firstly, we consider the first term and the greatest contribution of the second one, and we find

$$\begin{cases} 1 - 2G(E_1 D_1)^{-n} \alpha^{n-1-nE_1} \simeq 0 \\ E_1 = 1 - 1/n, (E_1 D_1)^n = -2G. \end{cases} \quad (3.12)$$

The necessity of the exponential Ansatz (3.8) found in the case $n > 1$, reflects here in that E_1 is positive if $n > 1$.

Secondly, we consider the first iterate of the second term, and the greatest contribution of the third one by using (3.10) and (3.11). For the first iterate of the second term with $E_2 \neq 0$ we have

$$\frac{n}{2q_1 E_1^2} \left[E_1 (n+1) - \left(2 \sum_{i \neq j} q_j E_j + 2C + 3 + n \right) \right] \quad (3.13)$$

Replacing the q_1, q_2 dependence in (3.13) with the correspondence formula (3.11) we have

$$- \frac{nE_2}{E_1} \frac{q_2}{q_1} \sim -n \frac{E_2 D_2}{E_1 D_1} \alpha^{E_2 - E_1} . \quad (3.13a)$$

For the highest contribution of the third term we have

$$\left(\frac{A}{G} + \mu G \right) \frac{\alpha^{-1/n}}{(E_1 D_1)^{n-1}} . \quad (3.14)$$

The sum of these two terms must be zero, then

$$\begin{cases} E_2 = 1 - 2/n \\ E_2 D_2 = 1/n \frac{\left(\frac{A}{G} + \mu G \right)}{(E_1 D_1)^{n-2}} \end{cases} . \quad (3.15)$$

The condition $E_2 > 0$ now implies $n > 2$.

We now consider as examples the particular cases $n = 3/2$ and $n = 2$ for which $E_2 = 0$. From (3.13) we now have

$$- \frac{2Cn + 3n + 1}{2E_1 D_1} \alpha^{-E_1} . \quad (3.13b)$$

Moreover, we consider the last term of the bracket in (3.9) and we have

$$\frac{n}{q_1 E_1} \sim \frac{n}{E_1 D_1} \alpha^{-E_1} . \quad (3.16)$$

For $n = 3/2$, since $E_1 = 1/3$, the main contribution comes from (3.13b) and (3.16) and by comparison we have the value of C in (3.8)

$$C = - \frac{5}{6} .$$

For $n = 2$, the terms in (3.13a, 14, 16) being of the same order of magnitude, we find

$$2C + \frac{3}{2} = \frac{A}{G} + \mu G. \quad (3.17)$$

If we now suppose $E_3 \neq 0$, we have to take into account the third iteration of the second term in (3.9), the second of the third one, and the first of the fourth. This would give us $E_3 = 1 - \frac{3}{n}$ and D_3 . Once again we notice that $E_3 > 0$ implies $n > 3$. The same procedure can be repeated for all n 's in order to obtain all the parameters E_i, D_i and C . For instance, for $n = 3$,

$$3C + 2 = \frac{B}{G} + \frac{A^2}{4G^3} - \frac{G\mu^2}{4} - \frac{A\mu}{2G}.$$

IV. EXISTENCE OF THE JOST FUNCTION WITH THE USE OF THE LAPLACE TRANSFORM FOR V_2 YUKAWA GENERALIZED POTENTIALS

In this section we want to prove that the Jost function, as defined by (3.6) exists. To this aim we shall make use of the results of Chapter III concerning the asymptotic behaviour of ρ and of mathematical techniques to estimate the limit $r \rightarrow 0$ in (3.6). We shall prove that this limit is a finite constant.

Because of (3.6) and (3.8) this amounts to proving

$$I = \lim_{r \rightarrow 0} \int_{\mu/2}^{\infty} e^{-\alpha r} e^{\sum D_i \alpha^{E_i}} \alpha^C d\alpha < \infty \quad (4.1)$$

where the parameters E_i, D_i, C are as previously defined, for instance $E_i = 1 - i/n$.

Note that $D_i = [n/(n-1)] (2G)^{1/n} \eta = \lambda \eta$ where $\eta^n = -1$ and $\eta_j = e^{i\frac{\pi}{n}(1+2j)}$. For $\text{Re } \eta_j < 0$, (4.1) exists for $r = 0$ without the necessity of any limiting procedure.

We limit ourselves to $\text{Re } \eta_j \geq 0$. Here we consider explicitly the roots lying in the first quadrant. The formalism to be developed will also be immediately extended to the ones lying in the fourth quadrant. Thus we consider (4.1) by setting $\alpha = (t/r)^n$ and (4.1) becomes

$$I = \lim_{r \rightarrow 0} \frac{n}{r^{n(C+1)}} \int_{(\mu/2)^{1/n} \cdot r}^{\infty} e^{Yf(t,r)} t^{nC+n-1} dt \quad (4.2)$$

$$Y = r^{-n+1}; \quad f(t,r) = -t^n + \sum_{i=1}^{n-1} D_i t^{n-i} r^{i-1}. \quad (4.3)$$

We have to investigate the asymptotic behaviour of the integral in (4.2) when $Y \rightarrow \infty$. To this aim we shall use standard methods for obtaining asymptotic estimates of integrals with a large parameter, namely the so-called "saddle point" or "col" method⁽¹⁾. This roughly consists of

deforming the integration path in (4.2), in such a way that we go from a stationary point (saddle point) of $f(t,r)$ to another one along a "steepest path" on which $\text{Im } f(t,r) = \text{constant}$. We notice that in order to evaluate (4.2) in the complex t plane, we must, in general, put a cut starting from the origin. Along the steepest descent paths, we are led to the "Laplace method". This method consists in evaluating integrals over real intervals, where the integrand depends on a parameter Y and it has a stationary point (inside or at an end point of the interval) where it becomes more pronounced as Y becomes large. Since the method depends essentially on the choice of the path and no satisfactory general criteria to find it are known, to show with a simple example how it works, we first of all consider the case $n = 2$.

A) In this case ($n = 2$), $f(t) = -t^2 + 2i\sqrt{2G}t$ and the saddle point ($f'(t_0) = 0$) is $t_0 = i\sqrt{2G}$ with $f(t_0) = -2G$. Now if t is complex, $t = x + iy$, there are two steepest paths ($\text{Im } f(t) = 0$) crossing in t_0 : $y = \sqrt{2G}$ and $x = 0$. (4.2) becomes

$$I = \lim_{r \rightarrow 0} \frac{e^{-2G/r}}{r^{2C+2}} \int_{(\mu/2)^{1/2} r}^{\infty} e^{-Y(t-t_0)^2} t^{2C+1} dt. \quad (4.4)$$

We now deform the integration path in (4.2,4) as in Fig. 1. First of all because of the presence of the power t^{2C+1} we put a cut outside the sector $(\vec{Ox}, \vec{OD})$. Thus, in this case ($D = 2i\sqrt{2G}$), the cut is outside the first quadrant and we take the branch of t^{2C+1} real for t on the real axis.

