Lattice QCD without tuning, mixing and current renormalization

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Abstract

The classically perfect action of QCD requires no tuning to get the pion massless in the broken phase: the critical bare mass \( m_c^q \) is zero. Neither the vector nor the flavour non-singlet axial vector currents need renormalization. Further, there is no mixing between four-fermion operators in different chiral representations. The order parameter of chiral symmetry requires, however a subtraction which is given here explicitly. These results are based on the fact that the fixed point action satisfies the Ginsparg-Wilson remnant chiral symmetry condition. On chiral symmetry related questions any other local solution of this condition will produce similar results.

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1 Introduction

Lattice regularized, local QCD actions break chiral symmetry under general conditions [1]. Wilson fermions [2] have a dimension five symmetry breaking operator whose effect on the physical predictions goes to zero in the continuum limit. It leaves, however its trace behind in the form of additive quark mass renormalization, axial current renormalization and mixing between operators in nominally different chiral representations. There is a significant recent progress in calculating these renormalizations in a theoretically controlled non-perturbative way [3]. Nevertheless, the situation is not really pleasing theoretically and the technical difficulties are also significant.

Staggered fermions [4] keep a part of chiral symmetry intact which offers another possibility to study problems where chiral symmetry is essential [5]. On the other hand, staggered fermions do not solve the doubling problem and constructing operators with correct quantum numbers is far from trivial.

A rather different method to overcome the problems with chiral symmetry is the domain wall fermion [6, 7], or the overlap formalism [8]. Recent results on kaon matrix elements are nice and promising [7]. Similarly to the staggered fermions, the domain wall fermions were motivated by a single issue: to solve the problems of chiral symmetry in the fermion sector.

This paper is part of a project to construct a lattice formulation for QCD which performs well in every respect both in the gauge and in the fermion sectors, including classical solutions, topology, cut-off effects and chiral symmetry. The fixed point (FP) action \( (m_q = 0) \) and the actions on the trajectory along the mass direction \( (m_q \neq 0) \), which are local and determined by saddle point equations, are classically perfect [9, 10, 11, 12]. Among others, the FP action is perfect concerning the classical solutions leading to scale invariant instantons [9, 10, 13, 14, 15] and fermionic chiral zero modes satisfying the index theorem [16]. Even more, as we discuss here, these actions are quantum perfect what concerns chiral symmetry [17]. The problem of constructing the FP action and the mass trajectory by solving the corresponding classical saddle-point equations requires skill and patience. The real difficulty is, however to find a parametrization for these actions which is sufficiently precise and, at the same time, does not make the simulations too expensive. Examples in \( d = 2 \) show [9, 13, 18] that with a limited number of couplings a parametrization can be achieved which performs excellently in simulations and other numerical checks. In \( d = 4 \), although the preliminary results look promising, the parametrizations studied until now are admittedly rather primitive [10, 19].

As the discussion above shows, there exist several possibilities to cope with the problems of chiral symmetry. It might turn out on the long run that none of the new theoretical ideas can compete with the direct approach of using Wilson fermions and calculating renormalization factors, mixing coefficients in different processes and driving the system close to the continuum limit to kill the cut-off effects and to avoid other problems [20]. The author hopes that the solution will be more appealing.

It is somewhat surprising that the FP action respects chiral symmetry in its predictions, since – complying with the Nielsen-Ninomiya theorem [1] –, it breaks chiral symmetry explicitly. The basic observation is that this breaking is realized in a very special way. The FP action satisfies the equation [12, 16] \(^2\)

\[
\frac{1}{2} \{h^{-1}(U), \gamma^5 \} \gamma^5 = R_{n'n'}(U),
\]

or equivalently

\[
\frac{1}{2} \{h_{n'n'}, \gamma^5 \} = (h\gamma^5 Rh)_{n'n'}.
\]

Here \( h(U)^{-1} \) is the FP quark propagator over the background gauge field \( U \) and the matrix \( R_{n'n'} \) is trivial in Dirac space. Both \( h(U)_{nn'} \) and \( R(U)_{nn'} \) are local. The precise form of \( R \) depends on the block transformation whose FP we are considering.

