Abstract: In this talk I review both accomplished results and work in progress on the use of solvable Lie algebras as an intrinsic algebraic characterization of the scalar field sector of M–theory low energy effective lagrangians. In particular I review the application of these techniques in obtaining the most general form of BPS black hole solutions.

1 Introduction

In this talk I review a line of research [1],[2],[3] I have recently pursued in collaboration with Riccardo D’Auria, Sergio Ferrara and our Ph.D students Laura Andrianopoli and Mario Trigiante, whose contribution to the development of the entire project has been essential.

The main idea underlying this investigation originates from some results I had previously obtained in collaboration with Luciano Girardello, Igor Pesando and Mario Trigiante. In [4], extending work of Girardello, Ferrara and Porrati, we discovered that $N = 2$ supersymmetry can be spontaneously broken to $N = 1$ when the following conditions are met:

- The scalar manifold of supergravity, which is generically given by the direct product $SK \otimes QK$ of a special Kähler manifold with a quaternionic one, is a homogeneous non–compact coset manifold $G/H$.
- Some translational abelian symmetries of $G/H$ are gauged.

The basic ingredient in deriving the above result is the Alekseevskian description [5] of the scalar manifold $SK \otimes QK$ in terms of solvable Lie algebras, a Kähler algebra $K$ for the vector multiplet sector $SK$ and a quaternionic algebra $Q$ for the hypermultiplet sector $QK$. By means of this description the homogeneous non compact coset manifold $G/H$ is identified with the solvable group manifold $\exp[Solv]$ where

$$Solv = K \oplus Q$$

and the translational symmetries responsible for the supersymmetry breaking are identified with suitable abelian subalgebras

$$T \subset Solv$$

An obvious observation that easily occurs once such a perspective is adopted is the following one: for all extended supergravities with $N \geq 3$ the scalar manifold is a homogeneous non–compact coset manifold.
$G/H$. Hence it is very tempting to extend the solvable Lie algebra approach to such supergravity theories, in particular to the maximal extended ones in all dimensions $4 \leq D \leq 10$. This is what was done in the series of three papers [1, 2, 3].

2 R-R and NS-NS scalars

Relying on a well established mathematical theory which is available in standard textbooks (for instance [6]), every non compact homogeneous space $G/H$ is indeed a solvable group manifold and its generating solvable Lie algebra $\text{Solv}(G/H)$ can be constructed utilizing roots and Dynkin diagram techniques. This fact offers the so far underestimated possibility of introducing an intrinsic algebraic characterization of the supergravity scalars. In relation with string theory this yields a group–theoretical definition of Ramond and Neveu–Schwarz scalars. It goes as follows. The same supergravity lagrangian admits different interpretations as low energy theory of different superstrings related by duality transformations or of M–theory. The identification of the Ramond and Neveu Schwarz sectors is different in the different interpretations as low energy theory of different superstrings related by duality transformations of the supergravity scalars. In relation with string theory this yields a group–theoretical definition of Ramond and Neveu–Schwarz sectors. Each string theory admits a $T$–duality group whose product $S \otimes T$ constitutes a subgroup of the $U$–duality group, namely of the isometry group $U \equiv G$ of the homogeneous scalar manifold $G/H$. Physically $S$ is a non perturbative symmetry acting on the dilaton while $T$ is a perturbative symmetry acting on the "radii" of the compactification. There exist also two compact subgroups $H_S \subset S$ and $H_T \subset T$ whose product $H_S \otimes H_T \subset H$ is contained in the maximal compact subgroup $H \subset U$ such that we can write:

$$\text{Solv}(U/H) = \text{Solv}(S/H_S) \oplus \text{Solv}(T/H_T) \oplus \mathcal{W}$$

(3)

the three addends being all subalgebras of $\text{Solv}(U/H)$. The first two addends constitute the Neveu Schwarz sector while the last subalgebra $\mathcal{W}$ which is not only solvable but also nilpotent constitutes the Ramond sector relative to the chosen superstring interpretation.

