Quantum Numbers of the Θ Vacuum in (2+1)-Dimensional Spinor Electrodynamics: Charge and Magnetic Flux.*

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February 11, 1998

Abstract

A singular configuration of an external static vector field in the form of a magnetic string polarizes the vacuum of a second-quantized theory on the plane orthogonal to the string axis. The most general boundary conditions at the punctured singular point that are compatible with the self-adjointness of two-dimensional Dirac Hamiltonian are considered. The dependences of the induced vacuum quantum numbers on the parameter of the self-adjoint extension, on the string flux, and on the choice of irreducible reoresentation of the matrices in (2+1)-dimensional spacetime are disscussed.

1. It is well known that, in the fermion vacuum, a singular magnetic monopole

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induces the charge [1, 2, 3]

\[ Q^{(I)} = -\frac{1}{\pi} \arctg \left( \frac{\Theta}{2} \right), \]  

(1)

where \( \Theta \) is the self-adjoint-extension parameter that determines the boundary conditions at the punctured point corresponding to the monopole position. As a result, the monopole actually becomes a dion that violates \( CP \) symmetry and the condition of Dirac quantization.

In this study, we consider quantum numbers induced in the fermion vacuum by a singular static magnetic string. In relation to the elimination of a point, the elimination of a line leads to more substantial changes in the topology of space (the first homotopic group becomes nontrivial). Therefore, the properties of the \( \Theta \) vacuum are much richer in this case than in the case specified by (1). We will restrict our consideration to the plane orthogonal to the plane axis and study (2+1)-dimensional spinor electrodynamics on the plane with the punctured point corresponding to the string position. It can be shown [4, 5] that, in this case, charge and magnetic flux are induced in the vacuum. These quantities depend both on the parameter of the self-adjoint extension and on the magnetic flux of the string. In this study, we perform a more detailed analysis of these dependences.

2. Let us write the time-independent Dirac equation in an external vector field \( \mathbf{V}(\mathbf{x}) \) in the form

\[ \left\{ -i \alpha \left[ \frac{\partial}{\partial x} - i \mathbf{V}(\mathbf{x}) \right] + \beta m \right\} \psi(\mathbf{x}) = E \psi(\mathbf{x}), \]  

(2)

where

\[ \alpha = \gamma^0 \gamma, \quad \beta = \gamma^0, \]  

(3)

and \( \gamma \) and \( \gamma^0 \) are the Dirac \( \gamma \) matrices. In the (2+1)-dimensional spacetime \( (\mathbf{x}, t) = (x^1, x^2, t) \), the Clifford algebra does not have a faithful irreducible representation. Instead, it has two nonequivalent irreducible representations which differ from one another in the following way:

\[ i \gamma^0 \gamma^1 \gamma^2 = s, \quad s = \pm 1. \]  

(4)
Choosing the matrix $\gamma^0$ in the diagonal form

$$\gamma^0 = \sigma_3,$$  \hspace{1cm} (5)

we obtain

$$\gamma^1 = e^{i\sigma_3 \chi^s} i \sigma_1 e^{-i\sigma_3 \chi^s}, \quad \gamma^2 = e^{i\sigma_3 \chi^s} i \sigma_2 e^{-i\sigma_3 \chi^s},$$  \hspace{1cm} (6)

where $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the Pauli matrices, and $\chi_1$ and $\chi_{-1}$ are parameters that are varied in the interval $0 \leq \chi_s < 2\pi$ to go over to equivalent representations.

We choose the configuration of the external field $V(x) = (V_1(x), V_2(x))$ in a form of a singular magnetic vortex

$$x \times V(x) = \Phi^{(0)}, \quad B(x) \equiv \frac{\partial}{\partial x} \times V(x) = 2\pi \Phi^{(0)} \delta(x),$$  \hspace{1cm} (7)

where $\Phi^{(0)}$ is the total flux (in $2\pi$ units) of the vortex — that is, of the string that intersects the plane at the point $x = 0$. The wave function on the plane with the punctured singular point $x = 0$ obeys the most general condition (see. [4] for more details)

$$\psi(r, \phi + 2\pi) = e^{i2\pi \Upsilon} \psi(r, \phi),$$  \hspace{1cm} (8)

where $r = |x|$ and $\phi = \arctg(x^2/x^1)$ are polar coordinates, and $\Upsilon$ is a continuous real parameter.