It has been observed a long time ago that eq. (2) with a local \( R \) is the mildest way a local lattice action can

\[^2\] We denote the commutator and anticommutator by \([ , ]\) and \(\{ , \}\), respectively.

break chiral symmetry [21]. Along the mass trajectory \( m_q \neq 0 \) eq. (2) is modified as

\[
\frac{1}{2}\{\hat{h}_{nn'}, \gamma^5\} = (\hat{h} \gamma^5 R \hat{h})_{nn'} + m_q \Sigma^5_{nn'},
\]

where we denoted the action along the mass trajectory by \( \hat{h} \), \( \hat{h} \to h \) as \( m_q \to 0 \). The operator \( \Sigma^5 \) is a local pseudoscalar density which goes to \( \gamma^5 \delta_{nn'} \) in the formal continuum limit. The results of this paper are based on eq. (3), hence any local solution of this equation will lead to the same results. It is interesting in this context that in a recent paper [22] another solution (growing out of the overlap formalism) of eq. (1), which seems to be unrelated to renormalization group considerations, was presented. This is an interesting development, if it can be shown that the Dirac operator in [22] is local over any gauge configuration.

Since the locality of \( h, R \) and the currents is basically important in deriving the following results, let us discuss briefly what this means by this notion. An operator (like action density, topological charge density, current, etc.) on the lattice is local if it has an extension of \( O(a) \): the coupling between the fields in the operator at a distance \( n \) decays exponentially as \( \exp(-\gamma n) \), with \( \gamma = O(1) \), or it is identically zero beyond a certain \( O(a) \) range. In the continuum limit \( (a \to 0) \) the extension measured in physical units goes to zero. A quantum field theory with non-local interactions looses renormalizability and universality in general – this would be certainly too high a price for an 'improvement'.

The remark has been made already in [21] that the soft-pion theorems are expected to remain valid if eq. (2) is satisfied. The intuitive reason is that the two \( h \) factors on the r.h.s. of eq. (2) will cancel the two propagators which connect this term to other operators in the matrixelement producing only a contact term in Ward identities. On the other hand, in current algebra relations, in Ward identities related to mixing and elsewhere, contact terms are relevant, so it is necessary to study in detail what eq. (3) implies on chiral symmetry. Many of our considerations rely on the seminal paper of Bochicchio et al. [23] where the chiral symmetry properties of Wilson fermions are discussed.

### 2 The vector and axial currents

The classically perfect fermion action has the form

\[
S_{FP}^F(\bar{\psi}, \psi, U) = \sum_{m',n'} \bar{\psi}_{m'} \hat{h}_{m'n'}(U) \psi_{n'}. \tag{4}
\]

The Dirac, flavour and colour indices are suppressed in eq. (4). The Dirac operator \( \hat{h} \) is assumed to be flavour independent. The flavour generators are denoted by \( \tau^a, a = 1, \ldots, N_f^2 - 1, \) \( \text{tr}(\tau^a \tau^b) = \frac{1}{2} \delta_{ab} \).

As usual, the axial currents are constructed from the chiral symmetric part of the action using \( \hat{h}_{\text{SYM}} = \frac{1}{2}[\hat{h}, \gamma^5] \gamma^5 \). There is a certain freedom in constructing the vector currents. Using the full action, conserved vector currents are obtained. On the other hand, the asymmetry between the vector and axial case generates strange terms in current algebra relations and it requires extra work to show that they go away. We decided to follow [23] and construct both currents with \( h_{\text{SYM}} \). We get \(^3\)

\[
\nabla_\mu V_\mu^a(n) = \bar{\psi}_n \tau^a \left( \hat{h}_{\text{SYM}} \psi \right)_n - \left( \bar{\psi} \hat{h}_{\text{SYM}} \psi \right)_n \tau^a \psi_n, \tag{5}
\]

\[
\nabla_\mu A_\mu^a(n) = -\bar{\psi}_n \tau^a \left( \gamma^5 \hat{h}_{\text{SYM}} \gamma^5 \psi \right)_n - \left( \bar{\psi} \hat{h}_{\text{SYM}} \gamma^5 \right)_n \tau^a \psi_n. \tag{6}
\]

None of these currents are conserved, at least not without further considerations, if the equations of motion \( \hat{h} \bar{\psi} = 0 \) is used.