Γ_1 is parallel to the y axis and has $\text{Re}(t)$ going to infinity. Γ_2 is the steepest path $y = \sqrt{2G}$ and goes from infinity to the saddle point $t = i\sqrt{2G}$. Γ_3 and Γ_4 are paths depending on small r . In order to calculate the corresponding contributions of I we fixed r small and after the integrations are made we must take the limit $r \rightarrow 0$. Γ_3 is another steepest path on the y axis and goes from the saddle point $t_0^{(1)}$ to $i(\mu/2)^{1/2} r$. Γ_4 is a straight line which goes from the imaginary axis at $i(y\mu/2)^{1/2} r$ to the real axis at $(\mu/2)^{1/2} r$.

It is then necessary to investigate the contribution of (4.2) along the different paths Γ_i ($I = \sum_{i=1}^4 I_i$).

- 1) The contribution over Γ_1 is manifestly zero as is seen from (4.4).
- 2) On this path the end point t_0 is a maximum so we can apply the Laplace method.

We put $t - t_0 = v r^{1/2}$ and we have

$$I_2 = \lim_{r \rightarrow 0} \frac{2 e^{-2G/r}}{r^{2C+3/2}} \int_0^{\infty} e^{-v^2} (t_0 + r^{1/2} v)^{2C+1} dv.$$

We note firstly that the integral is bounded for any C and secondly that with the value of C given by (3.17) we obtain for I_2 when $r \rightarrow 0$ the behaviour quoted in Table 1 for the case $n = 2$.

- 3) On this path ($t = i\tau$), t_0 is a minimum so we cannot directly apply the Laplace method.

$$I_3 \approx \lim_{r \rightarrow 0} \frac{1}{r^{2C+2}} \int_{(\mu/2)^{1/2} r}^{\sqrt{2G}} e^{Y(\tau^2 - 2\sqrt{2G}\tau)} \tau^{2C+1} d\tau.$$

But $(\tau - 2\sqrt{2G})\tau < -\sqrt{2G}\tau$ for $\tau \in [0, \sqrt{2G}]$. We make the change $\tau = vr$ and we obtain

$$|I_3| \leq \lim_{r \rightarrow 0} \int_{(\mu/2)^{1/2}}^{\sqrt{2G}/r} e^{-v\sqrt{2G}} v^{2C+1} dv < \infty.$$

- 4) Along the path Γ_4 we have

$$t = x + iy = x + i(-x + (\mu/2)^{1/2} r) \quad \text{and}$$

$$|I_4| < \lim_{r \rightarrow 0} \text{const } r^{-1} \int_0^{(\mu/2)^{1/2} r} e^{+2\sqrt{2G}x/r} dx < \infty.$$

Finally for the case $n = 2$ we obtain $|I| < \text{const.}$

B) Now we want to generalize the result obtained for the case $n = 2$ to any $2n$. [The cut in the t plane starts at the origin, and is outside the sector $(\vec{0x}, \vec{0D_1})$].

In the case of non-integer n we note that the function $\psi(r)$ given by (2.5) has a branch point at $r = 0$, and when we take the Laplace transform of (2.4) $\rho(\alpha, k)$ is associated to a given determination of $F(k, r)$ (for complex r), such that $F(k, r)$ be a real function of r for real positive r ; we can then choose the cut along the real negative r axis. This requires some care in the determination of $\text{Arg } D_1$ in (3.12); for instance in the case $n = 3/2$ we have $\text{Arg } D_1 = \pm 2i\pi/3$.

Firstly, we consider the approximation (of zero order in r) $f(t, r) \simeq f(t) \equiv -t^n + D_1 t^{n-1}$. This can be done since r is arbitrarily small and what one neglects is $O(r)$ as seen from (4.3). After having considered the case $f(t, r) \simeq f(t)$, we shall shortly consider the corresponding modifications for the whole function $f(t, r)$.

Considering now $f(t)$, we always have a saddle point at $t_0^{(1)} = \frac{n-1}{n} D_1$ and in general (if $n > 2$) another one $t_0^{(2)}$ at the origin. Now we want to study the curves $\text{Im } f(t) = 0$ in the complex t plane ($t = |t| e^{i\theta}$). These curves will be studied in polar co-ordinates $(|t|, \theta)$. We have

$$\text{Im } f(t) \equiv |t|^{n-1} \{-|t| \sin n\theta + \lambda \sin [(2j+1) \frac{\pi}{n} + (n-1) \theta]\} = 0. \quad (4.5)$$

First of all the straight line $(t_0^{(1)} t_0^{(2)}) \left(\theta = \frac{(2j+1)\pi}{n} \right)$ is a steepest path. We now consider the other steepest paths given by

$$|t| = \frac{\lambda \sin \left[\frac{(2j+1)\pi}{n} + (n-1) \theta \right]}{\sin n\theta} \quad (4.6)$$

limiting ourselves to θ in the range $\theta \in \left[0, \frac{(2j+1)\pi}{n} \right]$ [where $2j+1 \leq n/2$].

Calling

$$\bar{\vartheta}_k = \frac{k\pi}{n} - \frac{(2j+1) - k}{n(n-1)}$$

the roots of the numerator of (4.6) and $\bar{\vartheta}_k = k\pi/n$ the roots of the denominator, we give the variations of the function $|t(\vartheta)|$

ϑ	0	$\bar{\vartheta}_1$	$\bar{\vartheta}_2$	$\bar{\vartheta}_2 - \dots - \bar{\vartheta}_{2j-1}$	$\bar{\vartheta}_{2j}$	$\bar{\vartheta}_{2j}$	$\bar{\vartheta}_{2j+1} \equiv \bar{\vartheta}_{2j+1}$
$ t(\vartheta) $	$+\infty \searrow 0$		$+\infty \searrow 0$	-----	$+\infty \searrow 0$		$+\infty \searrow t_0^{(1)}$

We notice that when $\vartheta \rightarrow 0$, the curve has an asymptote

$$|t| \sin \vartheta \underset{\vartheta \rightarrow 0}{\simeq} \lambda/n \sin \left(\frac{(2j+1)\pi}{n} \right) \equiv y_0.$$

In Fig. 2 we give the behaviour of the different sections of the curve for the case $j \neq 0$. The arrows indicate the values of increasing ϑ . For the peculiar case $j = 0$, the variation is given by

ϑ	0	π/n
$ t $	$+\infty \searrow t_0^{(1)}$	

and correspondingly we have the curve of Fig. 3. For any $n < 6$ the only possibility is $j = 0$ and therefore there always exists a steepest path Γ_2 coming from infinity (asymptotic to y_0) and going to $t_0^{(1)}$. When $n \geq 6$, the corresponding path Γ_2 (asymptotic to y_0) goes either to the steepest path $t_0^{(1)}$ if $j = 0$, or to the other one $t_0^{(2)} = 0$ if $j \neq 0$. In this approximation $f(t)$ of $f(t, r)$, (4.2) is

$$I = \lim_{r \rightarrow 0} \frac{1}{r^{n(C+1)}} \int_{\Sigma \Gamma_i} e^{Y \left(-t^n + \lambda e^{i(2j+1)\pi/n} t^{n-1} \right) t^{n(C+1)-1}} dt. \quad (4.2a)$$

B₁) First of all we consider the case $j = 0$ and we want to evaluate (4.2) by deforming the integration path in a similar way as was previously done for $n = 2$. The path is shown in Fig. 4.

