de Rham cohomology of $SO(n)$ and some related manifolds by supersymmetric quantum mechanics

Kazuto Oshima

*Gunma College of Technology, Maebashi 371-0845, Japan*

**Abstract**

We study supersymmetric quantum mechanics on $\mathbb{R}P^n$, $SO(n)$, $G_2$ and $U(2)$ to examine Witten’s Morse theory concretely. We confirm the simple instanton picture of the de Rham cohomology that has been given in a previous paper. We use a reflection symmetry of each theory to select the true vacuums. The number of selected vacuums agrees with the de Rham cohomology for each of the above manifolds.
1 INTRODUCTION

In a previous paper,\(^1\) the author has investigated Witten’s Morse theory\(^2\) for SO(\(n\)).\(^3\) Instanton effects between adjacent classical vacuums have been studied. Using reflection symmetries of the theory, a selection rule for true vacuums has been given. This selection rule works well, at least for \(n \leq 5\);\(^1\) the number of selected vacuums agrees with the de Rham cohomology of SO(\(n\)). The main purpose of this paper is to show that the selection rule works well for arbitrary \(n\). We also apply the selection rule to some related manifolds \(\text{RP}_n, G_2\) and \(U(2)\); \(\text{RP}_1 \simeq \text{SO}(2), \text{RP}_3 \simeq \text{SO}(3), G_2 \subset O(7)\) and \(U(2) \simeq \text{SO}(2) \times \text{SU}(2)\).

This paper is organized as follows: In Sec.II we review supersymmetric quantum mechanics on a manifold \(M\). The selection rule for true vacuums is stated in this section. The manifold \(\text{RP}_n\) are studied in the third section. The manifold \(\text{RP}_n\) are simple and this section will be helpful to understand the following sections. In Sec.IV the de Rham cohomology of \(\text{SO}(n)\) is derived for arbitrary \(n\) by the selection rule. The manifolds \(G_2\) and \(U(2)\) are discussed in Sec.V and VI, respectively. The results are summarized in the last section.

2 SUPERSYMMETRIC QUANTUM MECHANICS ON A MANIFOLD

The supersymmetric hamiltonian on a manifold with Morse function \(h\) is given by

\[
\hat{H} = -\frac{1}{2}(d_h d_h^\dagger + d_h^\dagger d_h),
\]

where \(d_h = e^{-h}d e^h, d_h^\dagger = e^h d^\dagger e^{-h}\), \(d\) is the exterior derivative and \(d^\dagger\) is its adjoint operator. In (any) coordinates \(\{x^\mu\}\), the exterior multiplication \(e_{dx^\nu}\) and the interior multiplication \(i_{\partial_{x^\nu}}\) can be identified with the fermion creation operator \(\hat{\psi}^\ast_{\nu}\) and the annihilation operator \(\hat{\psi}_{\mu}\) and we have

\[
d = \hat{\psi}^\ast_{\mu} \nabla_{\mu}, \quad d^\dagger = g^{-\frac{1}{2}} \nabla_{\mu} g^{\frac{1}{2}} g^{\mu\nu} \hat{\psi}_{\nu},
\]
where $g$ is the determinant of the metric tensor $g_{\mu\nu}$ and $\nabla_{\mu}$ is the covariant derivative

$$\nabla_{\mu} = \frac{\partial}{\partial x^{\mu}} - \Gamma_{\mu}^{\lambda\nu} \hat{\psi}^{*\nu} \hat{\psi}_{\lambda}.$$ (3)

The hamiltonian $\hat{H}$ [(1)] takes the form:

$$2\hat{H} = -\frac{1}{2} \nabla_{\mu} g^{\mu\nu} \nabla_{\nu} + R_{\mu\nu\sigma\tau} \hat{\psi}^{\sigma} \hat{\psi}^{*\tau} \hat{\psi}^{*\nu} \hat{\psi}_{\mu} + g^{\mu\nu} \frac{\partial h}{\partial x^{\mu}} \frac{\partial h}{\partial x^{\nu}} + H_{\mu\nu} [\hat{\psi}^{*\mu}, \hat{\psi}^{\nu}],$$ (4)

where $R_{\mu\nu\sigma\tau}$ is the Riemann tensor, and $H_{\mu\nu}$ is the Hessian matrix

$$H_{\mu\nu} = (\partial_{\mu} \partial_{\nu} - \Gamma_{\mu}^{\lambda\nu}) h.$$ (5)

The corresponding Lagrangian is

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \frac{dx^{\mu}}{dt} \frac{dx^{\nu}}{dt} + \frac{1}{2} g^{\mu\nu} \frac{\partial h}{\partial x^{\mu}} \frac{\partial h}{\partial x^{\nu}} + \hat{\psi}^{*\mu}(\frac{d}{dt}\hat{\psi}_{\mu} - \Gamma_{\mu}^{\lambda\nu} \hat{\psi}_{\lambda} \frac{dx^{\nu}}{dt}) + H_{\mu\nu} \hat{\psi}^{*\nu} \hat{\psi}^{\mu} + \frac{1}{4} R_{\mu\nu\sigma\tau} \hat{\psi}^{\mu} \hat{\psi}^{\nu} \hat{\psi}^{*\sigma} \hat{\psi}^{*\tau}. $$ (6)

The gradient flow equation of (6) is

$$\frac{dx^{\mu}}{dt} = \pm g^{\mu\nu} \frac{\partial h}{\partial x^{\nu}}.$$ (7)

A relevant instanton solution satisfies (7) and connects adjacent critical points.