\(^3\)We use the standard sign convention for the currents as opposed to that in [21, 16]. We use the notation \( \nabla_\mu f(n) = f(n) - f(n - \mu) \).
In order to find the currents themselves we introduce the flavour gauge matrices $W_\mu (n) = 1 + i w_\mu (n) \tau^a + \ldots$ and extend the product of colour $U$ matrices in eq. (4) along the paths between the fermion offsets $m', n'$ by the product of $W$ matrices along the same paths. Then the vector current is defined by

$$ V_\mu^a (n) = -i \sum_{m', n'} \bar{\psi}_{m'} \frac{\delta}{\delta w_\mu (n)} (\hat{h}_{\text{SYM}})_{m', n'} (U, W) \psi_{n'} \bigg|_{w=0}. $$

This current satisfies eq. (5) \cite{24}. Every path in $(\hat{h}_{\text{SYM}})_{m', n'}$ which goes through the link $(n, n + \hat{\mu})$ gives a contribution to $V_\mu^a (n)$ which is equal to the contribution to $(\hat{h}_{\text{SYM}})_{m', n'}$ times $(s_+ - s_-) \tau^a$, where $s_+$ $(s_-)$ is the number this path runs through $(n, n + \hat{\mu})$ in positive (negative) direction. Eq. (7) can also be written as

$$ V_\mu^a (n) = \sum_{m', n'} \bar{\psi}_{m'} \tau^a \Gamma^5_\mu (m', n' ; n ; U) \psi_{n'}, $$

where $\Gamma_\mu$ is trivial in flavour space. Similar considerations lead to the axial current

$$ A_\mu^a (n) = \sum_{m', n'} \bar{\psi}_{m'} \tau^a \Gamma^5_\mu (m', n' ; n ; U) \psi_{n'}, $$

where

$$ \Gamma^5_\mu = \Gamma_\mu \gamma^5 = -\gamma^5 \Gamma_\mu. $$

3 The basic Ward identity

We shall repeatedly use the following Ward identity which can be obtained by changing integration variables in the QCD path integral (see, for example \cite{23})

$$ i \left\langle \frac{\delta}{\delta \epsilon_n} \mathcal{O}(x_1, \ldots, x_j) \right\rangle = \left\langle \mathcal{O}(x_1, \ldots, x_j) \nabla_\mu A_\mu^a (n) \right\rangle - \left\langle \mathcal{O}(x_1, \ldots, x_j) X^a (n) \right\rangle - \left\langle \mathcal{O}(x_1, \ldots, x_j) 2m_q P^a (n) \right\rangle, $$

where $\mathcal{O}$ is some product of local operators, $\delta \mathcal{O}$ is its change under a local chiral transformation with the infinitesimal parameter $\epsilon_n^a$, $P^a (n)$ is a pseudoscalar density

$$ P^a (n) = \frac{1}{2} \left[ \bar{\psi}_n \tau^a (\Sigma^5 \psi)_n + (\bar{\psi} \Sigma^5)_n \tau^a \psi_n \right], $$

while $X^a$ comes from the remnant chiral symmetry condition eq. (2)

$$ X^a (n) = \bar{\psi}_n \tau^a (\hat{h} \gamma^5 R \hat{h} \psi)_n + (\psi \gamma^5 R \hat{h})_n \tau^a \psi_n. $$

In deriving eq. (11) we used eq. (6), $\hat{h}_{\text{SYM}} = \hat{h} - \frac{1}{2} \{ \hat{h}, \gamma^5 \} \gamma^5$ and eq. (3). For later use we introduce the flavour singlet scalar density $S(n)$

$$ S(n) = \frac{1}{2} \left[ \bar{\psi}_n (\Sigma \psi)_n + (\bar{\psi} \Sigma)_n \psi_n \right] $$

with

$$ \Sigma = \frac{1}{2} \{ \Sigma^5, \gamma^5 \}. $$
4 The limit \( m_q \to 0 \) and the order parameter of chiral symmetry

The following considerations are valid in the broken phase in general, not only in the continuum limit. We shall consider the Ward identity eq. (11) with \( Q = P^b(x) \) and sum over \( n \). After some algebra we obtain

\[
\sum_n i\left( \frac{\delta}{\delta a_n} P^b(x) \right) = -\delta_{ab} \frac{1}{N_f} \langle S(x) \rangle,
\]

(16)

\[- \sum_n \langle P^b(x) X^a(n) \rangle = \delta_{ab} \langle \text{tr}^{\text{DC}} \frac{1}{2} \{ \Sigma^5(U), \gamma^5 R(U) \} \rangle \equiv \alpha,
\]

(17)

where the trace \( \text{tr}^{\text{DC}} \) in eq. (17) is over Dirac and colour space. We used the notations introduced in eqs. (12-15).