- 1) The contribution I_1 is still zero as in the $n = 2$ case.
- 2) Along Γ_2 , $|t(\vartheta)|$ is given by (4.6) and we have

$$I_2 = \lim_{r \rightarrow \infty} \frac{1}{r^{n(C+1)}} \int_0^{\pi/n} \exp \left[-\lambda^n Y \frac{\left[\sin \left(\frac{\pi}{n} + (n-1)\vartheta \right) \right]^{n-1} \sin \left(\frac{\pi}{n} - \vartheta \right)}{(\sin n\vartheta)^n} \right] t(\vartheta)^{nC+n-1} dt(\vartheta). \quad (4.7)$$

In order to evaluate (4.7) we cut the integration path into three parts.

- i) $\vartheta \in [0, \pi/2n]$; $\sin n\vartheta$ vanishes for $\vartheta = 0$ and is positive everywhere else. All the other ϑ dependent functions are always positive and have lower and upper positive bounds. We put

$$v = \frac{1}{(\sin n\vartheta)^n} - a$$

with $0 < a < 1$ and we find a contribution

$$\text{const} \frac{e^{-Na/r^{n-1}}}{r^{nC+n}} \int_{1-a}^{\infty} e^{-\frac{Nv}{r^{n-1}}} \frac{(a+v)^{C+1/n}}{\sqrt{1-(a+v)^{-2/n}}} < \infty \quad (N > 0).$$

- ii) $\vartheta \in [\pi/2n, \pi/n - \epsilon]$ with ϵ small but finite. In this case all ϑ dependent functions have positive lower and upper bounds and we find a contribution less than

$$\text{const} \frac{e^{-N'/r^{n-1}}}{r^{nC+n}} \quad (N' > 0).$$

- iii) $\vartheta \in [\pi/n - \epsilon, \pi/n]$. After the substitution $\varphi = \frac{\pi}{n} - \vartheta$, expanding all φ dependent functions and putting $\varphi = vr^{\frac{n-1}{2}}$ we have a contribution less than

$$\text{const} \left(\frac{e^{-\frac{2G}{(n-1)r^{\frac{n-1}{2}-1/2}}}}{r^{nC+\frac{n}{2}+1/2}} \right)^{\epsilon/r^{\frac{n-1}{2}}} \int_0^{\epsilon/r^{\frac{n-1}{2}}} e^{-Gnv^2} (\text{const} + v^2 r^{n-1})^{nC+n-1} dv < \infty.$$

Concluding, we have $I_2 < \infty$.

- 3) On the path Γ_3 , putting $t = \eta \tau$ we have

$$|I_3| < \text{const} \lim_{r \rightarrow 0} \frac{1}{r^{n(C+1)}} \int_{(\mu/2)^{1/n} r}^{\lambda \frac{n-1}{n}} e^{Y(\tau^n - \lambda \tau^{n-1})} \tau^{n(C+1)-1} d\tau.$$

But $\tau^n - \lambda \tau^{n-1} \leq -\frac{\lambda}{n} \tau^{n-1}$ for $\tau \in [0, \lambda \frac{n-1}{n}]$ and putting $\tau = vr$, similarly to the $n = 2$ case, we have $|I_3| < \infty$.

- 4) Working exactly as in the $n = 2$ case we find $|I_4| < \infty$.

B₂) We now consider $j \neq 0$ and deform the integration path following Fig. 5.

We still take the cut outside the first quadrant. The difference between this case $j \neq 0$ and $j = 0$ is that Γ_2 now goes towards the saddle point at the origin instead of $t_0^{(1)}$. This fact will give a different contribution along Γ_2 . Because of the presence of r small Γ_2 does not go exactly to 0. The path Γ_3 is similar to the Γ_4 of the previous case $j = 0$.

- 1) The contribution along Γ_1 is negligible as before.
- 2) Along Γ_2 , $|t(\vartheta)|$ is still given by (4.6) and we have

$$I_2 = \lim_{r \rightarrow 0} \frac{1}{r^{nC+n}} \int_0^{\bar{\vartheta}_1 - \xi} \exp \left[-\lambda^n Y \frac{\left[\sin \left[(2j+1) \frac{\pi}{n} + (n-1)\vartheta \right] \right]^{n-1} \sin \left[(2j+1) \frac{\pi}{n} - \vartheta \right]}{(\sin n\vartheta)^n} \right] \cdot t(\vartheta)^{nC+n-1} dt(\vartheta)$$

where ξ is such that $|t(\bar{\vartheta}_1 - \xi)| = \text{const}_1 r$ which gives, for r small, $\xi \simeq \text{const}_2 r$. Now we divide the integration path in two:

- i) $\vartheta \in [0, \pi/2n]$ we have similar bounds as in the $j = 0$ case (B, i)
- ii) $\vartheta \in [\pi/2n, \bar{\vartheta}_1 - \xi]$ The only ϑ dependent function which plays a role is

$$\sin \left[\frac{(2j+1)\pi}{n} + (n-1)\vartheta \right]$$

which we put equal to vr . Then we find a contribution less than

$$\text{const} \int_{\text{const}}^{1/r} e^{-N'' v^{n-1}} \frac{v^{nC+n-1}}{\sqrt{1 - (vr)^2}} dv < \infty \quad (N'' > 0).$$

- 3) As previously found, along Γ_3 the contribution is still bounded.

We finally conclude that (4.2a) is bounded for any n and any $j \leq n/4 - 1/2$.

We now come back to the approximations of $f(t, r)$ and want to show briefly the modifications to the previous analysis. The next approximation is $f(t, r) \simeq f(t) + r D_2 t^{n-2}$. Now

$$\begin{aligned} \text{Im } f(t, r) \simeq |t|^{n-2} & \left[-|t|^2 \sin n\vartheta + |t|\lambda \sin \left((2j+1)\frac{\pi}{n} + (n-1)\vartheta \right) + \right. \\ & \left. + r |D_2| \sin \left((2j+1)\frac{2\pi}{n} + (n-2)\vartheta \right) \right]. \end{aligned}$$

First of all the straight line $\vartheta = (2j+1)\pi/n$ is always a steepest path. If we look for the other steepest paths, ($\vartheta \in [0, (2j+1)\pi/n]$) we find:

$$\begin{aligned} |t| = \frac{\lambda \sin \left[(2j+1)\frac{\pi}{n} + (n-1)\vartheta \right]}{2 \sin n\vartheta} & \left[1 + \right. \\ & \left. + \sqrt{\frac{1 + 4r |D_2| \sin n\vartheta \sin \left((2j+1)\frac{2\pi}{n} + (n-2)\vartheta \right)}{\left[\lambda \sin \left((2j+1)\frac{\pi}{n} + (n-1)\vartheta \right) \right]^2}} \right]. \quad (4.8) \end{aligned}$$

For the location of the saddle point we now find that $t_0^{(1)}$ is moved along the straight line $\vartheta = (2j+1)\pi/n$ of a quantity of the order of r (for r small).