There is one classical vacuum for each critical point of $h$. Around each critical point, $h$ has the expansion

$$h = h_0 + \sum_{i} \lambda_i \xi_i^2,$$ (8)

where $\lambda_i$ are eigenvalues of the Hessian matrix and $\xi_i$ are the corresponding local coordinates. The classical vacuum around a critical point corresponds to an $l-$form:

$$|l\rangle = \prod_{\lambda_i < 0} \hat{\psi}_{\xi_i} |0\rangle,$$ (9)

where $l$ represents the number of the excited fermions (the negative eigenvalues) and $|0\rangle$ is the bosonic vacuum.

The true vacuums are determined by the quantum tunneling between adjacent classical vacuums (9). According to Witten, one has

$$\langle l + 1 | d_h | l \rangle = \sum_{\gamma} n_{\gamma} e^{-h(P^{(l+1)}) - h(P^{(l)})},$$ (10)
where $n_{\gamma}$ is an integer assigned for each instanton path $\gamma$ from $P(l)$ to $P(l+1)$. If a state $|l\rangle$ does not couple with any adjacent classical vacuums, that is if $d_h |l\rangle = 0$ and $\langle l|d_h = 0$, $|l\rangle$ is a true vacuum.

If there are plural instanton paths connecting a pair of adjacent critical points, it is not so easy to determine whether the matrix element (10) is zero or not, because of the notorious minus signs associated with fermions. In fact, for SO($n$), there are exactly two instanton solutions between each pair of adjacent classical vacuums.\(^1\) The selection rule given in Ref.1 is as follows. Find a reflection transformation that leaves $h, d_h, d_h^\dagger$, and consequently $\hat{H}$ invariant and that interchanges the two instanton paths. If the parities of a pair of adjacent classical vacuums are the same, the corresponding matrix element does not vanish, and the two classical vacuums are not true vacuums. If the parities of the two classical vacuums are different, the corresponding matrix element vanishes, and the two classical vacuums can be true vacuums. Repeat this for each pair of adjacent classical vacuums, and one obtains all true vacuums.

In the following sections, we apply this selection rule to SO($n$) and some related manifolds. We see that the number of true vacuums agrees with the de Rham cohomology for each manifold.

3 \ RP\(_n\)

In this section, we discuss the real projective space RP\(_n\), which is very simple but non-trivial. The manifold RP\(_n\) can be identified with $(n+1) \times (n+1)$ real matrices $A$ with conditions $A^t = A, A^2 = A$ and $\text{tr} A = 1$.\(^4\) Concretely, we have

$$A = (x_i x_j), \quad i, j = 0, 1, 2, \ldots, n,$$

(11)

where $\sum_{i=1}^{n} x_i^2 \leq 1$ and $x_0 = \sqrt{1 - \sum_{i=1}^{n} x_i^2}$. Let us introduce the following polar coordinates

$$x_1 = \sin \theta_1 \cos \theta_2,$$
\[ x_2 = \sin \theta_1 \sin \theta_2 \cos \theta_3, \]
\[ \vdots \]
\[ x_{n-2} = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1}, \]
\[ x_{n-1} = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1} \cos \theta_n, \]
\[ x_n = \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \sin \theta_{n-1} \sin \theta_n, \]
(12)

with \(-\frac{\pi}{2} \leq \theta_i \leq \frac{\pi}{2}\).

Fix real numbers \(c_0, c_1, \cdots, c_n\) with \(c_{i+1} > 2c_i > 0\). Then
\[
h = \sum_{i=0}^{n} c_i x_i^2
\]
(13)
is a Morse function on \(\mathbb{R}P^n\) with critical points \(P^{(l)}\)
\[
P^{(l)} = \text{diag}(0, 0, \cdots, 0, x_l^2 = 1, 0, \cdots, 0).
\]
(14)

Around \(P^{(l)}\), \(h\) has the expansion
\[
h = (c_0 - c_l)(x_0 x_l)^2 + \cdots + (c_{l-1} - c_l)(x_{l-1} x_l)^2 + c_l
\]
\[+ (c_{l+1} - c_l)(x_{l+1} x_l)^2 + \cdots + (c_n - c_l)(x_n x_l)^2.
\]
(15)
The classical vacuum around \(P^{(l)}\) is \(l\) fermions excited state
\[
|l\rangle = \hat{\psi}_0^* \hat{\psi}_1^* \cdots \hat{\psi}_{l-1}^* |0\rangle,
\]
(16)
where \(\hat{\psi}_{ij}^*\) is the fermionic mode corresponding to \(x_i x_j\).