A solution that satisfies the Dirac equation (2) in the field of the singular string (7) and condition (8) can be represented as

$$\psi(x) = \sum_{n=-\infty}^{\infty} \begin{pmatrix} f_n(r) e^{i(n+\Upsilon)\phi} \\ g_n(r) e^{i(n+\Upsilon+s)\phi} \end{pmatrix},$$  \hspace{1cm} (9)

where radial functions $f_n(r)$ and $g_n(r)$ obey the system of equations

$$e^{-i\chi^s}[-\partial_r + s(n - \Phi^{(0)} + \Upsilon)r^{-1}]f_n(r) = (E + m)g_n(r),$$

$$e^{i\chi^s}[-\partial_r + s(n - \Phi^{(0)} + \Upsilon + s)r^{-1}]g_n(r) = (E - m)f_n(r).$$  \hspace{1cm} (10)

For integer values of $\Phi^{(0)} - \Upsilon$, the condition of square integrability makes it possible to construct solutions for the top and bottom components ($f_n(r)$
and $g_n(r)$, respectively) of the spinor wave function in such way that they are regular at the point $r = 0$. When $\Phi^{(0)} - \Upsilon$ is fractional, the same situation occurs only for $n \neq n_0$, where

$$n_0 = \left\lceil \Phi^{(0)} - \Upsilon \right\rceil + \frac{1}{2} - \frac{1}{2}s; \quad (11)$$

here $\lceil u \rceil$ is the integral part of $u$ (the largest integer that is less than or equal to $u$). For $n = n_0$, each of linear independent solutions satisfies the requirement of square integrability. If one solution is chosen to have a regular top and a singular bottom component, the other has a regular bottom and a singular top component. In other words, the partial Dirac Hamiltonian, corresponding to $n \neq n_0$, represents the family of self-adjoint extensions parametrized by one continuous real variable ($\Theta$). It follows that, instead of satisfying the regularity condition at the point $r = 0$, the radial functions for $n = n_0$, obey the condition [6]

$$\lim_{r \to 0} (|m| r)^F \cos \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) f_{n_0}(r) = -e^{i\chi_s} \lim_{r \to 0} (|m| r)^{1-F} \sin \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) g_{n_0}(r), \quad (12)$$

where

$$F = \frac{1}{2} + s \left( \{ \Phi^{(0)} - \Upsilon \} - \frac{1}{2} \right); \quad (13)$$

here $\{ u \} = u - \lceil u \rceil$ is the fractional part of $u$, $0 \leq \{ u \} < 1$. It is worth noting, in $(2+1)$-dimensional space-time, the mass parameter $m$ in the Dirac equation (2) can take both positive and negative values. We emphasize once again that relation (12) holds only for $0 < F < 1$, because? in the case of $F = 0$ (integral values of $\Phi^{(0)} - \Upsilon$), the radial functions are regular both for $n = n_0$ and for $n \neq n_0$.

Suppose that $0 < F < 1$. When the condition

$$-\infty < \text{sgn}(m) \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) < 0, \quad (14)$$

is satisfied, the spectrum of states involves not only a continuum ($|E| > |m|$) but also a bound state ($|E| < |m|$) whose energy is determined by the equation [5]

$$\frac{(1 + m^{-1}E)^{1-F}}{(1 - m^{-1}E)^{F}} = -\text{sgn}(m)A; \quad (15)$$
where
\[ A = 2^{1-2F} \frac{\Gamma(1-F)}{\Gamma(F)} \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right), \tag{16} \]
and
\[ \text{sgn}(u) = \begin{cases} 1, & u < 0 \\ -1, & u > 0 \end{cases}. \]
In equation (16), \( \Gamma(u) \) is the Euler gamma-function.

It can be seen from (15) that the energy of the bound state vanishes \( (E = 0) \) when
\[ \text{sgn}(m)A = -1. \tag{17} \]

