Sample variance of the cosmic velocity field

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ABSTRACT

Since the cosmic peculiar velocity field depends on small wave-number modes strongly, we cannot probe its universal properties unless we observe a sufficiently large region. We calculate the expected deviation (sample variance) of the peculiar velocity dispersion from its universal value in the case observed volume is finite. Using linear theory we show that the sample variance remains as large as $\sim 10\%$, even if the observed region is as deep as $100h^{-1}\text{Mpc}$ and that it seriously affects the estimation of cosmological parameters from the peculiar velocity field.

Subject headings: cosmology: theory — large-scale structure of the Universe — methods: statistical

1. Introduction

Observational analysis of the large-scale peculiar velocity field is considered as a very effective method to impose constraint on the cosmological parameters as well as the spectrum of the primordial density fluctuations in the universe. Since its power spectrum depends more strongly on small wave-number modes, which are less contaminated by nonlinear effects, than that of density fluctuations, linear perturbation theory suffices to reproduce the amplitude of the peculiar velocity even on a relatively small length scales. In fact, Bahcall et al. (1994) have calculated the one-point peculiar-velocity dispersion smoothed with a Gaussian filter over the smoothing scale $3h^{-1}\text{Mpc}$ in their large-scale $N$–body simulations with various cosmological models and have found an excellent agreement between their calculations and the predictions of linear theory. Here $h$ is as usual the present Hubble parameter in unit of $100\text{km/sec/Mpc}$. Validity of the linear analysis has also been confirmed analytically in the framework of higher order Eulerian perturbation theory by Makino, Sasaki & Suto (1992), who showed that second-order effects were negligible on the smoothed velocity dispersion in the case of cold-dark-matter power spectrum.

The stronger dependence on the smaller wave-number modes, on the other hand, implies that the correlation length of peculiar velocity field is larger than that of the density field and that we must survey a volume large enough to contain sufficient number of such modes. In that sense, Bahcall et al. (1994) was quite correct in adopting a simulation box whose dimension was as large as $800h^{-1}\text{Mpc}$. At present, however, we can by no means hope to observe the peculiar velocity field that far in the real Universe. For example, the observed depth of the recent high quality
catalogue by Giovanelli et al. (1997) has been limited to $100h^{-1}$Mpc mainly due to the difficulties to estimate the distance which is of course essential in obtaining the line-of-sight peculiar velocity (see also Giovanelli et al. 1996). Thus we should be careful in interpreting the observational data to compare with theoretical predictions of various cosmological models.

In this Letter we present a simple analysis of the uncertainty caused by the finiteness of the sampled volume, which we call the sample variance. We concentrate on the one-dimensional velocity dispersion of the smoothed peculiar velocity field which is the sole quantity to characterize its distribution on scales linear theory suffices, provided that primordial fluctuations are Gaussian distributed, which we assume throughout the paper. Uncertainties related to the determination of the peculiar velocity dispersion include various factors, such as the observational errors in the estimation of distances to the objects, sparseness of the number of the clusters/galaxies probed, and so on. Monte Carlo calculations using mock catalogues including all these errors have been done in the literatures (e.g. Borgani et al. 1997), which are very complicated and require high computational costs. In contrast to these approaches, our purpose in this article is to clarify the fundamental limitations due to the finiteness of the sample, which is very important in the sense that the uncertainty is independent of how accurate we could measure the peculiar velocities of galaxies or clusters.

We develop a simple analytical formula similar to the same kind of analysis on the CMB measurement (Scott et al. 1993). We show that the sample variance causes an uncertainty as large as 10% on the measurement of velocity dispersion, even if we take a full sphere of radius $100h^{-1}$Mpc around us as the observational volume. This is comparable to the current observational error (Bahcall & Oh 1996).

2. Formulation

First we define the peculiar velocity field, $V(x)$, smoothed over a radius $R$ by

$$V_R(x) = \int V(x) W_R(x - x') d^3x',$$  
(1)

where $W_R(x - x')$ is the Gaussian window function given by $W_R(x) = (2\pi)^{-3/2}R^{-3} \exp(-x^2/2R^2)$. We define the line-of-sight component by

$$\hat{V}_R(x) = V_R(x) \cdot \frac{x}{|x|}.$$  
(2)

It is true that, in reality, we must first project and then smooth, but we ignore this incommutability since we consider a survey region whose dimension is much larger than $R$. Otherwise the statistical average would become nonsense. To be specific we assume that the observed region is a sphere with radius $L$ and denote its volume by $V_L \equiv \frac{4}{3} \pi L^3$ with $L \gg R$.