We introduce a metric on \(\mathbb{R}P_n\) and find reflection symmetries of the theory and instanton solutions concretely. We introduce the following metric
\[
g_{ij} = \frac{1}{2} \left( \frac{\partial A^i}{\partial \theta_j} \right) \left( \frac{\partial A^j}{\partial \theta_i} \right)
\]
(17)
The non-zero components of \(g_{ij}\) and Christoffel symbols are
\[
g_{11} = 1, g_{ii} = \sin^2 \theta_1 \sin^2 \theta_2 \cdots \sin^2 \theta_{i-1}, i = 2, \cdots, n
\]
\[
\Gamma^j_{ij} = \cot \theta_i, i = 1, \cdots, n-1, j = i+1, \cdots, n.
\]
(18)
The Morse function \( h \) \([13]\), the metric \( g_{ij} \) and the corresponding supercharge

\[
d = \hat{\psi}^* \nabla = \hat{\psi}^* \left( \frac{\partial}{\partial \theta_i} - \cot \theta_i \sum_{j=i+1}^{n} \hat{\psi}^j \hat{\psi}_j \right),
\]

are invariant under the following transformations

\[
[i] : \quad \theta_i, \hat{\psi}^* \psi_i \rightarrow -\theta_i, -\hat{\psi}^* \psi_i, \quad i = 1, \cdots, n. \tag{20}
\]

Subsequently, the theory is invariant under these reflection transformations. Under the transformation \([i]\), \( x_j \) transform to \( -x_j \) if \( j \geq i \), and \( x_j \) are invariant if \( j < i \).

The fermionic modes \( \hat{\psi}^* jk \) have the same transformation properties as \( x_jx_k \)

From the gradient flow equation (6), we have

\[
\frac{d\theta_i}{dt} = (c_0 - c_{i-1} + (c_i - c_0) \cos^2 \theta_{i+1} + (c_{i+1} - c_0) \sin^2 \theta_{i+1} \cos^2 \theta_{i+2}
+ \cdots + (c_n - c_0) \sin^2 \theta_{i+1} \sin^2 \theta_{i+2} \cdots \sin^2 \theta_n) \sin 2\theta_i, \quad i = 1, \cdots n - 1,
\]

\[
\frac{d\theta_n}{dt} = (c_n - c_{n-1}) \sin 2\theta_n. \tag{21}
\]

One finds that there are exactly a pair of instanton paths that connect \(|l - 1\rangle\) and \(|l\rangle\) \([16]\]

\[
A = \begin{pmatrix}
0 & \cdots & \\
& \ddots & \\
& \cdots & 0 \\
0 & \cdots & 0
\end{pmatrix},
\]

where \(0 \leq \theta_l \leq \frac{\pi}{2}\). Each of these two instanton solutions causes non-zero instanton effect between \(|l - 1\rangle\) and \(|l\rangle\).
Let us consider the transformation $l$. This transformation interchanges the two instanton paths. Under this transformation the two classical vacuums transform as

$$|l-1\rangle \rightarrow |l-1\rangle, \quad |l\rangle \rightarrow (-1)^l |l\rangle. \quad (23)$$

Thus if $l$ is even, the two classical vacuums $|l-1\rangle$ and $|l\rangle$ have the same parities, which means that the classical vacuums $|l-1\rangle$ and $|l\rangle$ are quantum mechanically coupled, and are not true vacuums. If $l$ is odd, the two classical vacuums $|l-1\rangle$ and $|l\rangle$ have the opposite parities and the two instanton effects cancel each other.

One obtains the following results. For $n$ = even there is only one true vacuum $|0\rangle$, and for $n$ = odd there are two true vacuums $|0\rangle$ and $|n\rangle$. These results agrees with the de Rham cohomology of $\text{RP}_n$; for $n$ = even, there is one non-zero betti number $b_0 = 1$, and for $n$ = odd, there are two non-zero betti numbers $b_0 = b_n = 1$.

4 \textbf{SO}(n)

In Ref.1, all the classical vacuums, all the instanton solutions and symmetry transformations are identified for a certain Morse function on $\text{SO}(n)$. We review them to find true vacuums for arbitrary $n$.

Let $A = (a_{ij})$ be a group element of $\text{SO}(n)$, and fix real numbers $c_1, c_2, \ldots, c_n$ with $c_i > 2c_{i+1} > 0$. Then

$$h = \sum_{i=1}^{n} c_i a_{ii}, \quad (24)$$

is a Morse function$^4$ on $\text{SO}(n)$ with critical points $P(l)$,

$$P(l) = \text{diag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_n), \quad \epsilon_i = \pm 1, \prod_i \epsilon_i = 1. \quad (25)$$

The Morse index of $P(l)$ is $\sum_{i=1}^{n} (n - i) \frac{1 + \epsilon_i}{2}$. Around each critical point, $h$ has the expansion

$$h = \sum_i \epsilon_i c_i + \sum_{i<j} (\lambda_{ij} \xi_{ij}^2 + \mu_{ij} \eta_{ij}^2), \quad (26)$$
where
\[
\lambda_{ij} = -\frac{\epsilon_j - \epsilon_i}{4}(c_j - c_i), \quad \xi_{ij} = a_{ij} + a_{ji},
\]
(27)
\[
\mu_{ij} = -\frac{\epsilon_j + \epsilon_i}{4}(c_j + c_i), \quad \eta_{ij} = a_{ij} - a_{ji}.
\]
(28)

