Elements of the Continuous Renormalization Group

Tim R. Morris

Physics Department,
University of Southampton,
Highfield
Southampton, SO17 1BJ
U.K.

(Received February 9, 1998)

These two lectures cover some of the advances that underpin recent progress in deriving continuum solutions from the exact renormalization group. We concentrate on concepts and on exact non-perturbative statements, but in the process will describe how real non-perturbative calculations can be done, particularly within derivative expansion approximations. An effort has been made to keep the lectures pedagogical and self-contained. Topics covered are the derivation of the flow equations, their equivalence, continuum limits, perturbation theory, truncations, derivative expansions, identification of fixed points and eigenoperators, and the role of reparametrization invariance. Some new material is included, in particular a demonstration of non-perturbative renormalisability, and a discussion of ultraviolet renormalons.

§1. Introduction

As stated above, these lectures will concentrate on exact statements, the conceptual advances, in the exact renormalization group, a.k.a. Wilson’s continuous renormalization group.¹ This is motivated by the belief that these are ultimately the most important aspects of the recent progress, but at the same time this viewpoint lends itself to a (hopefully) elegant and pedagogical introduction to this area. This means however that applications will not be reviewed, or practical matters such as the accuracy of approximations discussed per se. Individuals interested to learn more about these issues, are encouraged to consult our reviews²,³ and the lectures by Aoki and Wetterich in this volume. Suffice to say here that there are approximations, in particular the derivative expansion, which give fair to accurate numerical results in practice. The motivation fueling the recent progress is the need to derive better analytic approximation methods for truly non-perturbative quantum field theory, i.e. where there are no small parameters⁴ one can fruitfully expand in. There is a clear need for such approaches, of course within the archetypical example – low energy QCD, but perhaps more importantly in the need to better understand (even qualitatively) the possibilities offered by the full parameter space of non-perturbative quantum field theories, such as may explain some of the mysteries of the symmetry breaking sector of the Standard Model (for example), and/or Planck scale physics. On the other hand, the issue of renormalisability, which in many approaches ap-

¹ Typical small parameters that are sometimes useful are small coupling, i.e. perturbation theory, or 1/N where N is the number of components of a field, or $\epsilon = 4 - D$ where D is the space-time dimension.
pears as a subtle problem – particularly so for approximations that do not rely on expansion in some small parameter \(^4\), is essentially trivial within the exact Renormalization Group (RG), as we will see. This means that within the framework of the continuous RG, almost all approximations preserve a crucial property of quantum field theory, namely the existence of a continuum limit. Moreover, as we will see, this framework allows us to find systematically\(^*) all possible non-perturbative continuum limits within the infinite dimensional parameter space of all possible quantum field theories with a given field content and symmetries.

As mentioned briefly in the abstract, the topics covered are as follows. In sect. 2, we cover the derivation of the exact RG flow equations, Polchinski’s version and the Legendre flow equation, in such a way that it is clear that they correspond to integrating out modes and that they are equivalent to each other. In particular we emphasize the important consequence that the Green functions of the theory may be extracted directly from the Wilsonian effective action. In sect. 3, we show how solutions for the effective action corresponding to continuum limits may be accessed directly in renormalised terms, and sketch a non-perturbative proof of self-similarity (renormalisability) of continuum solutions. We delineate the rôle of fixed points, eigenperturbations and renormalised trajectories. We show that the direct solution in renormalised variables follows particularly simply in perturbation theory via a process of iteration. We use this to discuss the (non)existence of the continuum limit of four dimensional scalar field theory and show how this is related to the appearance of ultraviolet renormalons, whose existence follows very naturally in this formalism. In sect. 4, we briefly cover non-perturbative approximations, noting that these preserve renormalisability. In sects. 5 and 6, we use the local potential approximation to show how fixed points are determined by the requirement that they are non-singular, and eigenperturbations are determined through the requirement of self-similarity. Finally, in sect. 7, we show how the fields anomalous dimension is determined through the property of reparametrization invariance. We demonstrate the existence of this symmetry in a simple example (the Gaussian fixed point), and explain how it may be maintained in derivative expansions, and the problem that arises if it is broken.

\section*{2. The RG flow equations}

The basic idea behind the (continuous) RG is illustrated in fig.1. Rather than integrate over all momentum modes \(q\) in one go, one first integrates out modes between a cutoff scale \(\Lambda_0\) and a very much lower energy scale \(\Lambda\). Both of these scales are introduced by hand. The remaining integral from \(\Lambda\) to zero may again be expressed as a partition function, but the bare action \(S^\text{tot}_{\Lambda_0}\) (which is typically chosen to be as simple a functional as possible) is replaced by a complicated effective action \(S^\text{eff}_\Lambda\) and the overall cutoff \(\Lambda_0\) by the effective cutoff \(\Lambda\), in such a way that all physics i.e. all Green functions, are left invariant. It may seem at first sight that such a partial integration step merely complicates the issue. For example, we have

\footnote{\(^*)\) in some approximation scheme, e.g. derivative expansion}
had to replace the (generally) simple $S_{\Lambda_0}^{\text{tot}}$ by a complicated $S_{\Lambda}^{\text{tot}}$. However, for the most part the complicated nature of $S_{\Lambda}^{\text{tot}}$ merely expresses the fact that quantum field theory itself is complicated. Indeed from fig.1, we see that we can regard the cutoff $\Lambda$ as an infrared cutoff for the modes $q$ that have already been integrated out. Thus we should expect that $\exp -S_{\Lambda}^{\text{tot}}$ is (more or less) the original partition function for the quantum field theory, but modified by an infrared cutoff $\Lambda$. This means in particular, that we can recover all Green functions of the theory from $\lim_{\Lambda \to 0} S_{\Lambda}^{\text{tot}}$. This statement is surprising if one adheres to the view that Wilsonian RG steps involve a loss of information (and thus a complete loss of information when $\Lambda \to 0$), but it lies at the heart of why the present techniques allow valuable approximation methods for quantum field theory. Therefore let us sketch a proof.