One dimensional velocity dispersion estimated from the line-of-sight component in the surveyed volume $V_L$ is written as

$$X(L, R) \equiv \frac{1}{V_L} \int_{V_L} \hat{V}_R(x)^2 d^3x.$$  
(3)

The ensemble average of this dispersion is of course independent of the size of the observed volume, and it is simply the one-dimensional velocity dispersion with smoothing radius $R$, which is denoted as

$$\langle X(L, R) \rangle = \sigma_{1D}^2(R).$$  
(4)
Here and hereafter, the angular bracket represents an average over ensembles of the "universes". The error associated with the estimation of $X(L, R)$ due to the finiteness of the sampled volume is characterized by the variance,

$$\left(\{X(L, R) - \langle X(L, R) \rangle\}^2\right) = \langle X^2(L, R) \rangle - \langle X(L, R) \rangle^2.$$  \hspace{1cm} (5)

We assume that primordial density fluctuation is isotropic random Gaussian, and apply linear theory, since we adopt large enough smoothing scales suggested by Bahcall et al. (1994). Using properties of the Gaussian distribution, we obtain

$$\langle X^2(L, R) \rangle = \frac{1}{V_L^2} \int_{V_L} d^3x \int_{V_L} d^3y \langle \hat{V}_R(x)^2 \hat{V}_R(y)^2 \rangle$$

$$= \frac{1}{V_L^2} \int_{V_L} d^3x \int_{V_L} d^3y \left( \langle \hat{V}_R(x)^2 \rangle \langle \hat{V}_R(y)^2 \rangle + 2 \langle \hat{V}_R(x) \hat{V}_R(y) \rangle \right).$$

Hence the variance of $X(L, R)$ reads

$$\left(\{X(L, R) - \langle X(L, R) \rangle\}^2\right) = \frac{2}{V_L^2} \int_{V_L} d^3x \int_{V_L} d^3y \langle \hat{V}_R(x) \hat{V}_R(y) \rangle^2.$$  \hspace{1cm} (6)

Here $\langle \hat{V}_R(x) \hat{V}_R(y) \rangle$ depends only on the geometry decided by $x$ and $y$, namely, on the three variables $|x| = r_1, |y| = r_2$ and the angle $\theta$ between $x$ and $y$. Let us normalize the above dispersion by $\langle X(L, R) \rangle^2 = \sigma_{1D}^4(R) = \langle \hat{V}_R(x) \hat{V}_R(x) \rangle^2$ to define

$$E(L, R) = \frac{\langle X^2(L, R) \rangle - \langle X(L, R) \rangle^2}{\langle X(L, R) \rangle^2} = \frac{16\pi^2}{V_L^2} \int_0^L dr_1 \int_0^L dr_2 \int_0^\pi d\theta r_1^2 r_2^2 \sin \theta f(r_1, r_2, \theta)^2,$$  \hspace{1cm} (7)

with

$$f(r_1, r_2, \theta) \equiv \frac{\langle \hat{V}_R(x) \hat{V}_R(y) \rangle}{\langle \hat{V}_R(x) \hat{V}_R(x) \rangle}.$$  \hspace{1cm} (8)

Next we calculate the correlation function $f(r_1, r_2, \theta)$. After some algebra, we can relate it with parallel and perpendicular velocity covariance functions, $\Phi_\parallel$ and $\Phi_\perp$, defined by Górski (1988) as

$$f(r_1, r_2, \theta) = \Phi_\parallel(r_1, r_2, \theta = n_x n_y \sin \theta) + \Phi_\perp(r_1, r_2) \left\{ \cos \theta n_y^2 - \sin \theta n_x n_y \right\},$$  \hspace{1cm} (9)

where, $r_{12}, n_x$ and $n_y$ are defined by

$$r_{12}^2 = |x - y|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos \theta, \hspace{0.5cm} n_x = \frac{r_1 \cos \theta - r_2}{r_{12}}, \hspace{0.5cm} n_y = \frac{r_1 \sin \theta}{r_{12}}.$$  \hspace{1cm} (10)

$\Phi_\parallel(r)$ and $\Phi_\perp(r)$ are the velocity correlation functions normalized to unity at $r = 0$, or $\Phi_\parallel(0) = \Phi_\perp(0) = 1$, and they are written in terms of the power spectrum of linear density fluctuations, $P(k)$, as follows (Górski 1988).

$$\Phi_\parallel(r) \propto \int_0^\infty dk P(k) W_R(k)^2 \left( j_0(kr) - 2 \frac{j_1(kr)}{kr} \right),$$  \hspace{1cm} (11)

$$\Phi_\perp(r) \propto \int_0^\infty dk P(k) W_R(k)^2 \frac{j_1(kr)}{kr}.$$  \hspace{1cm} (12)
In the above expressions $W_R(k) \equiv \exp(-k^2R^2/2)$ is the Fourier transform of $W_R(x)$, and $j_m(z)$ is the spherical Bessel functions of the $m$-th order.