The last term in eq. (11) is dominated by the pion state for small \( m \). Integrating out the fermions we get

\[- \langle S(x) + N_f \text{tr}^{\text{DC}} \frac{1}{2} \{ \Sigma^5, \gamma^5 R \} \rangle = 2N_f \frac{m_q}{m_\pi^2} |\alpha|^2, \quad (m_q \to 0),
\]

(19)

We shall show now that the combination on the l.h.s. is one of the possible order parameters of chiral symmetry: it is zero in perturbation theory, or more generally in the limit \( \text{lim}_{\nu \to \infty} \text{lim}_{m_q \to 0} \) (in this order). If this order parameter picks up a non-zero expectation value in the limit \( \text{lim}_{m_q \to 0} \text{lim}_{\nu \to \infty} \) due to non-perturbative effects then eq. (19) implies

\[ m_\pi^2 \sim m_q, \quad (m_q \to 0),
\]

(20)

i.e. the critical quark mass is zero. There is no tuning.

Let us first construct the simplest order parameter. Consider

\[ \langle \bar{\psi} x \psi x \rangle_{\text{sub}} = \langle \bar{\psi} x \psi x + 4N_f \text{tr}^{\text{C}} R(U) \rangle \]

(21)

in the limit \( m_q \to 0 \) in a finite volume. Integrating out the fermions we get

\[ \langle \bar{\psi}_x \psi_x \rangle_{\text{sub}} = \langle \text{tr}^{\text{DFC}}(- h_{xx}^{-1} + R_{xx}) \rangle \]

(22)

where the trace is over Dirac, flavour and colour space. The r.h.s. of eq. (22) is zero as can be seen easily by taking the trace of eq. (1). For specific block transformations ('blocking out of continuum' [25]) \( R_{xx} = 1/\kappa \), where \( \kappa \) is a parameter entering the blocking procedure. In this case the order parameter is obtained from \( \langle \bar{\psi}_x \psi_x \rangle \) by simply adding the constant \( 4N_f N_c/\kappa \).

Eq. (1) allows to construct other order parameters also. Multiplying eq. (1) by \( \Sigma^5 \gamma^5 \) from the left and adding to it the product from the right, after taking the trace the expectation value on the l.h.s. of eq. (19) is obtained as another definition of the order parameter. This is what we wanted to show.

5 Current renormalization

Current algebra relations which are non-linear in the currents, are very convenient for the study the renormalization of the currents. Consider the Ward identity, studied also by Bochicchio et al. [23] in this context, in the continuum

\[
\partial^\mu \langle A^a_\mu(x) A^b_\mu(y)V^c(z) \rangle = 2m_q \langle P^a(x) A^b_\mu(y)V^c(z) \rangle + i f^{abc} \delta(x-y) \langle V^a_\mu(y)V^c(z) \rangle + i f^{abc} \delta(x-z) \langle A^b_\mu(y)A^c_\mu(z) \rangle.
\]

(23)
Integrating over $x$, we obtain
\[
0 = \int dx \, 2m_q \langle P^a(x) A^b_v(y) V^c_\rho(z) \rangle + i f^{a bd}(V^d_\nu(y) V^c_\rho(z)) + i f^{ac d}(A^b_v(y) A^d_\rho(z)). \tag{24}
\]

We shall consider eq. (24) for $|y - z|$ being a physical distance. With this condition we assure that the communication between the points $y$ and $z$ goes through physical intermediate states. We work out the Ward identity corresponding to eq. (24) on the lattice. Consider the general Ward identity in eq. (11) with
\[
\mathcal{O} \rightarrow A^b_v(y) V^c_\rho(z), \tag{25}
\]
where $|y - z|$ is much larger than the lattice unit $a$. We get 4
\[
\sum_n \frac{\delta}{\delta e_n} V^c_\rho(z) = i f^{a cd} A^d_\rho(z), \tag{26}
\]
\[
\sum_n \frac{\delta}{\delta e_n} A^b_v(y) = i f^{ab d} V^d_\nu(y). \tag{27}
\]

Using eqs. (25,26,27) in eq. (11), the continuum Ward identity eq. (24) is reproduced if
\[
\langle \sum_n X^a(n) A^b_v(y) V^c_\rho(z) \rangle = 0. \tag{28}
\]

Eqs. (8,9,13) give after integrating out the fermions
\[
0 = \sum_n 2m_q \langle P^a(n) A^b_v(y) V^c_\rho(z) \rangle + i f^{a bd}(V^d_\nu(y) V^c_\rho(z)) + i f^{ac d}(A^b_v(y) A^d_\rho(z)), \tag{30}
\]
where $P^a(n)$ is defined in eq. (12).