We will introduce the effective ultraviolet cutoff by modifying propagators $1/q^2$ to $\Delta_{UV} = C_{UV}(q,\Lambda)/q^2$, where $C_{UV}$ is a profile that acts as an ultra-violet cutoff, i.e. $C_{UV}(0,\Lambda) = 1$ and $C_{UV} \to 0$ (sufficiently fast) as $q \to \infty$. Similarly, we introduce an infrared cutoff by modifying propagators $1/q^2$ to $\Delta_{IR} = C_{IR}(q,\Lambda)/q^2$, where $C_{IR}$ is a profile with the properties $C_{IR}(0,\Lambda) = 0$ and $C_{IR} \to 1$ as $q \to \infty$. We require that the two cutoffs are related as follows,

$$C_{IR}(q,\Lambda) + C_{UV}(q,\Lambda) = 1 \quad (2.1)$$

Then we have the identity (up to a constant of proportionality$^*$),

$$Z[J] = \int D\varphi \exp \{-\frac{1}{2} \varphi \cdot q^2 \cdot \varphi - S_{\Lambda_0}[\varphi] + J \cdot \varphi\} \quad (2.2)$$

$$= \int D\varphi_<> D\varphi_> \exp \{-\frac{1}{2} \varphi_> \cdot \Delta_{IR}^{-1} \varphi_> - \frac{1}{2} \varphi_< \cdot \Delta_{UV}^{-1} \varphi_< \}$$

$$- S_{\Lambda_0}[\varphi_<> + \varphi_<] + J.(\varphi_> + \varphi_<) \} \quad (2.3)$$

$^*$ From now on we will drop these uninteresting constants of proportionality.
where $S_{A_0}$ is the interaction part of the bare action. In view of their respective propagators, we interpret the $\varphi>$ field as the momentum modes higher than $\Lambda$, and the $\varphi<$ field as the modes that are lower than $\Lambda$. [But please note that the truth is fuzzier unless the cutoff is $C_{UV} = \theta(A - q)$, i.e. sharp. When the cutoff is smooth, modes lower (higher) than $\Lambda$ in $\varphi>$ ($\varphi<$) are only damped.]

To see that the identity (2.3) is true perturbatively, note that (as illustrated in fig.2) as a consequence of the sum form in the interactions, every Feynman diagram constructed from (2.2) now appears twice for every internal propagator it contains: once with $1/q^2$ replaced by $\Delta_{UV}$ and once with $1/q^2$ replaced by $\Delta_{IR}$. Thus for every such propagator line, what actually counts is the sum, which is however just $1/q^2$ again, by (2.1). The non-perturbative proof is almost as trivial: one simply makes some shifts on the fields and performs a Gaussian integration.  \[4\]

\[
\Delta_{UV} + \Delta_{IR} = 1/q^2
\]

Fig. 2. Feynman diagram representation of the identity (2.3).

Now consider only integrating over the higher modes:

\[
Z_{A}[J, \varphi<] = \int \mathcal{D}\varphi> \ exp\left\{-\frac{1}{2}\varphi> \cdot \Delta_{IR}^{-1} \cdot \varphi> - S_{A_0}[\varphi> + \varphi<] + J(\varphi> + \varphi<)\right\}. \tag{2.4}
\]

By a similar shift of variables, it is straightforward to show that $Z_A$ does not depend on both $J$ and $\varphi<$ independently, but essentially only on the sum

\[
\varphi = \Delta_{IR} \cdot J + \varphi<. \tag{2.5}
\]

The exact statement is \[4\]

\[
Z_{A}[J, \varphi<] = \exp\left\{\frac{1}{2}J \cdot \Delta_{IR} \cdot J + J \cdot \varphi< - S_{A}[\Delta_{IR} \cdot J + \varphi<]\right\}, \tag{2.6}
\]

for some functional $S_A$. What is the meaning of $S_A$? In contrast to some works \[7\] we have not restricted the support of $J$ to low energy modes only. Had we done so, we would have had $\Delta_{IR} \cdot J = 0$. In this case (2.6) simplifies, but from (2.4) and (2.3),

\[
Z[J] = \int \mathcal{D}\varphi< Z_{A}[J, \varphi<] \ exp\left\{-\frac{1}{2}\varphi< \cdot \Delta_{UV}^{-1} \cdot \varphi<\right\}. \tag{2.7}
\]

We see that $S_A$ is nothing but the interaction part of the Wilsonian effective action $S_{A_{\text{tot}}}$. This is a nice result. There is no price to pay for letting $J$ couple to all modes:
although the effective dependence on \( J \) is now non-linear, its dependence is no worse than that already contained in \( S_A \). As we see from (2.5), the dependence on \( J \), is essentially carried by the higher modes of \( \varphi \).

The exact RG equations, or ‘flow equations’, follow readily from the fact that (2.4) depends on \( \Lambda \) only through the \( \varphi > \cdot \Delta_{IR}^{-1} \cdot \varphi > \) term. Thus, differentiating \( Z_A \) with respect to \( \Lambda \) we obtain immediately the flow equation for \( Z_A \):

\[
\frac{\partial}{\partial \Lambda} Z_A[\varphi <, J] = \frac{1}{2} \left( \frac{\delta}{\delta J} - \varphi < \right) \cdot \left( \frac{\partial}{\partial \Lambda} \Delta_{IR}^{-1} \right) \cdot \left( \frac{\delta}{\delta J} - \varphi < \right) Z_A .
\] (2.8)

And substituting (2.6), yields Polchinski’s version of Wilson’s flow equation:

\[
\frac{\partial}{\partial \Lambda} S_A[\varphi] = \frac{1}{2} \left( \frac{\delta S_A}{\delta \varphi} \cdot \frac{\partial \Delta_{UV}}{\partial \Lambda} \cdot \frac{\delta S_A}{\delta \varphi} - \frac{1}{2} \text{tr} \frac{\partial \Delta_{UV}}{\partial \Lambda} \cdot \frac{\delta^2 S_A}{\delta \varphi \delta \varphi} \right) \cdot \varphi > .
\] (2.9)

(Of course here \( \partial / \partial \Lambda \) is to be taken at constant \( \varphi \).) On the other hand, if we recognize (2.4) as a partition function for an infrared cutoff theory (in an ‘external’ field \( \varphi < \)), we can by the standard formulae, construct the Legendre effective action \( \Gamma_{tot}[^c, \varphi <] \).

Here \( \varphi ^c \) is the classical field (defined in the usual way by \( \varphi ^c = \delta \ln Z_A / \delta J \)). The fact that \( Z_A \)’s dependence is prescribed through (2.6), turns out to imply that \( \Gamma_{tot} \)’s dependence on \( \varphi < \) is very simple:

\[
\Gamma_{tot}[^c, \varphi <] = \frac{1}{2} (\varphi ^c - \varphi <) \cdot \Delta_{IR}^{-1} \cdot (\varphi ^c - \varphi <) + \Gamma_A[\varphi ^c] .
\] (2.10)

\( \Gamma_A \) contains the effective interactions of the Legendre effective action. Substituting the Legendre transform equations into (2.8), one readily obtains its flow equation:

\[
\frac{\partial}{\partial \Lambda} \Gamma_A[\varphi ^c] = -\frac{1}{2} \text{tr} \left[ \frac{1}{\Delta_{IR}} \cdot \left( 1 + \Delta_{IR} \cdot \frac{\delta^2 \Gamma_A}{\delta \varphi ^c \delta \varphi ^c} \right)^{-1} \right] .
\] (2.11)

From (2.6), it is clear that \( S_A \) is (essentially) the generator of connected Green functions in the infrared cutoff theory. Therefore, there is a Legendre transform relation that maps between the Wilsonian effective action and (infrared cutoff) Legendre effective action:

\[
S_A[\varphi] = \Gamma_A[\varphi ^c] + \frac{1}{2} (\varphi ^c - \varphi) \cdot \Delta_{IR}^{-1} \cdot (\varphi ^c - \varphi) .
\] (2.12)

We see that the \( \Lambda \to 0 \) limit of the Wilsonian effective action can be related to the standard Legendre effective action through \( \Gamma[^c] = \lim_{\Lambda \to 0} \Gamma_A[\varphi ^c] \), and hence to Green functions, S matrices, classical effective potentials, and so forth. On the other hand, since the infrared cutoff Legendre effective action is just a Legendre transform of the Wilsonian effective action, we can expect it also to have fixed point and self-similar RG behaviour.