A vacuum state which is localized around a critical point with the Morse index \(l\) is
\[
|l\rangle = \prod_{i<j} \prod_{\epsilon_i = \epsilon_j} \hat{\psi}_{\epsilon_{ij}}^* \prod_{\epsilon_i > \epsilon_j} \hat{\psi}_{\epsilon_{ij}}^* |0\rangle.
\]
(29)

There are exactly a pair of instanton solutions between the critical points \(\text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_{i-1}, -1, 1, \epsilon_{i+2}, \ldots, \epsilon_n)\) and \(\text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_{i-1}, 1, -1, \epsilon_{i+2}, \ldots, \epsilon_n)\):
\[
\begin{pmatrix}
\epsilon_1 \\
\vdots \\
\epsilon_{i-1} \\
\cos \theta & \mp \sin \theta \\
\mp \sin \theta & \cos \theta \\
\epsilon_{i+2} \\
\vdots \\
\epsilon_n
\end{pmatrix}
\]
(30)

Some changes of the signs are needed in (30) for the instanton solutions between the critical points \(\text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-2}, -1, -1)\) and \(\text{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-2}, 1, 1)\).

The theory is invariant under each of the supersymmetric generalizations of the following reflection transformations
\[
a_{ij} \rightarrow -a_{ij} \quad \text{and} \quad a_{ji} \rightarrow -a_{ji}, \quad j \neq i, i = 2, 3, \ldots, n.
\]
(31)
and the transposition
\[
a_{ij} \rightarrow a_{ji}.
\]
(32)

We call the corresponding supersymmetric transformations as \([i]\) and \([t]\), respectively. Supersymmetric generalization means that the corresponding fermionic modes are also transformed. A transformation that is useful to elucidate true vacuums is the transformation that interchanges the two instanton solutions (30).

For the instanton process corresponding to (30), the transformation \([i]\) is relevant. From (29), one sees what kind of fermionic modes are newly exited or
suppressed in this process. In this process, only the fermionic modes including the indices \( i \) or \( i + 1 \) can newly be exited or suppressed. At first, the fermionic modes \( \hat{\psi}^*_{k_i} \) and \( \hat{\psi}^*_{k_{i+1}} \), \( k = 1, \ldots, i - 1 \), are excited in pair \( \hat{\psi}^*_{k_i} \hat{\psi}^*_{k_{i+1}} \) if \( \epsilon_k = 1 \). They change into \( \hat{\psi}^*_{n_{k_i}} \hat{\psi}^*_{n_{k_{i+1}}} \) in this process, but do not cause any changes in their parities. In this process, the mode \( \hat{\psi}^{*}_{\xi_{i+1}} \) is newly exited. As for the fermionic modes \( \hat{\psi}^*_{i_k} \) and \( \hat{\psi}^*_{i_{k+1}} \), \( k = i + 1, \ldots, n \), the corresponding exited modes are \( \hat{\psi}^*_{n_{i_k}} \) and \( \hat{\psi}^*_{n_{i_{k+1}}} \), for \( \epsilon_k = 1 \) and \( \epsilon_k = -1 \), respectively. In this process they change into \( \hat{\psi}^*_{n_{i_k}} \) and \( \hat{\psi}^*_{n_{i_{k+1}}} \), respectively. Thus, in this process the number of the index \( i \) increases by \( n - i \), and the number of the index \( i + 1 \) diminishes by \( n - i - 2 \) in the exited fermionic modes. The number of exited \( \eta \)-modes does not change, so this process is not prohibited by the transformation \([t]\) ( the instanton process below (30) is prohibited by \([t]\)). Taking the transformation \([i]\) or \([i + 1]\) into consideration, we see that the two instanton effects cancel each other if \( n - i \) is odd, and they add up if \( n - i \) is even.

Applying this simple rule to all the instanton solutions, one can easily select all true vacuums.

Let us first consider \( \text{SO}(2n - 1) \). We mark the elements of the critical points off two by two

\[
\text{diag}(\epsilon_1, \epsilon_2 \mid \epsilon_3, \epsilon_4 \mid \cdots | \epsilon_{2n-3}, \epsilon_{2n-2} \mid \epsilon_{2n-1}).
\]

From the above rule, one sees that there are instanton effects between the two states around the following critical points

\[
\text{diag}(\epsilon_1, \epsilon_2 \mid \cdots | 1, -1 \mid \cdots | \epsilon_{2n-1}) \leftrightarrow \text{diag}(\epsilon_1, \epsilon_2 \mid \cdots | -1, 1 \mid \cdots | \epsilon_{2n-1}).
\]

On the contrary, if \( \epsilon_{2i-1} = \epsilon_{2i} (i = 1, 2, \cdots n - 1) \), the corresponding state does not affected by instanton effects, and it is decided to be a true vacuum. Thus, the critical points that correspond to true vacuums are

\[
\text{diag}(\epsilon_1, \epsilon_1 \mid \epsilon_3, \epsilon_3 \mid \cdots | \epsilon_{2n-3}, \epsilon_{2n-3} \mid 1).
\]