In the $j = 0$ case we see that $|t(\vartheta, r)| \simeq |t(\vartheta)| + r T(\vartheta)$ where $|t(\vartheta)|$ is the same as (4.6) and $T(\vartheta)$ is finite for $\vartheta \in [0, \pi/n]$. In the evaluation of (4.2) the only change is in (B₁ iii) whose contribution is now less than

$$\text{const} \lim_{r \rightarrow 0} \frac{1}{r^{n(C+1/2)+1/2}} \exp \left\{ \frac{1}{r^{n-1}} \left[\frac{-2G}{r^{n-1}} + \frac{r}{n-2} \left(\frac{A}{G} + \mu G \right) \right] \right\}.$$

For $j \neq 0$, analysing (4.8) for $\vartheta \in [0, \bar{\vartheta}_1 - \xi]$ we see that there still exists a steepest path Γ_2 and everything goes parallel to what was done in B₂. By taking further terms in $f(t, r)$ the situation modifies in the same line, as sketched before. We have therefore proved that (4.1,2) is bounded and thus, that the Jost function as defined by (3.6) exists.

V. METHOD OF CONVERGENCE OF THE JOST FUNCTION FOR V_2 YUKAWA GENERALIZED POTENTIALS

We shall consider here the problem of constructing explicitly the Jost function as given by (3.6) since we have already proved its existence (4.1). To this aim we shall make use of the results of the preceding two sections concerning ρ , the inverse Laplace transform of $F(k, r)$ and in particular (3.4, 8). In order to completely clarify the procedure, we shall first consider the simplest cases $2n = 3, 4$ and then we shall generalize the result to any n .

a) $n = \frac{3}{2}$

Recalling that

$$\rho(\alpha, k) \underset{\alpha \rightarrow \infty}{\sim} e^{D\alpha^E} \alpha^C \quad (5.1)$$

and that in this case $D = 3(2G)^{2/3} e^{\pm 2i\pi/3}$, $E = 1/3$, $C = -5/6$, since $\text{Re } D < 0$, we may safely put $r = 0$ in (4.1) and from (3.6) we have

$$F(k, 0) = 1 + \int_{\mu/2}^{\infty} \rho(\alpha, k) d\alpha. \quad (5.2)$$

For practical purposes we define the following sequence which will be useful in the present case for the explicit construction of the Jost function

$$F_p = 1 + \int_{\mu/2}^{p\mu/2} \rho(\alpha, k) d\alpha = 1 + \int_{\mu/2}^{p\mu/2} \left(\sum_{q=1}^{p-1} \rho_q(\alpha, k) \right) d\alpha. \quad (5.3)$$

In (5.3) we have made use of the great advantage coming from the application of the iterative property of ρ , namely (3.4). In this case the sequence (F_p) is convergent and we have

$$\lim_{p \rightarrow \infty} (F_p) = F(k, 0) . \quad (5.4)$$

Notice that if $1 < n < 2$ we are in the same situation as before since $\text{Re } D < 0$ and (5.4) is still valid.

b) $n = 2$

In this case we recall that the corresponding values of C, D, E in (5.1) are $C = -3/4 + 1/2 (A/G + \mu G)$, $D = \pm 2i \sqrt{2G}$, $E = 1/2$.

- 1) If $C < 0$, we have exactly the same kind of convergence as in the previous case a), i.e., (5.4) is still valid.
- 2) We now consider the more delicate case $C > 0$.

In this case the sequence F_p as defined by (5.3) does not converge. We are then forced to modify our method for constructing the Jost function. It will be seen that the method to be given below is still valid in the previous cases ($n \leq 2$) and that it will also work in the general case ($n > 2$).

We introduce a new sequence defined as

$$G(p, r) = 1 + \int_{\mu/2}^{p\mu/2} e^{-\alpha r} \rho(\alpha, k) d\alpha = 1 + \int_{\mu/2}^{p\mu/2} e^{-\alpha r} \left(\sum_{q=1}^{p-1} \rho_q(\alpha, k) \right) d\alpha . \quad (5.5)$$

Firstly, due to (3.6) we have

$$\lim_{r \rightarrow 0} \left(\lim_{p \rightarrow \infty} G(p, r) \right) = F(k, 0) . \quad (5.6)$$

Secondly, because of (3.5) for any finite p we have

$$\lim_{r \rightarrow 0} G(p, r) \equiv G(p, 0) \equiv F_p . \quad (5.7)$$

Furthermore, due to the fact that

$$\lim_{p \rightarrow \infty} F_p \equiv \lim_{p \rightarrow \infty} G(p, 0) \quad (5.8)$$

does not exist we are not allowed to interchange the order of the two limits in (5.6).

The existence of the limit (5.6) suggests that we can connect the two limiting procedures ($p \rightarrow \infty$, $r \rightarrow 0$) by replacing the double limit with a single one. To this aim we attribute a p dependence to r (or an r dependence to p) in such a way as to have the following identities:

$$\lim_{r \rightarrow 0} G(p(r), r) \equiv \lim_{p \rightarrow \infty} G(p, r(p)) = F(k, 0) . \quad (5.9)$$

Since we have seen that the two limits $p \rightarrow \infty$ and $r \rightarrow 0$ cannot be interchanged, we argue that $p(r)$ and $r(p)$ cannot be completely arbitrary.

In the following, in order to satisfy (5.9), we shall find that there exist two limiting dependences $r_L(p)$ and $p_L(r)$ such that $r(p)$ can go arbitrarily to zero less rapidly than $r_L(p)$ [and $p(r)$ more rapidly to infinity than $p_L(r)$]. It could be shown that $r_L(p)$ and $p_L(r)$ are the inverse functions of each other. Thus it is sufficient, for instance, to consider only $r(p)$.

We formally put

$$r(p) \underset{p \rightarrow \infty}{\simeq} b p^{-\gamma} (\ln p)^\delta, \quad (b > 0, \gamma > 0) . \quad (5.10)$$

In order to determine the value of the parameters b, γ, δ giving the limiting dependence $r_L(p)$ we shall ask for the weakest convergence of (5.9). More explicitly

$$\lim_{p \rightarrow \infty} \int_{\mu/2}^{p\mu/2} e^{-\alpha r(p)} \rho(\alpha, k) d\alpha < \infty \quad (\text{for } r(p) \geq r_L(p)) . \quad (5.11)$$

We consider the integrand in (5.11) for large $\alpha \simeq p\mu/2$

$$\exp \left[-\frac{\mu}{2} p r(p) + C \ln \left(\frac{\mu}{2} p \right) + 2i \sqrt{\mu G p} \right] . \quad (5.12)$$

Taking into account that the last factor in (5.12) is an oscillating one in the variable $\sqrt{p}$ we find

$$\begin{cases} \gamma_L = 1 \\ \delta_L = 1 \\ b_L > (C + \frac{1}{2})2/\mu \end{cases} \quad (5.13)$$

We shall now prove that for any $r(p) \geq r_L(p)$, (5.9) is satisfied, i.e.,

$$\lim_{p \rightarrow \infty} G(p, r(p)) = F(k, 0), \quad (r(p) \geq r_L(p)) \quad (5.14)$$

First we note that for any p we have identically

$$G(p, r(p)) + R(p, r(p)) = F(k, r(p)) \quad (5.15)$$

where

$$R(p, r(p)) = \int_{p\mu/2}^{\infty} e^{-\alpha r(p)} \rho(\alpha, k) d\alpha \quad (5.16)$$

is the remainder of the sequence $G(p, r(p))$. The proof of (5.14) is then equivalent in proving that $\lim_{p \rightarrow \infty} R(p, r(p)) = 0$.