To see this RG behaviour however, it is necessary to add the other essential ingredient of an RG ‘blocking’ step: scaling the cutoff back to its original size. Simpler and equivalent, is to ensure that all variables are ‘measured’ in units of \( \Lambda \), i.e. we change variables to ones that are dimensionless, by dividing by \( \Lambda \) raised to the power of their scaling dimensions. From now on, we will assume that this has been done.
§3. Continuum limits

This RG method of describing quantum field theory becomes advantageous when we consider the continuum limit. We will indicate how one can solve the flow equations in this case, directly in the continuum, dispensing with the standard, but for quantum field theory, actually artificial and extraneous, scaffolding of imposing an overall cutoff \( \Lambda_0 \), finding a sufficiently general bare action \( S_{\Lambda_0} \), and then tuning to a continuum limit as \( \Lambda_0 \to \infty \).

The simplest case corresponds to a fixed point of the flow, \( S_A = S_* \) with

\[
A_0 \frac{\partial}{\partial A} S_\Lambda[\varphi] = 0 .
\]

(3.1)

(We will write \( S_A \) here and later, but the same comments apply for its Legendre transform \( \Gamma_A \).) Because all dimensionful variables have been exchanged for dimensionless ones using \( \Lambda \), independence of \( \Lambda \) implies that \( S_* \) depends on no scale at all, i.e. corresponds to physics that is scale invariant, and thus in particular describes a massless continuum limit. (You can see that it must be massless, for otherwise the mass would set the scale. On the other hand, it must be a continuum limit, corresponding to \( \Lambda_0 \to \infty \), for otherwise \( \Lambda_0 \) would set the scale.) Nota bene, massless continuum limits thus follow directly from fixed points (3.1); no tuning or bare actions required!

In the massive case, as a consequence of the fact that the flow equations are sensitive only to momenta of order \( \Lambda \), we can obtain the continuum solutions directly in terms of the renormalized variables, viz. the renormalised field \( \varphi \), and the relevant and marginally-relevant couplings, say \( g^1 \) to \( g^n \), and the anomalous dimension (wavefunction renormalization) \( \gamma \):

\[
S_A[\varphi] = S[\varphi] \left( g^1(A), \cdots, g^n(A), \gamma(A) \right) .
\]

(3.2)

Note that all the scale dependence appears only in the couplings and \( \gamma \). This self-similar evolution is equivalent to the statement of renormalisability (because it shows that \( S_A \) does not explicitly depend on \( \Lambda_0/A \)). Let us show, as advertised, that this property, the subject of long and subtle perturbative proofs, follows essentially trivially in this framework, even non-perturbatively.

3.1. Renormalisability

Before doing this however, we need to recall the standard lore\(^1\) on how a non-perturbative massive continuum limit is obtained in this framework. This is illustrated in fig.3. In the infinite dimensional space of bare actions, there is the so-called critical manifold, which consists of all bare actions yielding a given massless continuum limit. Any point on this manifold – i.e. any such bare action – flows under a given RG towards its fixed point; local to the fixed point, the critical manifold is spanned by the infinite set of irrelevant operators. The other directions emanating out of the critical manifold at the fixed point, are spanned by relevant and relevant and

---

\(^{1}\) i.e. scaled to give a normalised kinetic term

\(^{**}\) masses are included as one of the relevant couplings
marginally relevant perturbations (with RG eigenvalues $\lambda_i > 0$ and $\lambda_i = 0$, respectively). Choosing an appropriate parametrization of the bare action, we move a little bit away from the critical manifold. The trajectory of the RG will to begin with, move towards the fixed point, but then shoot away along one of the relevant directions towards the so-called high temperature fixed point which represents an infinitely massive quantum field theory.

To obtain the continuum limit, and thus finite masses, one must now tune the bare action back towards the critical manifold and at the same time, reexpress physical quantities in renormalised terms appropriate for the diverging correlation length. In the limit that the bare action touches the critical manifold, the RG trajectory splits into two: a part that goes right into the fixed point, and a second part that emanates out from the fixed point along the relevant directions. This path is known as a Renormalised Trajectory (RT). The effective actions on this path are ‘perfect actions’, \( \text{10) } \) In terms of renormalised quantities, the far end of this path obtains a finite limit, namely the effective action of the continuum quantum field theory.

Therefore, to obtain the massive continuum limit directly we must first describe the RT. Clearly from the above discussion, the RT is fixed by specifying that it emanates from the fixed point and giving the ‘rates’ in the relevant and marginally relevant directions:

\[
S_A = S_\star[\varphi] + \sum_{i=1}^{n} \alpha^i (\mu/A)^{\lambda_i} \mathcal{O}_i[\varphi] \quad \text{as} \quad A \to \infty . \quad (3.3)
\]

Here the $\mathcal{O}_i[\varphi]$ are the eigenperturbations conjugate to the couplings $g^i$, the $\alpha^i$ are integration constants – the ‘rates’, (which should be taken finite\(^\text{11)}\)) and $\mu$ is as usual, an arbitrary finite mass scale. Since the RG equations are first-order in $A$, it is sufficient for a trajectory to give a boundary value for $S_A$ (at some point $A$) except at a fixed point, which is a so-called singular point for the differential equation. Here this slightly more subtle boundary condition is required. The power law behaviour
in $\Lambda$ follows simply from expansion of the flow equations (2.9,2.11) to first order in
the perturbation, and by separation of variables

(Actually, for marginal perturbations, $\lambda = 0$, it is necessary to follow the evolution
to second order and the power law behaviour in (3.3) is replaced by logarithmic
evolution. For marginally relevant perturbations the multiplying factor still decays
back into the fixed point as $\Lambda \to \infty$. We will ignore these minor complications in
this subsection and persist with formulae appropriate for the strictly relevant cases
$\lambda > 0$. There is however a more subtle issue buried in our assumption that the per-
turbations can be treated to first order. Since (3.3) incorporates a limit as $\Lambda \to \infty$,
this assumption looks innocent enough for $\lambda > 0$, but we will see later that it is true
only for certain quantized perturbations.)