An example of this way of reasoning is provided by maximal supergravities in $D = 10 - r$ dimensions. For such lagrangians the scalar sector is given by $M_{\text{scalar}} = E_{r+1(r+1)}/H_{r+1}$ where the group $E_{r+1(r+1)}$ is obtained exponentiating the maximally non compact real form of the exceptional rank $r+1$ Lie algebra $\text{E}_{r+1}$ and $H_{r+1}$ is the corresponding maximal compact subgroup. If we interpret supergravity as the low energy theory of Type IIA superstring compactified on a torus $T^r$, then the appropriate $S$-duality group is $O(1,1)$ and the appropriate $T$–duality group is $SO(r,r)$. Correspondingly we obtain the decomposition:

$$\text{Solv}(E_{r+1(r+1)}/H_{r+1}) = O(1,1) \oplus \text{Solv}\left(\frac{SO(r,r)}{SO(r) \times SO(r)}\right) \oplus W_{r+1}$$

(4)

where the Ramond subalgebra $W_{r+1} \equiv \text{spin}[r,r]$ is nothing else but the chiral spinor representation of $SO(r,r)$. In the four dimensional case $r = 6$ equation (4) takes the exceptional form:

$$\text{Solv}(E_{7(7)}/SU(8)) = \text{Solv}(SL(2,R)/O(2)) \oplus \text{Solv}\left(\frac{SO(6,6)}{SO(6) \times SO(6)}\right) \oplus W_7$$

(5)

The 38 Neveu Schwarz scalars are given by the first two addends in (5), while the 32 Ramond scalars in the algebra $W_7$ transform in the spinor representation of $SO(6,6)$ as in all the other cases.

Alternatively we can interpret maximal supergravity in $D = 10 - r$ as the compactification on a torus $T^r$ of Type IIB superstring. In this case the ST–duality group is different. We just have:

$$S \otimes T = O(1,1) \otimes GL(r)$$

(6)

Correspondingly we write the solvable Lie algebra decomposition:

$$\text{Solv}(E_{r+1(r+1)}/H_{r+1}) = O(1,1) \oplus \text{Solv}\left(\frac{GL(r)}{SO(r)}\right) \oplus \tilde{W}_{r+1}$$

(7)
where $\mathcal{W}_{r+1}$ is the new algebra of Ramond scalars with respect to the Type IIB interpretation. Actually, as it is well known, Type IIB theory already admits an $SL(2,R)$ U–duality symmetry in ten dimensions that mixes Ramond and Neveu Schwarz states. The proper S–duality group $O(1,1)$ is just a maximal subgroup of such $SL(2,R)$. Correspondingly eq. (7) can be restated as:

$$\text{Solv}_{r+1} \equiv \text{Solv} (E_{r+1(r+1)}/H_{r+1}) = \text{Solv} (SL(2,R)/O(2)) \oplus \text{Solv} \left( \frac{\text{GL}(4)}{\text{SO}(4)} \right) \oplus \mathcal{W}_{r+1} \quad (8)$$

Finally a third decomposition of the same solvable Lie algebra can be written if the same supergravity lagrangian is interpreted as compactification on a torus $T^3$ of M–theory. For the details on this and other decompositions of the scalar sector that keep track of the sequential compactifications on multiple torii we refer the reader to the original papers [1, 2].

3 BPS black holes

Another interesting application of the solvable Lie algebra parametrization of the scalar sector is provided by the systematic construction of completely general BPS black hole or BPS black brane solutions. In this problem a fundamental role is played by the scalar evolution from arbitrary values at infinity to fixed values at the horizon. Such an evolution is best understood and dealt with when the scalars are algebraically characterized as generators of a solvable algebra. In ref.[3] we considered the $N=8,D=4$ case and we solved the problem of writing the most general BPS black hole solution that preserves $1/8$ of the original supersymmetries. We are presently pursuing the solution of the same problem in the case where the preserved supersymmetries are either $1/2$ or $1/4$ [7]. Let us briefly illustrate these three cases from our viewpoint.