In order to study the identity (5.15) we make the same change of variable as in Chapter IV giving (4.4) for $F(k, 0)$. Since the asymptotic behaviour of ρ is the same as the integrand in (4.1) which defines I , in order to evaluate $F(k, 0)$, we deform the integration path as made for I (Fig. 1). We accordingly call $F^{(i)}$ the corresponding contribution along Γ_i . We are then left to evaluate the integrals appearing in the identity (5.15) in the complex t plane along the contour of Fig. 1. We recall that along $\Gamma_1 + \Gamma_2$ the contribution to $F(k, 0)$

is bounded by an exponentially vanishing function of r when $r \rightarrow 0$. On the contrary along $\Gamma_{3,4}$ the respective contributions $F^{(3)}, F^{(4)}$ are less than finite constant. But in fact along Γ_3 there is only a small part of the integration path which gives a non-vanishing contribution ($F^{(3a)}$). To show this we put

$$d(p, r(p)) = (p\mu/2)^{1/2} r(p), \quad d_0 = (\mu/2)^{1/2} r(p). \quad (5.17)$$

By using the same notations as in Chapter IV (A3) we consider $F^{(3a)}$, ($\tau \in [d_0, d_L]$) and $F^{(3b)}$ ($\tau \in [d_L, \sqrt{2G}]$). Following the same procedure we find:

$$|F^{(3a)}| \leq \text{const} \lim_{r(p) \rightarrow 0} \frac{(p\mu/2)^{1/2}}{(\mu/2)^{1/2}} \int e^{-\sqrt{2G}v} v^{2C+1} dv < \infty, \quad (5.18)$$

$$|F^{(3b)}| < \text{const} \lim_{r(p) \rightarrow 0} \frac{\exp[-(p\mu/2)^{1/2}\sqrt{2G}]}{(r(p))^{2C+2}}.$$

Therefore while $F^{(3a)}$ is bounded by a constant, $F^{(3b)}$ is exponentially vanishing for $p \rightarrow \infty$ (or $r \rightarrow 0$). Finally we have

$$F(k, 0) = F^{(1)} + F^{(2)} + F^{(3b)} + F^{(3a)} + F^{(4)} \quad (5.19)$$

and when $p \rightarrow \infty$ ($r(p) \rightarrow 0$), only the last two terms in (5.19) give a non-zero contribution. We recall that we study (5.15) for $r(p) \geq r_L(p)$, then from (5.10, 18) we have

$$d = b(\mu/2)^{1/2} p^{1/2 - \gamma}.$$

$$\text{i) } \begin{matrix} d \rightarrow 0 \\ p \rightarrow \infty \end{matrix} \begin{cases} \gamma > 1/2 \\ \gamma = 1/2, \delta < 0 \end{cases} \quad \text{ii) } \begin{matrix} d \rightarrow \infty \\ p \rightarrow \infty \end{matrix} \begin{cases} \text{const } (\gamma = 1/2, \delta = 0) \\ (\gamma = 1/2, \delta > 0) \\ (\gamma < 1/2) \end{cases}$$

To investigate $G(p, r(p))$ and $R(p, r(p))$ we proceed as in Chapter IV (A) and deform the integration path following Fig. 6i) and Fig. 6ii). In Fig. 6i) we have

$$F^{(3b)} = F'^{(3b)} + F''^{(3b)}$$

$$\begin{cases} G = F^{(5)} + F^{(4)} + F'^{(3b)} + F^{(3a)} \\ R = -F^{(5)} + F''^{(3b)} + F^{(2)} + F^{(1)} \end{cases} \quad (5.20i)$$

and

$$\begin{cases} G = F^{(5)} + F^{(3)} + F^{(4)} \\ R = -F^{(5)} + F^{(2)} + F^{(1)} \end{cases} \quad (5.20ii)$$

In order to prove (5.14) or alternatively to find the asymptotic behaviour of (5.16) for $p \rightarrow \infty$ it is therefore enough to evaluate $F^{(5)}$.

We find

$$|F^{(5)}_{(i)}| \leq \text{const} \lim_{r(p) \rightarrow \infty} \left(\frac{d}{r(p)} \right)^{2C+1} \left(e^{-\frac{d^2}{r(p)}} - e^{-\frac{d^2}{r(p)}} - \frac{2\sqrt{2G} d}{r(p)} \right) \quad (5.21i)$$

i.e.,

$$|F^{(5)}_{(i)}| \leq \text{const} \lim_{p \rightarrow \infty} \begin{cases} p^{-\epsilon} & [\gamma_L = 1, \delta_L = 1, b_L = \mu/2 (C+1/2) + \epsilon, \epsilon > 0] \\ e^{-bp^{1-\gamma}(\ln p)^\delta} p^{C+1/2} & (1/2 < \gamma < 1, \text{ any } \delta; \gamma = 1/2, \delta < 0). \end{cases} \quad (5.21i)$$

Similarly

$$|F_{(ii)}^{(5)}| \leq \text{const} \lim_{r(p) \rightarrow 0} \frac{e^{-\frac{2G}{r(p)}}}{[r(p)]^{2C+2}} \int_0^d \frac{x^2}{e^{r(p)}} \left(\frac{2G}{d^2} - 1 \right) \left[x \left(1 - \frac{\sqrt{2G}}{d} \right) + \sqrt{2G} \right]^{2C+1} dx \quad (5.21ii)$$

and

$$|F_{(ii)}^{(5)}| \leq \text{const} \lim_{p \rightarrow \infty} \begin{cases} e^{-b\sqrt{p}} p^{1/2} (\mu G + A/G), & (d \rightarrow \text{const} < \sqrt{2G}) \\ e^{-\frac{2G}{b}\sqrt{p}} p^{1/2} (\mu G + A/G), & (d \rightarrow \text{const} > \sqrt{2G}) \\ e^{-\frac{2G}{b} \frac{p^\gamma}{(\ln p)^\delta}} p^\gamma (\mu G + A/G), & (d \rightarrow \infty) \end{cases} \quad (5.21iii)$$

Summarizing, and taking into account (5.19) and (5.20), the above results concerning $F^{(5)}$ inform us that (5.14) is satisfied

$$G(p, r(p)) \xrightarrow{p \rightarrow \infty} F(k, 0), \quad (r(p) \geq r_L(p)) \quad (5.14)$$

or alternatively that the remainder R goes to zero for $p \rightarrow \infty$, at least as the bounds (5.21i), (5.21ii) given for $F^{(5)}$. This shows that all the sequences $G(p, r(p))$ (for $r(p) \geq r_L(p)$) converge towards the same limit and that this limit is the Jost function. This allows us, by choosing a suitable $r(p)$, to obtain approximate expressions for the Jost functions

$$F(k, 0) \simeq G(p, r(p)) \quad (5.22)$$

where no limiting procedure is involved, and where what is neglected becomes smaller and smaller by increasing p .