Consider finally the Ward identity eq. (11) with $\mathcal{O}(x) = P^b(x)$ as in Section 4 but do not sum over $n$. Assume that $x - n$ is much larger than the lattice unit. Using similar steps as above we obtain
\[
\langle P^b(x) \nabla_\mu A^a_\mu(n) \rangle = \langle P^b(x) 2m_q P^a(n) \rangle. \tag{31}
\]

From eq. (31) we conclude that $\nabla_\mu A^a_\mu$ and $2m_q P^a(n)$ have the same renormalization factor. Eq. (30) gives then
\[
Z_A = Z_V = 1. \tag{32}
\]

We remark that in the considerations above the $m_q \rightarrow 0$ limit was not necessary.

### 6 Mixing

In the study of weak, non-leptonic matrix elements it is essentially important to construct operators in definite chiral representations. In the case of Wilson fermions the nominal (tree level) chiral assignment of operators is invalidated by quantum corrections due to the chiral symmetry breaking terms in the action. Operators

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4The simple relations in eqs. (26,27) would receive additional terms had we used the conserved form of the vector current. Although they can be shown not to influence the final conclusions, their presence would complicate our considerations.
with definite chiral properties are linear combinations of the operators with nominal assignment (mixing). For present Monte Carlo calculations, as the case of the $B_K$ parameter shows, the mixing coefficients should be calculated non-perturbatively, which is a highly non-trivial problem.

Using the example of local k-fermion operators ($k=4,6,\ldots$) we are going to demonstrate now that k-fermion operators in a definite chiral representation (assignment on the tree level) do not mix with other k-fermion operators from a different chiral representation. In particular, for the $\Delta s = 2$, 4-fermion operator, whose matrixelement between $K^0$ and $\bar{K}^0$ defines the $B_K$ parameter, the nominal chiral assignment is equal to the full quantum chiral assignment – there is no mixing.

Consider the Ward identity eq. (11) with $O \rightarrow O(x)B(y_1,\ldots,y_j)$, where $O(x)$ is some k-fermion operator in a definite chiral representation, $B(y_1,\ldots,y_j)$ is some product of local operators, and $|y_1-x|,\ldots,|y_j-x|$ are much larger than the lattice unit. Choosing the point $n$ in eq. (11) equal to $x$, or, if the operator $O$ has an extension of $O(a)$, summing over $n$ in the neighbourhood of $x$, then on the l.h.s. the chiral variation of the k-fermion operator $O(x)$ enters. This is a linear combination of k-fermion operators in the same representation. On the r.h.s., the $\langle X^a(n)O(x)B(y_1,\ldots,y_j) \rangle$ matrixelement might also produce k-fermion terms local in $x-n$ in different representations. This is the source of mixing when using Wilson fermions [23]. In our case

$$\langle X^a(n)O(x)B(y_1,\ldots,y_j) \rangle =$$

$$\langle (\bar{\psi}_n\tau^a(\hat{h}\gamma^5R\hat{h})\psi)_n + (\bar{\psi}_n(\hat{h}\gamma^5R\hat{h})\tau^a\psi)_n \rangle O(x)B(y_1,\ldots,y_j) \rangle.$$

If one of the fermions in $X^a$ are paired with one of the fermions in $B$, the propagator is cancelled by $\hat{h}$ in $X^a$ and the result will be zero since $n \sim x$ is far (in lattice units) from $y_1,\ldots,y_j$ and $R$ and $\hat{h}$ are local. Hence, both fermions in $X^a(n)$ should be paired with the fermions in the k-fermion operator $O(x)$. The result is a (k-2)-fermion operator local in the point $x$. The dangerous term, therefore can not give a k-fermion contribution to the l.h.s. of eq. (11). In the example of the $B_K$ parameter, the matrixelement in eq. (33) is zero, since no 2-fermion operator exists with $\Delta s = 2$.

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References


[12] for a recent summary on the perfect actions, see P. Hasenfratz, hep-lat/9709110; for a pedagogical introduction, see P. Hasenfratz, in the proceedings of the 'NATO ASI - Confinement, Duality and Non-perturbative Aspects of QCD', 1997, Ed. P. van Baal, Plenum Press, to be published.


[17] Some of the results presented here has been discussed in the first ref. of [12].


[22] H. Neuberger, hep-lat/9801031.


[24] P. Hasenfratz, in the proceedings of 'Advanced School on Non-Perturbative Quantum Field Physics' (Peñíscola) and of the YKIS’97 'Non-Perturbative QCD - Structure of the QCD Vacuum', (Kyoto), in preparation.