In the same manner we can also calculate the sample variance of density fluctuations smoothed over a radius $R$ at the level of the linear theory. In this case we only have to replace the line-of-sight peculiar velocity $\hat{V}_R(x)$ by the density contrast smoothed over the same radius $R$, $\delta_R(x)$. Thus $\langle \hat{V}_R(x)\hat{V}_R(y) \rangle$ in equation (6) is replaced by $\langle \delta_R(x)\delta_R(y) \rangle$, and $f$ in equation (8) is replaced by $\Xi(r_{12}) \equiv \xi_R(r_{12})/\xi_R(0)$, where $\xi_R(r)$ is the two point correlation function of the smoothed density field.

In this article, we adopt the power spectrum of cold-dark-matter (CDM) models given in Efstathiou, Bond & White (1992) as

$$P(k) = B \left\{ 1 + \left[ \alpha k + (\beta k)^{3/2} + (\gamma k)^2 \right]^{\mu} \right\}^{-2/\mu}$$

(13)

where $\alpha = (6.4/\Gamma) h^{-1}\text{Mpc}$, $\beta = (3.0/\Gamma) h^{-1}\text{Mpc}$, $\gamma = (1.7/\Gamma) h^{-1}\text{Mpc}$, $\mu = 1.13$, and $B$ is the normalization factor. Since we are dealing with normalized quantities in the linear theory, the system is characterized by the shape parameter $\Gamma = \Omega_0 h$, and the smoothing radius $R$ only.

3. Results

First we plot $\Phi_\parallel(r), \Phi_\perp(r)$, and $\Xi(r)$ in Figure 1. Apparently, the peculiar velocity field has larger correlation length than the density field, in particular, $\Phi_\perp(r)$ has a very large correlation length. This is understandable because in linear theory the density contrast, $\delta$, and the peculiar velocity, $V$, satisfy the relation $V \propto \nabla \cdot \nabla^{-2} \delta$, which implies that the power spectrum of the peculiar velocity field, $P_v(k)$, is related to $P(k)$ as $P_v(k) \propto k^{-2}P(k)$ (Peebles 1980). Therefore, velocity field is more weighted to smaller $k$ and hence has a much larger correlation length, which causes a larger sample variance as we see below.

In Figure 2, we plot the sample variance of the one-dimensional velocity dispersion, $E^{1/2}(L,R)$, with smoothing radius $R = 3(0.2/\Gamma) h^{-1}\text{Mpc}$ and $6(0.2/\Gamma) h^{-1}\text{Mpc}$. For comparison we also plot the corresponding sample variance for the density field in linear theory. To simplify our presentation, the length scale in the horizontal axis is shown in unit of $(0.2/\Gamma) h^{-1}\text{Mpc}$.

Apparently, the error due to the finiteness of the observed volume is much larger for the velocity field than for the linear density field. This figure shows clearly that even with a spherical observed region with radius $100h^{-1}\text{Mpc}$, the error on the estimation of the one-dimensional velocity dispersion, which is approximately equal to $E^{1/2}(L,R)/2$, is as large as 10%. This is comparable to the observational error of $\sim 10\%$ in Bahcall and Oh (1996). Note that magnitude of the error we have obtained here is for an ideal case that a specific region was observed fully and homogeneously, and that the actual sample variance would be even larger with the same depth of the sample.

Note that linear theory is insufficient to reproduce the magnitude of density fluctuations on the scales we are dealing with. Hence we do not claim the above result reflects the correct sample variance of the density field. All we would like to stress here is that magnitude of the sample variance is sensitive to the functional shape of the power spectrum.
4. Effects of the peculiar velocity of the observer

So far, we have not specified the peculiar velocity of the observer, who is supposedly at the center of the surveyed sphere with radius $L$. The peculiar velocity of the observer can be estimated from the dipole anisotropy of the cosmic microwave background radiation. If we assume that it is totally due to our peculiar motion then COBE observation gives $V_{obs} = 627\text{km/s}$ (Kogut et al. 1993). One may suspect that if the correlation of the peculiar velocity field is so strong, the observed velocity dispersion in the sphere may strongly be affected by the value of $V_{obs}$.

Here we estimate how much the observed velocity dispersion is expected to shift as a function of $\nu \equiv V_{obs}/\sigma_{1D}(R)$.