We would like to emphasize that what has been done in this paper is by no means equivalent to a cut-off procedure in which the limit $r = 0$ has to be explicitly performed at a certain stage. In the present approach, once the $r(p) \geq r_L(p)$ dependence has been given, this $r(p)$ insures the convergence of the sequences G to $F(k,0)$ and we never have to put $r(p) \equiv 0$ at any stage. In other words, the convergence's function $r(p)$ comes naturally in the game from the mathematical analysis of the properties of the Laplace transform, and is always present in any approximate expression for the Jost function.

We notice that the same procedure also applies to the case where the most singular term of the potential at the origin is less or equally singular than $1/r^2$, as well as in the previously considered $n < 2$ case; the only change being that now we would have no limiting dependence $r_L(p)$.

c) Arbitrary $n > 2$

Considering the corresponding behaviour for p and remembering the value found for the leading term of p (3.12) since there are always roots of D_1 such that $\text{Re } D_1 > 0$, we are in a similar situation to the case $n = 2$, $C > 0$, namely, the sequence F_p does not converge. We therefore retain the same notations and definitions for the sequences $G(p,r)$ defined by (5.5) and satisfying (5.6,7,8). Similarly, we shall also connect the two limiting procedures $p \rightarrow \infty$, $r \rightarrow 0$ to satisfy (5.9) by making the same assumptions (5.10,11). The changes will reflect into the limiting dependence $r_L(p)$.

We therefore consider in the case $n > 2$, the corresponding integrand (5.11) for large $\alpha \simeq p\mu/2$

$$\exp \left[-\frac{\mu}{2} p r(p) + D_1 \left(\mu \frac{p}{2} \right)^{1-1/n} + D_2 \left(p \frac{\mu}{2} \right)^{1-2/n} + \dots + C \ln \left(\frac{\mu}{2} p \right) \right] . \quad (5.12')$$

In the same way as in the case $n = 2$, $C > 0$ (B_2), we ask for the weakest convergence of (5.11) and we find

$$\gamma_L = \frac{1}{n}, \quad \delta_L = 0, \quad b_L > \frac{\operatorname{Re} D_1}{\left(\frac{\mu}{2}\right)^{1/n}} \quad (\operatorname{Re} D_1 > 0) \quad (5.13')$$

$$\gamma_L = \frac{2}{n}, \quad \delta_L = 0, \quad b_L > \frac{\operatorname{Re} D_2}{\left(\frac{\mu}{2}\right)^{2/n}} \quad (\operatorname{Re} D_1 = 0, \operatorname{Re} D_2 > 0) . \quad (5.13'')$$

In order to prove (5.14), we define the remainder $R(p, r(p))$ as in (5.15, 16).

First we consider the case $j = 0$ ($\operatorname{Re} D_1 > 0$). We now have

$$d = \left(p \frac{\mu}{2}\right)^{1/n} r(p), \quad d_0 = \left(\frac{\mu}{2}\right)^{1/n} r(p) \quad (5.17')$$

and

$$d_L \geq \frac{n}{n-1} \cos(\pi/n) |t_0^{(1)}|, \quad (d_L \text{ constant}) .$$

By using the same notations as in Chapter IV (B_1) and deforming the integration path for $F(k, 0)$ as in Fig. 4, we then consider the integrals of the identity (5.15) $F = G + R$ in the complex t plane. We distinguish the two cases

$$\text{i) } d_L > |t_0^{(1)}|$$

$$\text{ii) } d_L \leq |t_0^{(1)}|$$

and accordingly we have Fig. 7, where Γ_{5i} and Γ_{5ii} refer to the two possibilities.

We have

$$\begin{cases} G = F^{(5)} + F^{(2b)} + F^{(3)} + F^{(4)} \\ R = -F^{(5)} + F^{(2a)} + F^{(1)} \end{cases} \quad (5.20' i)$$

and

$$\begin{cases} G = F^{(5)} + F^{(3a)} + F^{(4)} \\ R = -F^{(5)} + F^{(3b)} + F^{(2)} + F^{(1)} \end{cases} \quad (5.20' ii)$$

Similarly to what was previously found while $|F^{(3a)} + F^{(4)}| < \text{const}$, $|F^{(2a)}|$, $|F^{(2b)}|$ and $|F^{(3b)}|$ are bounded by exponentially vanishing functions and we find for $|F^{(5)}|$ and $|F^{(5)}|$

(i)
(ii)

$$\begin{matrix} |F^{(5)}| & |F^{(5)}| \\ (i) & (ii) \end{matrix} < \text{const } e^{-C_1 p^{1-\frac{1}{n}}} p^{C_2} \quad (5.21' i, ii)$$

(i)
(ii)
 $p \rightarrow \infty$

with $C_1 > 0$ and where C_1 and C_2 are, in general, different for the two cases i), ii).

We now consider the case $j \neq 0$ for which the integration path (similarly to what was done in Fig. 5) is given in Fig. 8., and we have

$$\begin{aligned} G &= F^{(5)} + F^{(3)} + F^{(2b)} \\ R &= -F^{(5)} + F^{(2a)} + F^{(1)} \end{aligned}$$

For the two possibilities $\text{Re } D_1 > 0$ and $\text{Re } D_1 = 0$, we have respectively

$$d_L \geq \lambda \cos \left(\frac{(2j+1)\pi}{n} \right) \quad (\text{Re } D_1 > 0)$$

$$d_L \geq \frac{\text{Re } D_2}{\left(p \frac{\mu}{2} \right)^{1/n}} \quad (\text{Re } D_1 = 0, \text{Re } D_2 > 0) .$$

Similarly to what was found in the previous cases, $|F^{(1)}|$ and $|F^{(2a)}|$ are bounded by exponentially vanishing functions while $|F^{(2b)}|$ and $|F^{(3)}|$ go to constants. We also find

$$|F^{(5)}| < \text{const } e^{-\text{const } p^{1-1/n}} \quad (\text{Re } D_1 > 0) \\ p \rightarrow \infty$$

$$|F^{(5)}| < \text{const } e^{-\text{const } p^{1-2/n}} \quad (\text{Re } D_1 = 0, \text{Re } D_2 > 0) . \\ p \rightarrow \infty$$

The same analysis could be repeated if one would work on the complete function $f(t,r)$ instead of on $f(t)$ as was done here. We would still arrive at the result (5.14) concerning the convergence of the sequences $G(p,r(p))$ to the Jost function. The only change however, would be that the above given bounds for the remainder $R(p,r(p))$ would contain other p dependent factors which, however, would always be of a smaller order with respect to the ones given.

All the considerations made at the end of the $n = 2$ case concerning the possibility of constructing the Jost functions by using the always convergent sequences $G(p,r(p))$ are still valid.