The boundary condition (3.3) and the RG flow equation, completely specify the
massive continuum limit, i.e. the continuum limit is fully specified as

$$S_{\Lambda}[\varphi] \equiv S_{\Lambda}[\varphi](\alpha^1, \cdots, \alpha^n) \quad . \quad (3.4)$$

Once again, this is achieved directly – without $A_0$, bare actions or tuning. Let us
define the renormalised couplings $g^i(\Lambda)$ such that

$$g^i \sim \alpha^i(\mu/\Lambda)^{\lambda_i} \quad \text{as} \quad \Lambda \to \infty \quad , \quad (3.5)$$

i.e. define renormalization conditions consistent with the form of the $O_i[\varphi]$. Evi-
dently, by applying the renormalization conditions directly to (3.4) we may explicitly
read off the renormalized couplings: $g^i \equiv g^i(\alpha, \Lambda)$. [For example in four dimensional
$\lambda \varphi^4$ theory, the renormalization condition might be to define $\lambda(\alpha, \Lambda)$] But given the functions $g^i(\Lambda)$ and the values of the cou-
plings, we can invert to find $\Lambda$, and the $\alpha^i = \lim_{\Lambda \to \infty}(\Lambda/\mu)^{\lambda_i}g^i(\Lambda)$. Therefore the
couplings $g(\Lambda)$ [and $\gamma(\Lambda)$] provide entirely equivalent information for specifying the
solution as the $\alpha$ and $\Lambda$. Exchanging the latter for the former in (3.4), gives (3.2),
and renormalisability is thus shown, as required.

Using the renormalization conditions on the flow equation (2.9) (which scaled,
does not depend explicitly on $\Lambda$), we read off from the left hand side the functions
$\beta^i = \Lambda \partial g^i / \partial \Lambda$, and from the right hand sides explicit non-perturbative expressions
for these $\beta^i \equiv \beta^i(\alpha, \gamma)$. This self-similar form follows directly from (3.2).

3.2. Perturbation Theory

In perturbation theory, this direct continuum solution of the flow equations,
follows particularly simply, by iteration. This was demonstrated in a model ap-
proximation of four dimensional $\lambda \varphi^4$ theory\(^4\) and it will serve here as a simple
illustration. To construct the model we expanded both sides of the flow equation in
$\varphi$ and momenta, and kept only the coefficients in front of $\varphi^4$ and a certain irrelevant
operator:

$$\beta = \beta^i(\alpha, \gamma) \equiv \frac{3}{(4\pi)^2}(\lambda + 2\gamma_1)^2 \quad \text{(3.6)}$$

$$\Lambda \frac{\partial \gamma_1}{\partial \Lambda} = \gamma_1 - \frac{1}{24\pi^3}(\lambda + 2\gamma_1)(\lambda - [3\pi + 2]\gamma_1) \quad . \quad (3.7)$$
Here $\lambda(A)$ is the four point coupling, and $\gamma_1(A)$ the irrelevant coupling. (The irrelevant operator’s precise form is immaterial for the present purposes. It arises in the momentum expansion of the sharp cutoff flow equations and is responsible for 99% of the two-loop $\beta$ function coefficient.\textsuperscript{4,12})

Solving these equations numerically, starting with $\lambda(A_0) = \lambda_0$ and $\gamma_1(A_0) = 0$, results in the curves shown in fig.4.\textsuperscript{4} As can be seen from the curves, the irrelevant coupling $\gamma_1$ decays into what appears to be a well defined Renormalised Trajectory for $\lambda \sim 4$. (Actually this term is a misnomer here, as we discuss in the next subsection.) In this regime, we may solve the equations (3.6,3.7) perturbatively as follows. We recognize from (3.6) that $\beta$ is at least $O(\lambda^2)$, which implies from (3.7) that $\gamma_1 = \lambda^2/(24\pi^3) + O(\lambda^3)$. But given this information, (3.6) yields the first two orders, $O(\lambda^2)$ and $O(\lambda^3)$, in the $\beta$ function. Substituting these results in (3.7) yields the $O(\lambda^3)$ part of $\gamma_1$ and so on:\textsuperscript{4}

$$\gamma_1 = \frac{1}{24\pi^3}\lambda^2 + \frac{1}{96\pi^5}\lambda^3 + \cdots$$

$$\beta = \frac{3\lambda^2}{(4\pi)^2} + \frac{8}{\pi}(\frac{\lambda^3}{(4\pi)^4} + \cdots$$

Similarly, the perturbative solution for the full theory may be worked out directly in renormalised terms.\textsuperscript{13,14}

3.3. \textit{Triviality and renormalons}

Since we can in this way compute directly the RT, without having to worry about constructing a bare action and bare couplings, does this mean that after all, an interacting continuum limit for four dimensional $\lambda\varphi^4$ theory exists? The answer is of course negative. But it is instructive to understand why. Firstly, one must understand that the solution (3.8,3.9) (and the equivalent perturbative solution of the full theory) does \textit{not} parametrize the RT.\textsuperscript{*} The coupling $\lambda$ being marginally

\textsuperscript{*} contrary to statements in the literature.\textsuperscript{14}
irrelevant, sinks back into the Gaussian fixed point $\lambda = \gamma_1 = 0$; there is actually only one direction out of the critical surface\cite{footnote1} along $O \sim \varphi^2$ (plus quantum corrections), corresponding to a RT which describes a massive but non-interacting, i.e. trivial, theory.

Nevertheless, the trajectory (3.8,3.9) corresponds to an apparently unique trajectory within the critical surface and one might wonder whether this provides a clue to a non-perturbative continuum limit for scalar theory. Suppose for example, that there exists a non-perturbative ultraviolet fixed point.\cite{footnote1} Then there would indeed be a unique trajectory, lying in the critical surface, from this fixed point to the Gaussian fixed point, as sketched in fig.5. The new fixed point would then provide

![Fig. 5. A nontrivial fixed point would define a unique trajectory in the critical surface.](image)

the basis for a continuum limit, whose infrared behaviour could still be controlled by the Gaussian fixed point, by tuning the bare action along the relevant directions out of the new fixed point. However this scenario is also false. As we will see later there are no other fixed points (at least within the local potential approximation). And importantly, the apparent uniqueness of (3.8,3.9) is illusory. Clearly, one can readily generate very high orders of perturbation theory in (3.8,3.9). As hinted at the end of the introduction in ref.\cite{4}, the resulting series is divergent and non-Borel summable, which means that the series does not in itself determine a unique trajectory. This is a consequence of ultraviolet renormalons.\cite{15} The ambiguity can be determined by solving for a linearised perturbation $\delta \gamma_1(\lambda)$ to (3.8,3.9) in (3.6,3.7) (indicated on fig.4). Neglecting multiplicative perturbative corrections, we find

$$\delta \gamma_1(\lambda) \propto \lambda^\tau e^{-1/(\beta_1 \lambda)} ,$$

(3.10)

where the one-loop beta-function coefficient $\beta_1 = 3/(4\pi)^2$, and in this approximation $\tau = 2/3 - 8/(9\pi)$. (It is important to recognize that nothing about the Gaussian

\footnote{Footnote 1} here neglected
fixed point determines the proportionality constant here: all $\lambda$ derivatives of $\delta \gamma_1$ are zero at the Gaussian fixed point, so the fact that the trajectory points uniquely along the marginal direction here is insufficient to rule out such perturbations.)