The conditional probability distribution function of the radial peculiar velocity has been calculated in Górski (1988, eq. [7]) with a specific value of the peculiar velocity of the observer, from which we find

$$\langle \hat{V}_R^2(r) \rangle_\nu = \sigma_{1D}^2(R) \left[ 1 - \Phi_{||}(r)^2(1 - \nu^2 \cos^2 \Theta) \right],$$

(14)

where $\langle \cdots \rangle_\nu$ denotes the ensemble average on condition that the observer has a peculiar velocity of $|V_{obs}| = \nu \sigma_{1D}(R)$, and $\Theta$ is the angle between $V_{obs}$ and $r$. We thus find the constrained ensemble average of $X(L, R)$ is given by

$$\langle X(L, R) \rangle_\nu = \frac{\sigma_{1D}(R)^2}{V_L} \int_{V_L} d^3r \left\{ 1 - \Phi_{||}(r)^2(1 - \nu^2 \cos^2 \Theta) \right\}.$$  

(15)

Therefore the fractional change of the average due to the additional constraint is given by

$$\frac{\langle X(L, R) \rangle_\nu - \langle X(L, R) \rangle}{\langle X(L, R) \rangle} = \frac{\nu^2 - 3}{L^3} \int_0^L dr \Phi_{||}(r)^2 r^2.$$  

(16)

For COBE-normalized CDM model (with $h = 0.75$, $\Gamma = 0.2$) we find $\nu = 1.58$ and the above change is only about 0.5% for $R = 3(0.2/\Gamma)h^{-1}\text{Mpc}$ and $L = 100(0.2/\Gamma)h^{-1}\text{Mpc}$, much smaller than the sample variance we have discussed. Hence the peculiar motion of the observer does not cause a serious problem.

5. Discussion

We have shown that the fact that peculiar velocity field is very sensitive to small wave-number modes implies that we must observe a sufficiently large region in order to extract useful information on cosmological models from observational data. We have dealt with a smoothed peculiar velocity field, taking the smoothing length large enough to warrant the validity of linear theory in which the only important quantity is the dispersion as long as the primordial fluctuations are distributed Gaussian. Hence we concentrated on the sample variance on the dispersion of the one-dimensional line-of-sight peculiar velocity and have shown that it is not negligible at all.

Although estimation of cosmological parameters from the analysis of peculiar velocity field mostly uses more contrived methods (e.g. Dekel 1994), it would be natural to expect that similar limitation applies in these approaches as well, because they essentially start with the same kind of observational measures, namely, the line-of-sight peculiar velocity, and because linear theory has been shown to suffice on the relevant scales (Bahcall et al. 1994). In order to give a rough idea of magnitude of the error on the estimation of cosmological parameters caused by the sample variance, let us take the smoothing radius large enough that the linear theory applies for the density contrast
as well, say $R = 12 h^{-1}\text{Mpc}$, and relate the root-mean-square density fluctuation obtained from galaxy distribution, $\sigma_{\text{gal}}(R)$, with the one-dimensional peculiar velocity dispersion $\sigma_{1D}(R)$. Using the formula of Colberg et al. (1997), we find $\sigma_{1D}^2(R) = \Omega_0^{1.2} \sigma_{\text{gal}}(R)^2 F(\Gamma, R)/b^2$, where $b$ is the bias parameter and $F(\Gamma, R)$ is a function decided only by the shape of the matter power spectrum and $R$. Even in the ideal and hypothetical case that both $\sigma_{\text{gal}}(R)$ and the coefficient $F(\Gamma, R)$ are known exactly, the above simple formula tells us that we have an inevitable uncertainty in $\Omega_0^{0.6}/b$ of about $25\%$ for $L = 60 h^{-1}\text{Mpc}$ and $20\%$ for $L = 100 h^{-1}\text{Mpc}$ (in the case CDM spectrum (13) with $\Gamma = 0.2$ is adopted).

In conclusion, the strong dependence of the peculiar velocity field on small wave-number modes has the two aspects; the advantage is that the linear theory suffices even on relatively small scales, while the disadvantage is that we must have a larger observational volume to measure it accurately.

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REFERENCES


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The normalized correlation functions of velocity and density with smoothing radius \(3(0.2/\Gamma)h^{-1}\text{Mpc}\).

The solid curve represents the two point correlation function of the linear density field, \(\Xi(r)\). The dotted line represents the parallel velocity covariance function, \(\phi_\parallel(r)\), and dashed line represents the perpendicular component, \(\phi_\perp(r)\).

\[ E^{1/2}(L,R), \text{ the expected error on the estimation of the universal velocity dispersion. The horizontal axis is the radius } L \text{ of the observed sphere in unit of } (0.2/\Gamma)h^{-1}\text{Mpc. The dotted line corresponds to } E^{1/2}(L,6), \text{ and the solid line to } E^{1/2}(L,3), \text{ both for the one dimensional peculiar velocity. The long-dashed and short-dashed lines represent the corresponding quantities for the linear density contrast.} \]