We now wish to add some remarks of explanation concerning the different kinds of convergence; namely that in the $1 < n < 2$ case and $n = 2$, ($C < 0$) we can put $r = 0$ in the integrand of (3.6), whereas in the other cases this is not possible. In fact these differences are directly connected to the Laplace transform used for the function $F(r,k)$ which has essential singularities at the origin. There is a correspondence between the singularities of F near the origin

$$Y_{+1}(r)/Y_{-1}(r) = \exp \left\{ \frac{-2G}{(n-1)r^{n-1}} [1 + o(r)] \right\}$$

(see (2.9), Table 1 and the Appendix), and the singularities of ρ when

$\alpha \rightarrow \infty$; $\rho(\alpha) \simeq \exp \left(D_1 \alpha^{1-1/n} + \dots \right) \alpha^C$ (see 3.8). In the Laplace transform the real variable α is associated with r complex, and the inverse Laplace transform which gives $\rho(\alpha)$ from $F(r)$ is taken in the $\text{Re}(r) > 0$ plane along a line parallel to the imaginary axis. Now, F has an essential singularity at the origin. Suppose that $n < 2$, the only essential singularity of F is the leading one

$$\simeq \exp \left[\frac{-2G}{(n-1)r^{n-1}} [\cos (n-1)\vartheta - i \sin (n-1)\vartheta] \right] \text{ where } r = \tau e^{i\vartheta}.$$

If we go to zero along any path in the right half plane, we have no trouble and the limit is 0. Then, in order to prove the convergence of

$$\lim_{r \rightarrow 0} \int_{\mu/2}^{\infty} e^{-\alpha r} \rho(\alpha) d\alpha$$

we can take the limit along any path in $\text{Re}(r) \geq 0$ and we can put $r = 0$ directly in the integrand.

Secondly, we consider the transition case $n = 2$. Along the imaginary axis the essential singularity has an oscillatory character. We must then consider the next singularity which is a power; then along the imaginary axis, when we go to zero, the limit can be infinite.

Thirdly we consider the general case $n > 2$. Because of the presence of the leading essential singularity if we go to zero along any path in the right half plane with $|\vartheta| > \pi/2(n-1)$ near the origin, the limit is infinite, whereas inside $|\vartheta| < \pi/2(n-1)$ the limit is 0. Because of this situation over the limit of the leading essential singularity, if we take for instance

$$\lim_{r \rightarrow 0} \int_{\mu/2}^{\infty} e^{-\alpha r} \rho(\alpha) d\alpha$$

along the real axis which is inside $|\vartheta| < \pi/2(n-1)$, the result is finite as we have previously seen.

Finally we recall that in general, we must cut the r plane along the real negative axis so that ϑ has a finite range. Correspondingly for $\rho(\alpha)$ for $\alpha \rightarrow \infty$ some care must be devoted in the determination of the arguments of different D_i .

VI. CONCLUDING REMARKS

We want to recall that for the family of short-range potentials considered here with the use of the Laplace transform, we have been able to show, first, the existence of the Jost function and moreover to give practical methods for constructing it without the introduction of any arbitrary cut-off procedure. Furthermore, by comparing the present case with the case in which one has the same family of short-range potentials less or equally singular than the centrifugal term at the origin, we have the following situation. We consider the expansion of the Jost function $F = \sum_p F_p$ where F_p is the contribution to the Jost function coming from the iteration of order p of the Laplace transform ρ of $F(k, r)$.

- 1) $V \sim 0 \ (1/r^{2-\epsilon})$. The expansion is absolutely and uniformly convergent for any G ⁶⁾.
- 2) $V \sim 0 \ (1/r^2)$. The expansion⁵⁾ is convergent in a certain range of G which includes $G = 0$.
- 3) In our case where $V_1(r)$ is more singular than $1/r^2$ we are in a very different situation because the considered expansion always diverges.

Due to this peculiar behaviour in case 3) it was necessary to consider other methods in order to obtain the Jost function and, in fact, this new approach is also valid in cases 1) and 2). We have seen that taking great advantage of the iterative properties of the Laplace transform it is convenient for potentials more singular than $1/r^2$ to consider the Jost function as the limit of sequences $G(p, r(p))$ where $(p, r(p))$ are the contributions coming only from the sum of the first $(p-1)$ iterations of the Laplace transform ρ of $F(k, r)$. This new approach is based on the possibility of connecting the two limiting procedures $p \rightarrow \infty$ and $r \rightarrow 0$ by choosing a given $r(p)$ (or $p(r)$) dependence.

We have been able to show that for any given $n \geq 2$ there exists a limiting dependence $r_L(p)$ or $(p_L(r))$ insuring the convergence of $G(p, r_L(p))$ (or $G(p_L(r), r)$) to the Jost function. Furthermore, for any $p \rightarrow \infty$ $r \rightarrow 0$ $r(p)$ going to zero less rapidly than $r_L(p)$ when $p \rightarrow \infty$ (or $p(r)$ going to infinity more rapidly than $p_L(r)$ when $r \rightarrow 0$), all the sequences $G(p, r(p))$ ($G(p(r), r)$) also converge to the Jost function. $p \rightarrow \infty$ $r \rightarrow 0$

We have also been able to show that for any choice of $r(p) \geq r_L(p)$ ($p(r) \leq p_L(r)$) there exist bounds for the remainder $R(p, r(p))$ (or $R(p(r), r)$) depending mainly on the most singular part of the interaction and on the choice of $r(p)$ ($p(r)$). The case $r(p) = r_L(p)$ ($p(r) = p_L(r)$) is the one which gives the weakest vanishing of $R(p, r(p))$ ($R(p(r), r)$). $p \rightarrow \infty$ $r \rightarrow 0$

To exemplify this, if we choose, for instance, two arbitrary dependences $r_1(p) > r_2(p) > r_L(p)$, the two corresponding functions G_1 and G_2 have the same limit (which is the Jost function) when $p \rightarrow \infty$. For any fixed p we have then two different approximations G_1 and G_2 and the two corresponding remainders R_1 and R_2 are such that for very large p , R_1 vanishes more rapidly than R_2 .

Finally we would like to emphasize that, within this approach, there is only one difference between the present case ($n \geq 2$) and the cases 1), 2) and 3) ($n < 2$). Namely, in the latter ones there is no limiting dependences $r_L(p)$ ($p_L(r)$) and therefore any $r(p)$ going to zero ($p(r)$ going to infinity) insures the convergence of the sequences $G(p, r(p))$ ($G(p(r), r)$) to the corresponding Jost function.

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A P P E N D I X

In order to prove that $Z(r)$ near the origin behaves as the most singular part of $R(\pm k, r)$, according to the formalism developed in Chapter II, it is sufficient to show that

$$F(k, r) \sim C_1 + C_2 \bar{F}(r)$$

where $\bar{F}(r)$ vanishes when $r \rightarrow 0$ equivalently to (2.9).

Remembering (2.5, 7) by expanding $e^{-\mu r/2}$ near the origin (and putting $W(r) \sim W_0 r^{-m}$) the differential equation (2.4) for $F(k, r)$ becomes

$$\begin{aligned} F'' + F' \left[-\frac{2G}{r^n} + \left(\frac{A}{G} + \mu G \right) \frac{1}{r^{n-1}} + \left(\frac{B}{G} + \frac{A^2}{4G^3} - \frac{G\mu^2}{4} - \frac{\mu}{2} \frac{A}{G} \right) \frac{1}{r^{n-2}} + \right. \\ \left. + \dots + \frac{2n}{r} \right] + F(k^2 + W_0 r^{-m}) \simeq 0. \end{aligned} \quad (A.1)$$

As is easily seen, (A.1) is not of the Fuch's type, therefore we consider an Ansatz of products of essential singularities and we want to show that we can determine all the parameters

$$F(k, r) \underset{r \rightarrow 0}{\sim} \text{const } r^\lambda \exp \left[\sum_i \frac{\beta_i}{r^{\gamma_i}} \right] \quad (A.2)$$

(with all $\gamma_i > 0$ and $\gamma_1 > \gamma_2 > \dots$).