Integrating the $\beta$ function (3.9) shows that (3.10) is nothing but the irrelevant perturbation $\delta \gamma_1 \sim \Lambda/\Lambda_0$ (plus quantum corrections) rewritten in terms of $\lambda$, as again should be expected. On the other hand, if (3.10) delineates the leading singularity of the non-perturbative part of $\gamma_1(\lambda)$ in the complex $\lambda$-plane, then for high orders in $\lambda$, by Cauchy’s theorem:

$$\gamma_1(\lambda) \sim A\int_0^\infty dg \frac{g^{\tau - 1}}{g - \lambda} e^{-1/(\beta_1 g)} ,$$

(3.11)

for some coefficient $A$. Expanding this as a perturbation series in $\lambda$, we obtain

$$\gamma_1(\lambda) \sim A \sum_m (\beta_1 \lambda)^m \Gamma(m - \tau) .$$

(3.12)

This indeed matches the large order behaviour of the perturbation series (3.8). This is the phenomenon of resurgence in asymptotic series. Using the large order terms of (3.8), we can determine a value for $A$ which corresponds to the simplest Borel contour. However, this yields a complex renormalon coefficient for (3.10).

We see that the way renormalons appear, at least the ultraviolet ones, is particularly straightforward and intuitive in this framework, as might well have been expected. The same behaviour will appear in the full theory, with an infinite number of ultraviolet renormalon contributions, one from each irrelevant operator.

§4. Approximations

After this diversion into the world of perturbation theory, we return to our purely non-perturbative discussion. We have seen that all continuum limits follow directly from the fixed points $S_\star[\phi]$ and marginally relevant, and relevant eigenoperators $O_i[\phi]$, and their associated RG eigenvalues $\lambda_i \geq 0$.

Since any approximation which preserves the fact that the RG flow equations are non-linear will continue to have this structure of fixed points and perturbations, any such approximation will preserve the existence of continuum limits, and thus renormalisability and self similar flows: $S[\phi](g, \gamma)$. For a particular fixed point, it is only necessary to ensure that its desired qualitative features (e.g. the number of relevant and marginally relevant eigenperturbations) are reproduced in the approximation.

4.1. Truncations

The simplest form of approximation is to truncate the effective action $S_A$ so that it contains just a few operators. The $\Lambda$ dependent coefficients of these operators then have flow equations determined by equating coefficients on the left and right hand side of (2.9) [or (2.11)], after rejecting from the right hand side of (2.9) all terms that do not ‘fit’ into this set (this being the approximation). The difficulty with

---

*The peculiar scaling dimension is again a consequence of sharp cutoffs. For smooth cutoffs only even dimensions would appear.
this approximation is that it inevitably results in a truncated expansion in powers of the field $\phi$ (about some point), which can only be sensible if the field $\phi$ does not fluctuate very much, which is the same as saying that it is close to mean field,\textsuperscript{3} i.e. in a setting in which weak coupling perturbation theory\textsuperscript{*} is anyway valid. This is precisely the opposite regime from the truly non-perturbative one that concerns us here. (Generically in this regime, one finds that higher orders of truncation cease to converge and reliability is lost since many spurious fixed points are generated,\textsuperscript{16})

It is hard to see how this, admittedly qualitative, argument can fail in practice. However, truncations of powers of the field $\phi$ around the minimum of the effective potential in scalar field theory appear to provide an exception.\textsuperscript{17},\textsuperscript{21},\textsuperscript{22},\textsuperscript{2} At high orders of truncation it is possible to obtain as much as 9 digits accuracy,\textsuperscript{23} before succumbing to the generic pattern for finite truncations as outlined above.\textsuperscript{25},\textsuperscript{16} Recalling the above qualitative argument, the success of these truncations suggests to me that some sort of perturbation theory may in fact be applicable to this case in practice.

4.2. Derivative expansion

A less severe, more natural, and more accurate expansion, closely allied to the successful truncations in real space RG of spin systems,\textsuperscript{3},\textsuperscript{2} is rather to perform a ‘short distance expansion’\textsuperscript{3},\textsuperscript{12} of the effective action $S_A$, which for smooth cutoff profiles corresponds to a derivative expansion:\textsuperscript{9},\textsuperscript{11},\textsuperscript{26} - \textsuperscript{30}

$$S_A \sim \int d^Dx \left\{ V(\phi, \Lambda) + \frac{1}{2}(\partial_\mu \phi)^2 K(\phi, \Lambda) + O(\partial^4) \right\} .$$ \hspace{1cm} (4.1)

[In the $N \neq 1$ component case there is a second $O(\partial^2)$ term: $\frac{1}{2}(\phi^a \partial_\mu \phi^a)^2 Z(\phi, \Lambda).$]

The simplest such approximation is the so-called Local Potential Approximation (LPA), introduced by Nicoll, Chang and Stanley:\textsuperscript{31}

$$S_A \sim \int d^Dx \left\{ V(\phi, \Lambda) + \frac{1}{2}(\partial_\mu \phi)^2 \right\} .$$ \hspace{1cm} (4.2)

It has since been rediscovered by many authors,\textsuperscript{12} notably Hasenfratz and Hasenfratz.\textsuperscript{32} As a concrete example, consider the case of sharp cutoff. The flow equations may be shown to reduce to\textsuperscript{31},\textsuperscript{32},\textsuperscript{4},\textsuperscript{16},\textsuperscript{12}

$$\frac{\partial}{\partial t} V(\phi, t) + d \phi V' - DV = \ln(1 + V'') ,$$ \hspace{1cm} (4.3)

where $t \equiv \frac{\partial}{\partial \phi}$, $t = \ln(\mu/\Lambda)$ and $d = \frac{1}{2}(D - 2)$. In the $N$ component case, the right hand side of (4.3) has an extra term $+(N - 1)\ln(1 + V'/\phi)$ (and $\phi$ stands for the length of the $\phi^a$ vector), however for the most part it will be sufficient to consider a single component scalar. Actually, the $N = \infty$ case was already derived by Wegner and Houghton in their paper introducing the sharp-cutoff flow equation.\textsuperscript{18} In this limit the LPA is effectively exact.\textsuperscript{19} It follows from our earlier discussion\textsuperscript{19},\textsuperscript{11},\textsuperscript{4} that $V$ is an approximation to the Legendre effective potential in the limit $\Lambda \to 0$, i.e. $t \to \infty$.\textsuperscript{\textsuperscript{*}} or in some settings, large $N$ approximations\textsuperscript{17} - \textsuperscript{20}
§5. Fixed points

We have not yet addressed the question as to how the fixed points and eigenperturbations are determined within the exact RG. We know from other methods, including experiment, that there are generically a discrete set of RG fixed point solutions and a discrete set of eigenperturbations, but here these quantities are determined from functional differential equations (2.9) or (2.11) with thus, apparently, a continuum of solutions. The derivative expansion approximations, also have the property that the RG flow equations are differential equations with a continuum of solutions. We will show how one finds nevertheless (generically) only a discrete set of acceptable solutions within these approximations. Since this is true to all orders of the derivative expansion, we assume that there are only a discrete set of acceptable solutions of the exact RG, for the same reasons.