The $D=4$ supersymmetry algebra with $N=8$ supersymmetry charges can be written in the following form:

$$\{\mathcal{Q}_{aI}^{\alpha}, \mathcal{Q}_{bJ}^{\beta}\} = i (C \gamma^\mu)_{\alpha\beta} P_\mu \delta_{IJ} - C_{\alpha\beta} \epsilon_{ab} \times \mathbf{Z}_{IJ} \quad (a,b = 1,2; \quad I,J = 1,\ldots,4) \quad (9)$$

where the SUSY charges $\mathcal{Q}_{aI}^{\alpha} \equiv Q_{aI}^{\gamma_0} = Q_{aI}^{\gamma_0} C$ are Majorana spinors, $C$ is the charge conjugation matrix, $P_\mu$ is the 4–momentum operator, $\epsilon_{ab}$ is the two–dimensional Levi Civita symbol and the symmetric tensor $\mathbf{Z}_{IJ} = \mathbf{Z}_{JI}$ is the central charge operator. It can always be diagonalized $\mathbf{Z}_{IJ} = \delta_{IJ} Z_I$ and its 4 eigenvalues $Z_I$ are the central charges.

Consider the reduced superscharges:

$$\mathbf{S}_{aI}^{\pm} = \frac{1}{2} \left( \mathcal{Q}_{aI}^{\gamma_0} \pm i \epsilon_{ab} \mathcal{Q}_{bI}^{\gamma_0} \right) \quad (10)$$

They can be regarded as the result of applying a projection operator to the supersymmetry charges: $\mathbf{S}_{aI}^{\pm} = \mathcal{Q}_{bI}^{\pm} \mathbf{P}_{ba}^{\pm}$, where $\mathbf{P}_{ba}^{\pm} = \frac{1}{2} (1 \delta_{ba} \pm i \epsilon_{ba} \gamma_0)$. In the rest frame where the four momentum is $P_\mu = (M,0,0,0)$, we obtain the algebra: $\{\mathbf{S}_{aI}^{\pm}, \mathbf{S}_{bJ}^{\pm}\} = \pm \epsilon_{ac} C \mathbf{P}_{cb}^{\pm} (M \mp Z_I) \delta_{IJ}$ and the BPS states that saturate the bounds $(M \pm Z_I)$ as BPS state,$i = 0$ are those which are annihilated by the corresponding reduced superscharges:

$$\mathbf{S}_{aI}^{\pm} |\text{BPS state},i \rangle = 0 \quad (11)$$

Eq.(11) defines short multiplet representations of the original algebra (9) in the following sense: one constructs a linear representation of (9) where all states are identically annihilated by the operators $\mathbf{S}_{aI}^{\pm}$ for $I = 1,\ldots,n_{\text{max}}$. If $n_{\text{max}} = 1$ we have the minimum shortening, if $n_{\text{max}} = 4$ we have the maximum shortening. On the other hand eq.(11) can be translated into first order differential equations on the bosonic fields of supergravity whose common solutions with the ordinary field equations are the BPS saturated black hole configurations. In the case of maximum shortening $n_{\text{max}} = 4$ the black hole preserves $1/2$ supersymmetry, in the case of intermediate shortening $n_{\text{max}} = 2$ it preserves $1/4$, while in the case of minimum shortening it preserves $1/8$. 

3
3.1 The Killing spinor equation and its covariance group

In order to translate eq.(11) into first order differential equations on the bosonic fields of supergravity we consider a configuration where all the fermionic fields are zero and we set to zero the fermionic SUSY rules appropriate to such a background

\[ 0 = \delta \text{fermions} = \text{SUSY rule (bosons, } \epsilon_{\alpha i} \text{)} \]  \hspace{1cm} (12)

and to a SUSY parameter that satisfies the following conditions:

\[ \xi^\mu \gamma_\mu \epsilon_{\alpha i} = i \epsilon_{ab} \epsilon_{b i} ; \quad i = 1, \ldots, n_{\text{max}} \]
\[ \epsilon_{\alpha i} = 0 ; \quad i > n_{\text{max}} \]  \hspace{1cm} (13)

Here \( \xi^\mu \) is a time–like Killing vector for the space–time metric and \( \epsilon_{\alpha i}, \epsilon_{a I} \) denote the two chiral projections of a single Majorana spinor: \( \gamma_5 \epsilon_{a I} = \epsilon_{a I}, \gamma_5 \epsilon^{a I} = -\epsilon^{a I} \). We name eq.(12) the Killing spinor equation and the investigation of its group–theoretical structure was our main goal in ref [3]. There we restricted our attention to the case \( n_{\text{max}} = 1 \): we are presently considering the other two possibilities [7]. In all three cases eq.(12) has two features which we want to stress as main motivations for the developments we have pursued:

1. It requires an efficient parametrization of the scalar field sector
2. It breaks the original \( SU(8) \) automorphism group of the supersymmetry algebra to the subgroup \( Usp(2n_{\text{max}}) \times SU(8 - 2n_{\text{max}}) \times U(1) \)

The first feature is the reason why the use of the rank 7 solvable Lie algebra \( \text{Solv}_7 \) associated with \( E_7(7)/SU(8) \) is of great help in this problem. The second feature is the reason why the solvable Lie algebra \( \text{Solv}_7 \) has to be decomposed in a way appropriate to the decomposition of the isotropy group \( SU(8) \) with respect to the subgroup \( Usp(2n_{\text{max}}) \times SU(8 - 2n_{\text{max}}) \times U(1) \).

This decomposition of the solvable Lie algebra is a close relative of the decomposition of \( N = 8 \) supergravity into multiplets of the lower supersymmetry \( N' = 2n_{\text{max}} \). This is easily understood by recalling that close to the horizon of the black hole one doubles the supersymmetries holding in the bulk of the solution. Hence the near horizon supersymmetry is precisely \( N' = 2n_{\text{max}} \) and the black solution can be interpreted as a soliton that interpolates between ungauged \( N = 8 \) supergravity at infinity and some form of gauged \( N' \) supergravity at the horizon. The reason why we stress that the horizon theory is gauged is that its geometry is an anti de Sitter geometry.

Let us now study the explicit structure of the three cases at hand.

3.1.1 The 1/2 SUSY case

Here we have \( n_{\text{max}} = 4 \) and correspondingly the covariance subgroup of the Killing spinor equation is \( Usp(8) \subset SU(8) \). Indeed condition (13) can be rewritten as follows:

\[ \xi^\mu \gamma_\mu \epsilon_A = i C_{AB} \epsilon^B ; \quad A, B = 1, \ldots, 8 \]  \hspace{1cm} (14)

where \( C_{AB} = -C_{BA} \) denotes an \( 8 \times 8 \) antisymmetric matrix satisfying \( C^2 = -\mathbb{1} \). The group \( Usp(8) \) is the subgroup of unimodular, unitary \( 8 \times 8 \) matrices that are also symplectic, namely that preserve the matrix \( C \).

We are accordingly lead to decompose the solvable Lie algebra as written below:

\[ \text{Solv}_7 = \text{Solv}_6 \oplus O(1,1) \oplus \mathbb{D}_6 \]
\[ 70 = 42 + 1 + 27 \]  \hspace{1cm} (15) (16)
where, following the notation established in (8):

\[
\begin{align*}
\text{Solv}_7 & \equiv \text{Solv} \left( \frac{E_7(7)}{SU(8)} \right) \\
\text{Solv}_6 & \equiv \text{Solv} \left( \frac{E_6(6)}{Usp(8)} \right)
\end{align*}
\]

\[
\begin{align*}
\dim \text{Solv}_7 &= 70 ; \quad \text{rank } \text{Solv}_7 = 7 \\
\dim \text{Solv}_6 &= 42 ; \quad \text{rank } \text{Solv}_6 = 6
\end{align*}
\]

(17)

In eq.(15) \( \text{Solv}_6 \) is the solvable Lie algebra that describes the scalar sector of \( D = 5, N = 8 \) supergravity, while the 27–dimensional abelian ideal \( \mathbb{D}_6 \) corresponds to those \( D = 4 \) scalars that originate from the 27–vectors of supergravity one–dimension above [2]. Eq.(16) corresponds also to the decomposition of the 70 irreducible representation of \( SU(8) \) into irreducible representations of \( Usp(8) \). Indeed we have:

\[
70 \xrightarrow{Usp(8)} 42 \oplus 1 \oplus 27
\]

(18)

In order to single out the content of the first order Killing spinor equations we need to decompose them into irreducible \( Usp(8) \) representations. This is easily done. The gravitino equation is an 8 of \( SU(8) \) that remains irreducible under \( Usp(8) \) reduction. On the other hand the dilatino equation is a 56 of \( SU(8) \) that reduces as follows:

\[
56 \xrightarrow{Usp(8)} 48 \oplus 8
\]

(19)

Hence altogether we have the following three constraints 8, 8', 48 on the three subalgebras of scalar fields 42, 1 and 27. Working out the consequences of these constraints and deciding which scalars are set to constants and which are instead evolving is work in progress [7].