Their Morse indices are \( \sum_{i=\text{odd}}^{2n-3} (4n - 3 - 2i) \epsilon_i \). This formula agrees with the de Rham
cohomology of SO\((2n-1)\):\(\Lambda(x_3, x_7, \cdots, x_{4n-5})\).\(^5\)

In the same way, for SO\((2n)\), the critical points that correspond to true vacuums are found to be

\[
\text{diag}(\epsilon_{2n-1} | \epsilon_1, \epsilon_1 | \epsilon_3, \epsilon_3 | \cdots | \epsilon_{2n-3}, \epsilon_{2n-3} | \epsilon_{2n-1}).
\]

Their Morse indices are

\[
\sum_{i=\text{odd}}^{2n-3} (4n-3-2i)\epsilon_i + (2n-1)\frac{\epsilon_{2n-1} + 1}{2},
\]

which agrees with the de Rham cohomology of SO\((2n)\):\(\Lambda(x_3, x_7, \cdots, x_{4n-5}, x_{2n-1})\).\(^5\)

5 \quad G_2

The group G_2 is a 14-dimensional submanifold of O(7). An element \(A\) of G_2 is given by three 7-components vectors \(a_i (i = 1, 2, 4);\)

\[
M(7, 3; \mathbb{R}) \ni G_2 \ni A = (a_1, a_2, a_4),
\]

with conditions \(|a_i| = 1 (i = 1, 2, 4)\) and \((a_1, a_2) = (a_1, a_4) = (a_2, a_4) = (a_1a_2, a_4) = 0.\) The inner product and the absolute value are defined by \((a_i, a_j) = \sum_{k=1}^{7} a_{ki}a_{kj},\)

\(|a_i| = \sqrt{(a_i, a_i)}\) and the products \(a_i a_j\) are defined using octonians(see Appendix ).

Fix real numbers \(c_1, c_2\) and \(c_4\) with \(c_1 > c_2 > c_4 > 0.\) Then

\[
h = c_1a_{11} + c_2a_{22} + c_4a_{44}
\]

is a Morse function on G_2.\(^4\) This Morse function has eight critical points with the Morse indices \(l = 0, 3, 5, 6, 8, 9, 11\) and 14. We are interested in the following four classical vacuums \(|5\rangle, |6\rangle, |8\rangle\) and \(|9\rangle\) which are localized around the critical points \((a_{11}, a_{22}, a_{44}) = (-1, 1, -1), (1, -1, -1), (-1, 1, 1)\) and \((1, -1, 1)\), respectively.

Around each critical point \((\epsilon_1, \epsilon_2, \epsilon_4), h\) has the expansion \(^4\)

\[
h = \sum_i \epsilon_i c_i + \frac{1}{2} \sum_{i < j} (\lambda_{ij} x_{ij}^2 + \mu_{ij} n_{ij}^2) - \epsilon_1 c_1(a_{31}^2 + a_{51}^2 + a_{71}^2) - \epsilon_2 c_2(a_{32}^2 + a_{52}^2 + a_{72}^2)
\]

\[
- \epsilon_4 c_4(a_{54}^2 + a_{64}^2 + a_{74}^2) - \frac{1}{9}(a_{34} a_{52} a_{61})B(a_{34} a_{52} a_{61})^t.
\]

10
where $i,j = 1,2,4$, $\lambda_{ij}, \mu_{ij}, \xi_{ij}$ and $\eta_{ij}$ are defined by (27) and (28), and $B$ is the following $3 \times 3$ matrix

$$B = \begin{pmatrix}
4\epsilon_4 c_4 + \epsilon_2 c_2 + \epsilon_1 c_1 & 2\epsilon_2 c_4 + 2\epsilon_4 c_2 - \epsilon_1 \epsilon_2 \epsilon_4 c_1 & 2\epsilon_1 c_4 - \epsilon_1 \epsilon_2 \epsilon_4 c_2 + 2\epsilon_4 c_1 \\
2\epsilon_2 c_4 + 2\epsilon_4 c_2 - \epsilon_1 \epsilon_2 \epsilon_4 c_1 & \epsilon_4 c_4 + 4\epsilon_2 c_2 + \epsilon_1 c_1 & \epsilon_1 \epsilon_2 \epsilon_4 c_4 - 2\epsilon_1 c_2 - 2\epsilon_2 c_1 \\
2\epsilon_1 c_4 - \epsilon_1 \epsilon_2 \epsilon_4 c_2 + 2\epsilon_4 c_1 & \epsilon_1 \epsilon_2 \epsilon_4 c_4 - 2\epsilon_1 c_2 - 2\epsilon_2 c_1 & \epsilon_4 c_4 + \epsilon_2 c_2 + 4\epsilon_1 c_1
\end{pmatrix}.$$

From (39) one finds classical vacuums as

$$|5\rangle = |a_{32}, a_{62}, a_{72}, a_{24} + a_{42}, (a_{34}, a_{52}, a_{61})\rangle,$$

$$|6\rangle = |a_{31}, a_{51}, a_{71}, a_{12} + a_{21}, a_{14} + a_{41}, (a_{34}, a_{52}, a_{61})\rangle,$$

$$|8\rangle = |a_{32}, a_{62}, a_{72}, a_{24} - a_{42}, (a_{34}, a_{52}, a_{61}), a_{54}, a_{64}, a_{74}\rangle,$$

$$|9\rangle = |a_{31}, a_{51}, a_{71}, a_{12} - a_{21}, a_{14} - a_{41}, (a_{34}, a_{52}, a_{61}), a_{54}, a_{64}, a_{74}\rangle,$$

where the modes in the ket vectors denote exited fermionic modes and $(a_{34}, a_{52}, a_{61})$ represents a certain linear combination of $a_{34}, a_{52}$ and $a_{61}$.