To begin with, we need only consider the LPA, and for concreteness we will take the example of (4.3). In this case the fixed point potential \( V^* \) satisfies

\[
d \phi V^*_s(\phi) - DV_s(\phi) = \ln(1 + V''_s) .
\]

This equation has indeed a continuum of solutions, in fact a continuous two-parameter set. However, generically all but a countable number of these solutions are singular \(^16,33\) \((D = 2 \text{ dimensions is an exception},27)\). To illustrate this, let us choose \( V'_s(0) = 0 \). (This is anyway necessary if \( N \neq 1,2 \)). This fixes one boundary condition at \( \phi = 0 \). If we choose some numerical value for \( V_s(0) \), we then have the required two boundary conditions and can numerically integrate (5.1) out to positive \( \phi \). We find that almost without exception a singularity is encountered at some critical value of the field \( \phi = \phi_c \). (At a very basic level this is indicated by the failure of the numerical routine near this point, although it is possible to solve analytically for the singularity and use this information to do much better. \(^16,9,27\)) The value of \( \phi_c \) depends on \( V_s(0) \). In fig.6 we plot the results for two examples. The first graph is a plot for the case \( D = 4 \) and \( N = 4 \) (the Higgs field in the Standard Model). We see that only the (trivial) Gaussian fixed point solution \( V(\phi) \equiv 0 \) exists for all values of the field. If the same is done for the case \( D = 3 \) and \( N = 1 \), we get the second
graph. In this case there is also one non-trivial non-singular solution, corresponding to the famous Wilson-Fisher fixed point (Ising model universality class).

Note that this straightforward numerical procedure corresponds nevertheless, within the LPA, to an exhaustive search for continuum limits in the entire infinite dimensional space of all possible potentials \( V(\varphi) \). Similar entire searches are possible at higher orders of the derivative expansion. Clearly this is much more than is possible with other methods!

For large field \( \varphi \) the only consistent behaviour (with \( D > 2 \)) for the fixed point potential in (5.1) is

\[
V_*(\varphi) \sim A\varphi^{D/d},
\]

where \( A \) is a constant determined by the equations. This simply solves the left hand side of (5.1), these terms arising from purely dimensional considerations, and neglects the right hand side of the flow equation – which encodes the quantum corrections. Or in other words, (5.2) is precisely what would be expected by dimensions (since \( V \)'s mass-dimension is \( D \) and \( \varphi \)'s is \( d \)) providing only that any dependence on \( \Lambda \), and thus the remaining quantum corrections, can be neglected. Requiring the form (5.2) to hold for both \( \varphi \to \infty \) and \( \varphi \to -\infty \), provides the necessary two boundary conditions for the second order ordinary differential equation (5.1), so we should indeed generally expect at most a discrete set of globally non-singular solutions. These considerations generalise to any order of the derivative expansion, and indeed we thus expect them to hold also for the exact RG. (There is one modification: beyond LPA, \( d = \frac{1}{2}(D - 2 + \eta) \), where \( \eta \) is the anomalous dimension at the fixed point. We will discuss this later.)

§ 6. Eigenoperators

Now consider the determination of the eigenoperators. For this we perturb away from the fixed point:

\[
V(\varphi, t) = V_*(\varphi) + v(\varphi, t).
\]

For \( v \ll V_* \), we can expand (4.3) to first order in \( v \). Then by separation of variables,

\[
v(\varphi, t) = \alpha e^{\lambda t} u(\varphi),
\]

as in (3.3), where \( \alpha \) is a small parameter and \( u(\varphi) \) is some (normalised) solution of

\[
\lambda u + d\varphi u' - Du = \frac{u''}{1 + V_*^2}.
\]

We again have a two parameter continuum of solutions, but in this case each one is guaranteed globally well defined since (6.3) is linear, and this is true for every value of \( \lambda \). How can this be squared with the expectation of only a discrete spectrum of such operators? Firstly, one of the parameters corresponds just to the overall normalisation. The second parameter and \( \lambda \) are fixed however, for a much more subtle reason: only the discrete set of normalised solutions for \( u(\varphi) \) that behave as a power of \( \varphi \) for large field, can be associated with a corresponding renormalised
coupling \( g(t) \) and thus the universal self-similar flow (3.2) which is characteristic of the continuum limit.\(^3\),\(^1\),\(^2\)

Indeed we see from (6.3) and (5.2), that those solutions that behave as a power for large \( \varphi \) must do so as

\[
u(\varphi) \sim \varphi^{(D-\lambda)/d},\]

(6.4)

this being again the required power to balance scaling dimensions (with \([g(t)] = \lambda\)) if the remaining quantum corrections may be neglected in this regime. Once again for \( \varphi \to \pm \infty \), this supplies two boundary conditions, but this time, since (6.3) is linear, this overdetermines the equations, and generically allows only certain quantized values of \( \lambda \).

On the other hand if \( u \) does not behave as a power of \( \varphi \) for large \( \varphi \), then from (6.3) and (5.2), we obtain that instead for large \( \varphi \)

\[
u(\varphi) \sim \exp\{A(D - d)\varphi^{D/d}\}.
\]

(6.5)

(The precise form of the large \( \varphi \) dependence of the non-power-law perturbations depends on non-universal details including the level of derivative expansion approximation used, if any. Universally however, the non-power-law perturbations grow faster than a power, and this is all we will really need.)

To investigate whether these perturbations are associated with renormalised couplings, we must follow the evolution of the perturbed action (6.1) for a small but finite \( v \). Starting, say, with (6.2) at \( t = 0 \) as a boundary condition:

\[
v(\varphi,0) = \alpha u(\varphi),\]

(6.6)

we must show that the evolved solution \( v(\varphi,\alpha,t) \) can be expressed as a self-similar flow, \( v(\varphi,g(t)) \), for some renormalised coupling \( g \).