### 3.1.2 The 1/4 SUSY case

Here we have \( n_{\text{max}} = 2 \) and correspondingly the covariance subgroup of the Killing spinor equation is \( Usp(4) \times SU(4) \times U(1) \subset SU(8) \). Indeed condition (13) can be rewritten as follows:

\[
\begin{align*}
\xi^\mu \gamma_\mu \epsilon_a &= - i C_{ab} \epsilon^b ; \quad a, b = 1, \ldots, 4 \\
\epsilon_X &= 0 ; \quad X = 5, \ldots, 8
\end{align*}
\]

(20)

where \( C_{ab} = - C_{ba} \) denotes a 4 \( \times \) 4 antisymmetric matrix satisfying \( C^2 = -1 \). The group \( Usp(4) \) is the subgroup of unimodular, unitary 4 \( \times \) 4 matrices that are also symplectic, namely that preserve the matrix \( C \).

We are accordingly lead to decompose the solvable Lie algebra as follows. Naming:

\[
\begin{align*}
\text{Solv}_S & \equiv \text{Solv} \left( \frac{SL(2, R)}{U(1)} \right) \\
\text{Solv}_T & \equiv \text{Solv} \left( \frac{SO(6, 6)}{SO(6) \times SO(6)} \right)
\end{align*}
\]

\[
\begin{align*}
\dim \text{Solv}_S &= 2 ; \quad \text{rank } \text{Solv}_S = 1 \\
\dim \text{Solv}_T &= 36 ; \quad \text{rank } \text{Solv}_T = 6
\end{align*}
\]

(21)

we can write:

\[
\begin{align*}
\text{Solv}_7 &= \text{Solv}_S \oplus \text{Solv}_T \oplus W_7 \\
70 &= 2 + 36 + 32
\end{align*}
\]

(22) (23)
which is nothing else but (5). Indeed the solvable Lie algebras \( \text{Solv}_T \) and \( \text{Solv}_T \) describe the dilaton–axion sector and the six torus moduli, respectively, in the interpretation of \( N = 8 \) supergravity as the compactification of Type IIA theory on a six–torus \( T^6 \) [2]. The rank zero abelian subalgebra \( W_7 \) is instead composed by the Ramond-Ramond scalars as we have already explained.

Introducing the decomposition (22), (23) we have succeeded in singling out a holonomy subgroup \( SU(4) \times SU(4) \times U(1) \subset SU(8) \). Indeed we have \( SO(6) \equiv SU(4) \). This is a step forward but it is not yet the end of the story since we actually need a subgroup \( \text{Usp}(4) \times SU(4) \). This means that we must further decompose the solvable Lie algebra \( \text{Solv}_T \). This latter is the manifold of the scalar fields associated with vector multiplets in an \( N = 4 \) decomposition of the \( N = 8 \) theory. Indeed the decomposition (22) with respect to the \( S\)–\( T\)–duality subalgebra is the appropriate decomposition of the scalar sector according to \( N = 4 \) multiplets. The further decomposition we need is the following:

\[
\begin{align*}
\text{Solv}_T &= \text{Solv}_{T5} \oplus \text{Solv}_{T1} \\
\text{Solv}_{T5} &= \text{Solv} \left( \frac{SO(5,6)}{SO(5) \times SO(6)} \right) \\
\text{Solv}_{T1} &= \text{Solv} \left( \frac{SO(1,6)}{SO(6)} \right)
\end{align*}
\]

(24)

where we rely on the isomorphism \( Usp(4) \equiv SO(5) \). Hence, altogether we can write:

\[
\begin{align*}
\text{Solv}_7 &= \text{Solv}_S \oplus \text{Solv}_{T5} \oplus \text{Solv}_{T1} \oplus W_{32} \\
\text{dim} \text{Solv}_7 &= 70 = 2 + 30 + 6 + 32
\end{align*}
\]

(25)

which exactly corresponds to the decomposition of the 70 irreducible representation of \( SU(8) \) into irreducible representations of \( Usp(4) \times SU(4) \times U(1) \).

\[
\begin{align*}
70_{Usp(4) \times SU(4) \times U(1)} \longrightarrow (1 + \overline{1}, 1, 1) \oplus (1, 5, 6) \oplus (1, 1, 6)
\end{align*}
\]

(26)

Just as in the previous case we should now single out the content of the first order Killing spinor equations by decomposing them into irreducible \( Usp(4) \times SU(4) \times U(1) \) representations. This is also work in progress [7].