Let us discuss instanton solutions of $G_2$. Embedding $G_2$ into $O(7)$ (see Appendix), we see that the critical points that corresponds to $|5\rangle, |6\rangle, |8\rangle$ and $|9\rangle$ are elements of $SO(7)$. The Morse function (38) will be obtained from (24) by imposing some restrictions on the coordinates for $SO(7)$. Accordingly, instanton solutions of $G_2$ should be included in that of $SO(7)$. Consider the pair $|5\rangle$ and $|6\rangle$. There are four instanton solutions of $SO(7)$ that connect $(a_{11}, a_{22}, a_{44}) = (-1,1,-1)$ and $(1,-1,1)$ (these are diag($-1,1,-1,1,-1,1$) and diag(1,1,-1,-1,1,1) in $SO(7)$, respectively):

$$\begin{pmatrix}
\cos \alpha & \pm \sin \alpha \\
\pm \sin \alpha & -\cos \alpha \\
-1 & -1 \\
-\cos \beta & \pm \sin \beta \\
\pm \sin \beta & \cos \beta \\
1 & \pm \sin \beta
\end{pmatrix},\quad \text{Eq.}(41)$$

Setting $\alpha = \beta$, Eq.(41) represents two paths in $G_2$ and gives two instanton solutions in $G_2$. Thus, we have found that there are exactly two instanton solutions in $G_2$ that connect $|5\rangle$ and $|6\rangle$. Under the transformation $a_{12} \to -a_{12}, a_{21} \to -a_{21}$, which
induces the transformation $a_{56} \rightarrow -a_{56}$, $a_{65} \rightarrow -a_{65}$ in terms of $O(7)$, the two instanton solutions are interchanged. We use this transformation to elucidate true vacuums. In the same way, one finds that there are two similar instanton solutions between $|8\rangle$ and $|9\rangle$.

We will determine whether there are instanton effects or not between the classical vacuums (40). A reflection transformation that is useful to this purpose is such transformation under which the following each set of modes $(a_{12}, a_{21}), (a_{34}, a_{52}, a_{61})$, $(a_{14}, a_{41})$ and $(a_{24}, a_{42})$ has the same parities. For $SO(n)$, we have seen that a reflection transformation that transform a group element to a group element makes the theory invariant. As $G_2$ is a subgroup of $O(7)$, some reflection symmetries for $SO(n)$ will survive as symmetries for $G_2$ with suitable restrictions. For $G_2$, one finds eight reflection transformations that transform a group element to a group element. These eight transformations, which are certain combined transformations of $[i]$ for $SO(n)$, will be symmetries of the theory. Among these eight transformations, the four transformations in Table I interchange the two instanton solutions (41) with $\alpha = \beta$.

Table I.

The classical vacuums $|5\rangle$ and $|6\rangle$ have the same parities under each of the four transformations. Thus, the matrix element $(6|d_6|5\rangle$ is non-vanishing, and $|5\rangle$ and $|6\rangle$ are not true vacuums. In the same way, one finds that $|8\rangle$ and $|9\rangle$ are not true vacuums. We have found that true vacuums are $|0\rangle$, $|3\rangle$, $|11\rangle$ and $|14\rangle$ in accordance with de Rham cohomology of $G_2$; $b_p = 1$ for $p = 0, 3, 11, 14$ and $b_p = 0$ for the others.
In this section we discuss $U(2)$, which is topologically identical with $SO(2) \times SU(2)$.

We parametrize an element $A$ of $U(2)$ as

$$A = \begin{pmatrix} x_{11} + iy_{11} & x_{12} + iy_{12} \\ x_{21} + iy_{21} & x_{22} + iy_{22} \end{pmatrix} = \begin{pmatrix} \cos \theta e^{i(\alpha + \phi + \psi)} & -\sin \theta e^{i(\alpha + \phi + \psi)} \\ \sin \theta e^{i(\alpha + \phi + \psi)} & \cos \theta e^{i(\alpha + \phi + \psi)} \end{pmatrix}, \quad (42)$$

$$0 \leq \theta \leq \pi, 0 \leq \alpha, \phi, \psi \leq 2\pi.$$

Fix real numbers $c_1, c_2$ with $c_2 > 2c_1 > 0$. Then

$$h = c_1 x_{11} + c_2 x_{22} = (c_1 \cos(\alpha + \phi + \psi) + c_2 \cos(\alpha - \phi - \psi)) \cos \theta \quad (43)$$

is a Morse function on $U(2)$. This Morse function has four critical points

$$P^{(0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, P^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, P^{(3)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, P^{(4)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (44)$$

We denote the corresponding classical vacuums as $|0\rangle, |1\rangle, |3\rangle$ and $|4\rangle$, respectively.