We substitute (A.2) in (A.1) and consider the most singular terms near the origin

$$\begin{aligned}
& e^{\sum \beta_i r^{-\gamma_i}} r^{\lambda-2} \left\{ \sum_i (\gamma_i \beta_i)^2 r^{-2\gamma_i} + 2 \sum_{i \neq j} \gamma_i \gamma_j \beta_i \beta_j r^{-(\gamma_i + \gamma_j)} + \right. \\
& + \sum_i (2\lambda - \gamma_i - 1) (-\gamma_i \beta_i) r^{-\gamma_i} + \lambda(\lambda - 1) + \\
& + \left(\lambda + \sum_i (-\gamma_i \beta_i) r^{-\gamma_i} \right) \left[(-2G) r^{-(n-1)} + \left(\frac{A}{G} + \mu G \right) r^{-(n-2)} + \right. \\
& + \left. \left(\frac{B}{G} + \frac{A^2}{4G^3} - \frac{G\mu^2}{4} - \frac{\mu}{2} \frac{A}{G} \right) r^{-(n-3)} + \dots + n \right] + W_0 r^{-(m-2)} \left. \right\} \approx 0.
\end{aligned} \tag{A.3}$$

We want to determine the parameters γ_i , β_i , λ by imposing, that in the bracket (A.3), the coefficients of the most singular terms are zero.

- 1) We find for the most singular term

$$\begin{cases} \gamma_1 = n-1 \\ (\beta_1 \gamma_1)^2 = -2G(\beta_1 \gamma_1) \end{cases}$$

First of all the condition $\gamma_1 > 0$ is equivalent to $n > 1$, secondly we have either $\beta_1 = 0$ (corresponding to the constant solution for $F(k, r)$), or $\beta_1 = -2G/n-1$. The corresponding power behaviour of this most singular term in the bracket is of the order $r^{-(2n-2)}$. In order to determine the other parameters we consider, in the bracket, the terms of order $r^{-(2n-3)}$ and so on. We remark that the term $W(r)$ never comes into the game.

- 2) From the next singular term of the bracket we find

$$\gamma_2 = n - 2$$

$$\beta_2 = \frac{1}{n-2} (A/G + \mu G)$$

which gives another essential singularity in (A.2) for $n > 2$.

If $1 < n \leq 2$ we have only one essential singularity and a power behaviour. The result is quoted in Table 1 for the cases $n = \frac{3}{2}$ and $n = 2$ as examples.

- 3) If $n > 3$ we have another essential singularity

$$\gamma_3 = n - 3 .$$

For $2 < n \leq 3$ we have only two essential singularities and a power behaviour. The result is quoted in Table 1 for the cases $n = \frac{3}{2}, 3$ as examples.

For any $n > 1$ the above method can be used in order to evaluate all the parameters $\beta_i, \gamma_i, \lambda$. If $i < n < i+1$ (i integer) we have a product of i essential singularities and a power behaviour. Since the higher essential singularity is always

$$\exp \left(- \frac{2G}{(n-1)r^{n-1}} \right)$$

this ensures the vanishing of (A.2) when r goes to zero.

Table 1

$$V_1 = e^{-\mu r} \left(\frac{G^2}{r^{2n}} - \frac{A}{r^{2n-1}} - \frac{B}{r^{2n-2}} - \frac{C}{r^{2n-3}} - \dots \right)$$

n	$\ln Y_{-1}(r)$	$\ln Y_{+1}(r)$	$\ln [Y_{+1}(r)/Y_{-1}(r)]$
$3/2$	$\frac{2G}{r^{3/2}} + \frac{3}{4} \ln r$	$-\frac{2G}{r^{3/2}} + \frac{3}{4} \ln r$	$-\frac{4G}{r^{3/2}}$
2	$\frac{G}{r} + \left(1 + \frac{A}{2G} + \frac{\mu G}{2}\right) \ln r$	$-\frac{G}{r} + \left(1 - \frac{A}{2G} - \frac{\mu G}{2}\right) \ln r$	$-\frac{2G}{r} - \left(\frac{A}{G} + \mu G\right) \ln r$
$5/2$	$\frac{G}{3^{3/2} r^{3/2}} - \left(\frac{A}{G} + \mu G\right) \frac{1}{r^{1/2}} + \frac{5}{4} \ln r$	$-\frac{G}{3^{3/2} r^{3/2}} + \left(\frac{A}{G} + \mu G\right) \frac{1}{r^{1/2}} + \frac{5}{4} \ln r$	$-\frac{4G}{3^{3/2} r^{3/2}} + 2 \left(\frac{A}{G} + \mu G\right) \frac{1}{r^{1/2}}$
3	$\frac{G}{2r^2} - \frac{1}{2r} \left(\frac{A}{G} + \mu G\right) + \left(\frac{B}{2G} + \frac{A^2}{8G^3} - \frac{\mu^2 G}{8} - \frac{A\mu}{4G} + \frac{3}{2}\right) \ln r$	$-\frac{G}{2r^2} + \frac{1}{2r} \left(\frac{A}{G} + \mu G\right) + \left(-\frac{B}{2G} - \frac{A^2}{8G^3} + \frac{\mu^2 G}{8} + \frac{A\mu}{4G} + \frac{3}{2}\right) \ln r$	$-\frac{G}{r^2} + \frac{1}{r} \left(\frac{A}{G} + \mu G\right) - \dots \left(\frac{B}{G} + \frac{A^2}{4G^3} - \frac{\mu^2 G}{4} - \frac{A\mu}{2G}\right) \ln r$

For instance, in the case $n = 2$, we also have

$$X_{-1}(\mu, G, A) = \exp \left\{ G - \int_0^1 e^{-\frac{\mu}{2} r} \left[\frac{G}{r^2} - \left(1 + \frac{A}{2G}\right) \frac{1}{r} \right] dr - \int_1^\infty \left(e^{-\frac{\mu}{2} r} \left[\frac{G}{r^2} - \left(1 + \frac{A}{2G}\right) \frac{1}{r} \right] - \frac{G}{r^2} + \frac{1}{r} \left(1 + \frac{A}{2G} + \frac{\mu G}{2}\right) \right) dr \right\}$$

For the general case

$$\ln \frac{Y_{+1}(r)}{Y_{-1}(r)} \approx \frac{1}{r^{n-1}} \left[-\frac{2G}{n-1} + \frac{r}{n-2} \left(\frac{A}{G} + \mu G\right) + O(r^2) \right]$$

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Fig. 1 _

Fig. 2 _

Fig. 3 _

Fig. 4 _

Fig. 5 -

FIG. 6i

FIG. 6ii

FIG. 7

FIG. 8