Now by (5.2), for all perturbations behaving as (6.5), or (6.4) when \( \lambda \leq 0 \), there is potentially a problem because for any small but finite \( \alpha \), there will always be a \( \varphi \) large enough where we cannot treat \( v(\varphi,\varphi) \) as small compared to \( V_\varphi \). Therefore the linearised solution (6.2), and indeed more generally (3.3), needs reexamining in this regime. Fortunately, in the large \( \varphi \) regime, we may solve (4.3) non-perturbatively (and thus without making any assumption on the size of \( v/V_\varphi \)). This is because, just as before, we may neglect the quantum corrections in this regime. These are given by the right hand side of (4.3), and thus \( V(\varphi,\varphi) \) follows mean-field-like evolution:

\[
V(\varphi,t) \sim e^{Dt}V(\varphi e^{-dt},0) .
\]

(6.7)

Applying this to (6.1) and (6.6), we see that in the large \( \varphi \) regime these perturbations evolve as

\[
v(\varphi,\alpha,t) \sim \alpha e^{Dt}u(e^{-dt}) .
\]

(6.8)

For the power-law perturbations, (6.4) then implies

\[
v(\varphi,\alpha,t) \sim \alpha e^{\lambda t}\varphi^{(D-\lambda)/d} ,
\]

(6.9)

which is indeed self-similar with renormalised coupling \( g(t) \sim \alpha e^{\lambda t} \). (We also see that the linearised solution (6.2) is afterall still valid in this regime.) On the other
hand, using (6.5) we see that in the large $\varphi$ regime the non-power-law perturbations behave as

$$v(\varphi, \alpha, t) \sim \alpha \exp \left\{ Dt + A(D - d) e^{-D t} \varphi^{D/d} \right\}. \quad (6.10)$$

This cannot be rewritten as a universal self-similar flow $v(\varphi, g(t))$. Use of the ‘relevant’ non-power-law perturbations as the basis for a Renormalised Trajectory (3.3) would not be valid: on the one hand the linearised solution (6.2) is not valid when $\varphi$ is large, and on the other hand the full solution (6.10) does not fall back into the fixed point as $t \to -\infty$ ($\Lambda \to \infty$). Indeed this limit does not even exist. By Sturm-Liouville analysis, one can further show from (6.10) that these non-power-law perturbations collapse, on increasing $t$, into an infinite sum of the quantized power-law perturbations, and thus the non-power-law eigenperturbations are entirely irrelevant for continuum physics.

§7. Anomalous dimension and reparametrization invariance

So far we have been ignoring the determination of $\eta$, the fields anomalous dimension. Recall that this arises from the anomalous scaling of the field necessary in general to ensure that the kinetic term $\sim (\partial \mu \varphi)^2$ of the effective action is conventionally normalised, and more importantly to ensure that effective actions are finite and do actually achieve fixed points at scale invariant continuum limits. In the LPA, the lowest order of the derivative expansion, all momentum dependent corrections to the effective action are thrown away and thus $\eta$ is always zero in this approximation. Beyond LPA, and in the exact RG, we must scale out the field according to its full scaling dimension (as described at the end of sect.2): $\varphi \sim \Lambda^d$, where now

$$d = \frac{1}{2}(D - 2 + \eta). \quad (7.1)$$

(In this section, it will be sufficient for us to concentrate on the behaviour at fixed points and thus take $\eta$ as independent of $\Lambda$.) The flow equations at higher order in the derivative expansion (4.1), now take the generic form

$$\frac{\partial}{\partial t} V(\varphi, t) + d \varphi V' - D V = \cdots \quad (7.2)$$

$$\frac{\partial}{\partial t} K(\varphi, t) + \eta K = \cdots$$

(and so on for other coefficient functions). Here, the terms on the left of the equation arise once again purely from the assignment of scaling dimensions. The terms on the right of the equations arise from the right hand side of the exact flow equations (2.9,2.11), and as a consequence of the structure of (2.9,2.11), are non-linear and reduce to second order differential equations. As an example, the $O(\partial^2)$ approximation of the Legendre flow equations (2-11) in $D = 3$ dimensions (with a particular form of cutoff $C_{IR}$ which we will discuss later) yields

$$\frac{\partial V}{\partial t} + \frac{1}{2}(1 + \eta) \varphi V' - 3V = -\frac{1 - \eta/4}{\sqrt{K}\sqrt{V''} + 2\sqrt{K}}. \quad (7.3)$$
Elements of the Continuous Renormalization Group

\[ \frac{\partial K}{\partial t} + \frac{1}{2}(1 + \eta)\varphi'K' + \eta K = \left(1 - \frac{\eta}{4}\right) \left\{ \frac{1}{48} \frac{24K'K'' - 19(K')^2}{K^{3/2}(V'' + 2\sqrt{K})^{3/2}} \right\} \]

\[ - \frac{1}{48} \frac{58V'''K'\sqrt{K} + 57(K')^2 + (V'')^2K}{K(V'' + 2\sqrt{K})^{5/2}} + \frac{5}{12} \frac{(V'')^2K + 2V'''K'\sqrt{K} + (K')^2}{\sqrt{K}(V'' + 2\sqrt{K})^{7/2}} \]  

(The perceptive reader will note that the right hand sides of (7.4) actually contain \( V''' \)'s. However, differentiating (7.3) yields an equation for \( V''' \) in terms of expressions with lower derivatives, which when substituted into (7.4) reduces this to second order as claimed.)

Once again, one finds that the fixed point equations generically have at most a discrete set of non-singular solutions, and that this can be understood by studying the large \( \varphi \) behaviour. Indeed we find for large \( \varphi \) that the right hand sides of (7.2) are subleading for the power-law behaviour that solves the left hand sides:

\[ V \sim A_V \varphi^{D/d} \quad K \sim A_K \varphi^{-\eta/d} \]  

etc, where \( A_V \) and \( A_K \) are constants that get determined by the equations. Once again this large \( \varphi \) behaviour is precisely to be expected by scaling dimensions, providing only that any dependence on \( \Lambda \), and thus the remaining quantum corrections, can be neglected in this regime. Requiring that these hold for both \( \varphi \rightarrow \infty \) and \( \varphi \rightarrow -\infty \) provides the required two boundary conditions for each coefficient function, so that we should expect generically at most a discrete set of such globally non-singular solutions. However, this time this is true for each \( \eta \). How then, does \( \eta \) get determined?