Around each critical point $(\epsilon_1, \epsilon_2)$, $h$ has the expansion

$$8h = (c_1 + c_2) (8 - (\epsilon_1 x_{12} - \epsilon_2 x_{21})^2 - (\epsilon_1 y_{12} + \epsilon_2 y_{21})^2) - 4\epsilon_1 c_1 y_{11}^2 - 4\epsilon_2 c_2 y_{22}^2. \quad (45)$$

From this expansion one finds that

$$|0\rangle = |0\rangle, \quad |1\rangle = |\xi\rangle, \quad (46)$$

$$|3\rangle = |\eta, x_{12} + x_{21}, y_{12} - y_{21}\rangle, \quad |4\rangle = |\xi, \eta, x_{12} - x_{21}, y_{12} + y_{21}\rangle,$$

where $\xi = \alpha + \phi + \psi$ and $\eta = \alpha - \phi - \psi$. We introduce the following metric

$$g_{ij} = \frac{1}{2} \text{tr} \left[ \frac{\partial A^\dagger}{\partial \theta^i} \frac{\partial A}{\partial \theta^j} \right], \quad (47)$$

where the indices $i$ denote $\theta, \alpha, \psi$ and $\phi$. The non-zero components of $g_{ij}$ and Christoffel symbols are

$$g_{ii} = 1, g_{\phi\phi} = g_{\psi\psi} = \cos 2\theta, \quad (48)$$

$$\Gamma^\theta_{\phi\phi} = \Gamma^\theta_{\psi\psi} = \sin 2\theta, \Gamma^\phi_{\theta\phi} = \Gamma^\psi_{\theta\psi} = -\frac{1}{\sin 2\theta}.$$
The covariant derivatives are

\[
\nabla_\alpha = \partial_\alpha, \quad \nabla_\theta = \partial_\theta + \frac{1}{\sin 2\theta} (\hat{\psi}^* \hat{\psi} + \hat{\psi}^* \hat{\psi}), \\
\nabla_\phi = \partial_\phi - \sin 2\theta \hat{\psi}^* \hat{\psi} + \frac{1}{\sin 2\theta} \hat{\psi}^* \hat{\psi}, \\
\nabla_\psi = \partial_\psi - \sin 2\theta \hat{\psi}^* \hat{\psi} + \frac{1}{\sin 2\theta} \hat{\psi}^* \hat{\psi},
\]

(49)

Consider the following simultaneous transformation

\[
\theta^i, \hat{\psi}^i, \hat{\psi}^i \rightarrow -\theta^i, -\hat{\psi}^i, -\hat{\psi}^i, \theta^i = \alpha, \psi, \phi,
\]

(50)

which makes the Morse function \( h \) invariant. Under this transformation the covariant derivatives \( \nabla_\alpha, \nabla_\psi \) and \( \nabla_\phi \) reverse the sings and \( \nabla_\theta \) is invariant. Accordingly, the supercharges \( d_h, d_h^\dagger \) and the Hamiltonian is invariant under (50).

The gradient flow equations are

\[
\frac{d\theta}{dt} = -(c_1 \cos \xi + c_2 \cos \eta) \sin \theta, \\
\frac{d\alpha}{dt} = -(c_1 + c_2) \sin \alpha \cos(\phi + \psi) + (c_2 - c_1) \cos \theta \cos \alpha \sin(\phi + \psi), \\
\frac{d\phi}{dt} = \frac{d\psi}{dt} = ((c_2 - c_1) \sin \alpha \cos(\phi + \psi) - (c_1 + c_2) \cos \alpha \sin(\phi + \psi)) \frac{\cos \theta}{1 + \cos 2\theta}.
\]

(51)

The degrees of the freedom has reduced to three. Let us consider the equations in the \((\theta, \eta)\)–plane. There are four fixed points \((\theta, \eta) = (0, 0), (0, \pi), (\pi, 0)\) and \((\pi, \pi)\).

Around these fixed points the gradient flow equations (51) become simple:

\[
\theta \approx 0, \quad \frac{d\eta}{dt} = -2c_2 \sin \eta, \\
\theta \approx \pi, \quad \frac{d\eta}{dt} = 2c_2 \sin \eta, \\
\eta \approx 0, \quad \frac{d\theta}{dt} = -(c_2 + c_1 \cos \xi) \sin \theta, \\
\eta \approx \pi, \quad \frac{d\theta}{dt} = (c_2 - c_1 \cos \xi) \sin \theta.
\]

(52)

One sees that \((\theta, \eta) = (0, 0), (\pi, \pi)\) are stable fixed points and that \((\theta, \eta) = (0, \pi), (\pi, 0)\) are unstable fixed points irrespective to the value of \(\xi\).
The initial and the final conditions of a solution that connects $P^{(0)}$ and $P^{(1)}$ are both $\cos \theta \cos \eta = -1$. So, for solutions of (51) that connect $P^{(0)}$ and $P^{(1)}$, one concludes that $(\theta, \eta) \equiv (0, \pi)$ or $(\pi, 0)$. Under these conditions the gradient flow equation of $\xi$ becomes

$$\frac{d\xi}{dt} = \pm 2c_1 \sin \xi. \quad (53)$$

There are exactly two instanton solutions that connect $P^{(0)}$ and $P^{(1)}$

$$A = \begin{pmatrix} e^{\pm i\xi} & 0 \\ 0 & -1 \end{pmatrix}, \quad 0 \leq \xi \leq \pi. \quad (54)$$