3.1.3 The 1/8 SUSY case

Here we have \( n_{\text{max}} = 1 \) and \( \text{Solv}_7 \) must be decomposed according to the decomposition of the isotropy subgroup; \( SU(8) \longrightarrow SU(2) \times U(6) \). We showed in [3] that the corresponding decomposition of the solvable Lie algebra is the following one:

\[
\begin{align*}
\text{Solv}_7 &= \text{Solv}_3 \oplus \text{Solv}_4 \\
\text{rank} \text{Solv}_3 &= 3 \quad \text{rank} \text{Solv}_4 = 4 \\
\text{dim} \text{Solv}_3 &= 30 \quad \text{dim} \text{Solv}_4 = 40
\end{align*}
\]

(27)

The rank three Lie algebra \( \text{Solv}_3 \) defined above describes the thirty dimensional scalar sector of \( N = 6 \) supergravity, while the rank four solvable Lie algebra \( \text{Solv}_4 \) contains the remaining forty scalars belonging to \( N = 6 \) spin 3/2 multiplets. It should be noted that, individually, both manifolds \( \exp[\text{Solv}_3] \) and \( \exp[\text{Solv}_4] \) have also an \( N = 2 \) interpretation since we have:

\[
\begin{align*}
\exp[\text{Solv}_3] &= \text{homogeneous special Kähler} \\
\exp[\text{Solv}_4] &= \text{homogeneous quaternionic}
\end{align*}
\]

(29)
so that the first manifold can describe the interaction of 15 vector multiplets, while the second can
describe the interaction of 10 hypermultiplets. Indeed if we decompose the $N = 8$ graviton multiplet in
$N = 2$ representations we find:

$$N=8 \text{ spin } 2 \xrightarrow{N=2} \text{ spin } 2 + 6 \times \text{ spin } 3/2 + 15 \times \text{ vect. mult.} + 10 \times \text{ hypermult.} \quad (30)$$

Introducing the decomposition (27) we found in [3] that the 40 scalars belonging to $\text{Solv}_4$ are constants
independent of the radial variable $r$. Only the 30 scalars in the Kähler algebra $\text{Solv}_3$ can be radial
dependent. In fact their radial dependence is governed by a first order differential equation that can
be extracted from a suitable component of the Killing spinor equation. More precisely we obtained the
following result. Up to U–duality transformations the most general $N = 8$ black–hole is actually an $N = 2$
black–hole corresponding to a very specific choice of the special Kähler manifold, namely $\exp[\text{Solv}_3]$ as
in eq.(29),(28). Furthermore up to the duality rotations of $SO^*(12)$ this general solution is actually
determined by the so called $STU$ model studied in [8] and based on the solvable subalgebra:

$$\text{Solv} \left( \frac{SL(2, \mathbb{R})^3}{U(1)^3} \right) \subset \text{Solv}_3 \quad \quad (31)$$

In other words the only truly independent degrees of freedom of the black hole solution are given by
three complex scalar fields, $S,T,U$. The real parts of these scalar fields correspond to the three Cartan
generators of $\text{Solv}_3$ and have the physical interpretation of radii of the torus compactification from $D = 10$
to $D = 4$. The imaginary parts of these complex fields are generalised theta angles.

4 Uniqueness of the N=8 abelian gauging

Going back to the original motivations explained in the introduction, in [9] we have studied the possible
abelian gaugings of $N = 8$ supergravity. Using the solvable Lie algebra approach we have shown that
there exists a unique abelian subalgebra $\mathcal{A} \subset SL(8, \mathbb{R}) \subset E_7(7)$ that can be gauged and this is the
algebra $\text{CSO}(1, 7)$ already found by Hull [10] in his study of non–compact $N = 8$ gaugings. The number
of generators in $\mathcal{A}$ is seven. The scalar potential associated with this gauging has not been studied in full
detail and it is now our plan to study its properties in full generality by using the solvable Lie algebra
approach.

References