In the original partition function (2.2), physics does not depend on the normalization of \( \varphi \), i.e. we may substitute

\[ \varphi \rightarrow \Omega \varphi \]  

in the action while leaving the \( J \cdot \varphi \) term alone, or equivalently mapping only \( J \rightarrow J/\Omega \) and leaving the field alone. More generally we may exchange \( \varphi \) for \( \varphi \) plus any local expression in \( \varphi \). This is the “equivalence theorem”. The continuous RG has a similar “reparametrization” symmetry but in general it is given by a complicated functional integral transform. Thus all fixed points appear as lines of equivalent fixed points generated by an, in general complicated, exactly marginal perturbation. This means that the boundary conditions (7.5), which as we have seen are already sufficient to constrain the fixed point solutions down to a discrete set, actually over constrain the solution space and lead to quantization of \( \eta \) (in a similar way to the linear equations for eigenperturbations). Equivalently, note that we have the freedom to choose an extra boundary condition, a normalization condition, e.g. by requiring a conventionally normalized kinetic term \( K_*(0) = 1 \). For such representatives (of the equivalence classes under reparametrization) the fixed point equations are overconstrained, leading to quantization of \( \eta \). In this way, the reparametrization invariance turns the fixed point equations into non-linear eigenvalue equations for \( \eta \).
There is a problem however for approximations: the derivative expansion, indeed any truncation of the momentum or field dependence, generally breaks the reparametrization invariance, with the result that $\eta$ (and other universal quantities such as the RG eigenvalues) depend on some unphysical parameter such as $K_\star(0)$.\(^{26}\) Note that even without the reparametrization invariance we could choose to fix e.g. $K_\star(0) = 1$. The problem is that, if the invariance is broken, different values for $\eta$ and other universal quantities will be obtained for different normalisations $K_\star(0) \neq 1$, violating the equivalence theorem.\(^a\) Depending on how the equations are parametrized, this ambiguity can be shifted around into different quantities but it is always there if reparametrization invariance is absent. In this sense, the problem can be quite subtle to spot, and has been missed in a number of recent works. But the general question any practitioner must ask is if all fixed point solutions are investigated, where care is taken to ensure that only arbitrary normalisations are set in the solutions, are the results independent of these normalisations? Evidently this is only true if there exists some underlying reparametrization invariance.

We will demonstrate the existence of reparametrization invariance in a simple example. Recall from sect.2, that the Wilsonian effective action is given by

$$S_A^{\text{tot}}[\varphi] = \frac{1}{2} \varphi \cdot \Delta_{UV}^{-1} \cdot \varphi + S_A[\varphi] \quad ,$$

(7.7)

where $\Delta_{UV} = C_{UV}(q, A)/q^2$, and the effective interaction part $S_A$ is governed by the flow equation (2.9). Clearly the Gaussian fixed point is given by $S_A = 0$. But this is not the only Gaussian fixed point solution! Since (2.9) is given in terms of unscaled variables (the change to dimensionless variables at the end of sect.2 having not yet been done), it is easier for this example to work with the unscaled variables. We specialize to $C_{UV} \equiv C_{UV}(q^2/A^2)$.\(^{**}\) If we take as ansatz

$$S_A = \frac{1}{2} \varphi \cdot q^2 z(q^2/A^2) \cdot \varphi \quad ,$$

(7.8)

then any such solution will be a fixed point after the change to dimensionless variables $\varphi \mapsto A^{(D-2)/2} \varphi$, $q \mapsto Aq$. Substituting (7.8) into (2.9), we find that $z' = z^2 C'_{UV}$ (prime being differentiation with respect to its argument). This has general solution $z = 1/(a - C_{UV})$, for some integration constant $a > 1$.\(^{***}\) Thus the general fixed point solution takes the form

$$S_A^{\text{tot}} = \frac{1}{2} \varphi \cdot \frac{a q^2}{C_{UV}(a - C_{UV})} \cdot \varphi \quad .$$

(7.9)

This line of fixed points, parametrized by $a$, is a line of equivalent fixed points as can be confirmed by checking that the spectrum of RG eigenvalues (and thus in particular

\(^a\) Some techniques have been developed to suggest a ‘best’ choice amongst the one-parameter solution set for a given fixed point in these cases.\(^{26,30}\) Similar practical issues exist, in general, for the choice of cutoff function $C_{IR}$: universal quantities are independent of the detailed choice in the exact RG flow equations, but in general for approximations the results do depend on the choice and this raises the issue of choosing a ‘best’ function out of some set.\(^9,28,30\)

\(^{**}\) This is required by the canonical scaling of the Gaussian fixed point and Lorentz invariance.

\(^{***}\) The restriction on $a$ arises because $z$ must be nonsingular. We assume that $C_{UV} \leq 1$. Such is the case if $C_{UV}$ is monotonic for example.
the critical exponents) is still that of the Gaussian fixed point.\textsuperscript{30} The reparametrization invariance that maps between one representative and another clearly has a complicated momentum dependence for general $C_{UV}$. Also, a derivative expansion of (7.9) corresponds to a Taylor expansion in $q^2$, and thus clearly any derivative expansion (to finite order) will destroy the equivalence of these fixed points, and the reparametrization invariance, for general $C_{UV}$.

If we take the cutoff sharp $C_{UV} = \theta(\Lambda - q) = \theta(1 - q^2/\Lambda^2)$, then

\[
\frac{C_{UV}(a - C_{UV})}{a} = \left(\frac{a - 1}{a}\right) C_{UV} \quad .
\]

(7.10)

(as can easily be confirmed by comparing both sides for $q > \Lambda$ and $q < \lambda$). Since this is the inverse of the cutoff terms appearing in (7.9), we see that in the sharp cutoff case the reparametrization invariance is simply the linear momentum independent transformation (7.6). This can be shown to be true in general for the flow equations themselves.\textsuperscript{12,3,30} It is not possible however to use a derivative expansion in the case of a sharp cutoff; instead an expansion in ‘momentum scale’ may be used.\textsuperscript{12}

In general, one may show that the flow equations (2.9,2.11) enjoy a momentum independent and linearly realised reparametrization symmetry if and only if either $C_{IR} = \theta(q - \Lambda)$ i.e. the sharp case just discussed, or $C_{IR}^{-1} = 1 + (A^2/q^2)^k$ — a power-law cutoff.\textsuperscript{*9,27,3}

The power-law case can be used with derivative expansions, but it only regulates the Legendre flow equation (2.11). This is the cutoff that was used (with $k = 2$) to produce the examples (7.3,7.4). The reparametrization invariance takes the form $\varphi \mapsto \varphi t^{2k - 2}, Q \mapsto Q \Omega$. In the examples (7.3,7.4) this implies that the equations are invariant under $\varphi \mapsto t^{5/2} \varphi, V \mapsto t^3 V, K \mapsto t^{-4} K$. It is easy to check that (7.3,7.4) are indeed invariant under this symmetry.

Acknowledgements

It is a pleasure to thank Tsuneo Suzuki and the rest of the organizing committee for the invitation to speak at this pleasurable and well run international seminar, several participants – particularly Ken-Ichi Aoki, Haruhiko Terao and their students, and Peter Hasenfratz, for enjoyable and informative conversations, and finally the SERC/PPARC for financial support through an Advanced Fellowship.

\textsuperscript{*} $k$ should be chosen to be an integer. A function of the scale can be included in front of the monomial.\textsuperscript{9,27}
References

10) See P. Hasenfratz’s lectures in this volume.
25) See K.-I. Aoki’s lectures in this volume.
34) See e.g. E.L. Ince, Ordinary differential equations (1956).
35) See e.g. C. Itzykson and J.-B. Zuber, Quantum field theory (McGraw-Hill, 1980).