For solutions that connects $P^{(3)}$ and $P^{(4)}$, one sees that $(\theta, \eta) \equiv (0, 0)$ or $(\pi, \pi)$ and that there are exactly two instanton solutions

$$A = \begin{pmatrix} e^{\pm i\xi} & 0 \\ 0 & 1 \end{pmatrix}, \quad 0 \leq \xi \leq \pi. \quad (55)$$

By the transformation (50), $\xi$ reverses the sign and the two instanton solutions (54) (55) are interchanged each other. Under this transformation $|0\rangle$ and $|3\rangle$ have even parities and $|1\rangle$ and $|4\rangle$ have odd parities. Therefor, there are not instanton effects in the processes $P^{(0)} \rightarrow P^{(1)}$ and $P^{(3)} \rightarrow P^{(4)}$. Our result for $U(2)$ is that $|0\rangle, |1\rangle, |3\rangle$ and $|4\rangle$ are all true vacuums. This result agrees with the de Rham cohomology of $U(2)$; $b_p = 1$ for $p = 0, 1, 3, 4$ and $b_p = 0$ for the others. Our result is one of the examples of the Kunneth formula for a product manifold $M \times N$:

$$b_k(M \times N) = \sum_{p+q=k} b_p(M)b_q(N).$$

**7 SUMMARY**

We have studied supersymmetric quantum mechanics on $RP_n, SO(n), G_2$ and $U(2)$ as concrete examples of Witten’s Morse theory. We have identified all the true vacuums for each theory. The number of the true vacuums agrees with the de Rham cohomology for each manifold. We have seen that the simple instanton picture that the author proposed for $SO(n)^4$ holds for the above manifolds. For each pair of adjacent classical vacuums, there are exactly two instanton solutions which are
interchanged by a reflection transformation of the theory. Owing to the reflection symmetry, it is easy to decide whether the two instanton effects cancel each other or not.

**ACKNOWLEDGMENTS**

The author thanks Dr. Y. Yasui for introducing him to this subject. The author thanks Dr. H. Usui for useful discussions.

**APPENDIX: MULTIPLICATION RULE in \( G_2 \) and EMBEDDING**

We note the multiplication rule in \( G_2 \). For elements \( a_i \) of \( G_2 \), the following expansions by octonians \( e_j \) are taken into account

\[
a_i = \sum_{j=1}^{7} a_j e_j = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{i7} \end{pmatrix}.
\]

Note that \( e_0 = 1 \) does not appear in the expansion.

Table. II.

The multiplication rules of octonians are defined as usual (see Table II); for example, \( e_1 e_2 = e_3, e_5 e_3 = -e_6, \) and \( e_i^2 = -1, e_i e_j = -e_j e_i (i \neq j) \). Explicitly, \( a_3 = a_1 a_2 \) is given by

\[
a_3 = \begin{pmatrix}
-a_{31}a_{22} + a_{21}a_{32} - a_{51}a_{42} + a_{41}a_{52} - a_{71}a_{62} + a_{61}a_{72} \\
a_{31}a_{12} - a_{11}a_{32} + a_{61}a_{42} - a_{41}a_{62} - a_{71}a_{52} + a_{51}a_{72} \\
a_{31}a_{12} + a_{11}a_{22} - a_{71}a_{42} - a_{41}a_{52} + a_{51}a_{62} + a_{41}a_{72} \\
a_{51}a_{12} - a_{61}a_{22} + a_{71}a_{32} - a_{11}a_{52} + a_{21}a_{62} - a_{31}a_{72} \\
-a_{41}a_{12} + a_{71}a_{22} + a_{61}a_{32} + a_{11}a_{42} - a_{31}a_{62} - a_{21}a_{72} \\
a_{71}a_{12} + a_{41}a_{22} + a_{51}a_{32} - a_{21}a_{42} - a_{31}a_{52} - a_{11}a_{72} \\
-a_{61}a_{12} - a_{51}a_{22} - a_{41}a_{32} + a_{31}a_{42} + a_{21}a_{52} + a_{11}a_{62}
\end{pmatrix}.
\]

An element \( A \) of \( G_2 \) is embedded into \( O(7) \) as follows

\[
A = (a_1, a_2, a_4) \rightarrow (a_1, a_2, a_1a_2, a_4, a_1a_4, a_4a_2, a_1a_4a_2) \in O(7).
\]
References

TABLE I. Symmetry transformations that interchange the two instanton solutions (41) with $\alpha = \beta$. The minus signs mean that the corresponding elements $a_{ki}$, $i = 1, \ldots, 4$, $k = 1, \ldots, 7$, reverse the signs ($a_3$ is not independent).

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TABLE II. Multiplication table of octonians $e_j$.

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