3. The expressions for the density of the vacuum charge and for the angular component of the vacuum current are, respectively,
\[ \rho(r) = -\frac{1}{2} \sum_{E} \sum_{n=-\infty}^{\infty} \text{sgn}(E)[f_n^*(r)f_n(r) + g_n^*(r)g_n(r)] \tag{18} \]
and
\[ j^\phi(r) = -e^{i\chi_s} \sum_{E} \sum_{n=-\infty}^{\infty} \text{sgn}(E)f_n^*(r)g_n(r), \tag{19} \]
where the symbol \( \sum_{E} \) denotes summation over discrete values of the energy \( E \) and integration (with a certain measure) over its continuous values. The radial component of the vacuum current vanishes identically by virtue of the condition
\[ e^{i\chi_s} f_n^*(r)g_n(r) = e^{-i\chi_s} g_n^*(r)f_n(r). \tag{20} \]
Integrating (18) over the entire two-dimensional space, we obtain the total vacuum charge
\[ Q^{(I)} = 2\pi \int_{0}^{\infty} dr r \rho(r). \tag{21} \]
A global characteristic associated with the vacuum current is the total magnetic flux of the vacuum. In \( 2\pi \) units, this magnetic flux is given by
\[ \Phi^{(I)} = \frac{e^2}{2} \int_{0}^{\infty} dr r^2 j^\phi(r). \tag{22} \]
where \( e \) is the coupling constant having dimensions of \( \sqrt{|m|} \) in (2+1)-dimensional space-time. In deriving relation (22), we assumed that the vacuum magnetic field is related to the vacuum current by the Maxwell equation

\[
\partial_r B_{(I)}(r) = -e^2 j^\phi(r),
\]

and that the vacuum current decreases sufficiently fast (exponentially — see [5]) for \( r \to \infty \).

By using explicit expressions for the solutions to the system of equations (10), we can find the functional dependence of the vacuum quantum numbers (21) and (22) on the parameter of self-adjoint extension and on the magnetic flux of the string. It is obvious that, for a fixed value of \( \Theta \), the vacuum quantum numbers depend periodically on \( \Phi(0) - \Upsilon \), the period being equal to unity. For the integral values of \( \Phi(0) - \Upsilon \), the radial functions \( f_n(r) \) and \( g_n(r) \) are regular at the point \( r = 0 \), and this case is indistinguishable from that of the trivial vacuum \( \Phi(0) = \Upsilon = 0 \). As a result, we obtain

\[
Q^{(I)} = \Phi^{(I)} = 0, \quad F = 0.
\]

It was noted above that, for nonintegral values of \( \Phi(0) - \Upsilon \), the radial functions are regular for \( n \neq n_0 \) and satisfy condition (12) at \( n = n_0 \). Thus, we have [4, 5]

\[
Q^{(I)} = \begin{cases} 
-\frac{1}{2} \text{sgn}(m) F, & \Theta = \frac{\pi}{2} \mod 2\pi \\
\frac{1}{2} \text{sgn}(m) (1 - F), & \Theta = \left(-\frac{\pi}{2}\right) \mod 2\pi \\
-\text{sgn}(m) \left[ \frac{1}{2} \left( F - \frac{1}{2} \right) + S_1(F, \Theta) \right] - S_2(F, \Theta), & \Theta \neq \left( \pm \frac{\pi}{2} \right) \mod 2\pi
\end{cases}
\]

\[
\Phi^{(I)} = -\frac{e^2 F (1 - F)}{2\pi |m|} \left[ \frac{1}{6} \left( F - \frac{1}{2} \right) + S_1(F, \Theta) \right],
\]

where

\[
S_1(F, \Theta) = \frac{1}{4\pi} \int_1^\infty \frac{dv}{v^{\sqrt{v-1}}} \frac{Av^F - A^{-1}v^{1-F}}{Av^F + 2\text{sgn}(m) + A^{-1}v^{1-F}},
\]

\[
S_2(F, \Theta) = \frac{F - \frac{1}{2}}{\pi} \int_1^\infty \frac{dv}{v^{\sqrt{v-1}}} \frac{\sqrt{v-1}}{Av^F + 2\text{sgn}(m) + A^{-1}v^{1-F}},
\]
and the quality $A$ is determined by (16).

As might have been expected, the vacuum quantum numbers (25) and (26) remain unchanged upon going over to an equivalent representation (they do not depend on the parameter $\chi_s$) and, in general, change upon going over to a nonequivalent representation (the substitution $s \rightarrow -s$ is equivalent to the substitution $F \rightarrow 1 - F$). In the following, we treat the variables $\Theta$ and $F$ as independent ones. The substitution $m \rightarrow -m$ in the expressions for $Q^{(l)}$ and $\text{sgn}(m)\Phi^{(l)}$ is then equivalent to the simultaneous substitutions $\Theta \rightarrow \Theta + \pi$ and $F \rightarrow 1 - F$.

The function $S_2(F, \Theta)$ in (28) can be represented in the form

$$S_2(F, \Theta) = \begin{cases} 
-\frac{1}{\pi} \int_0^1 dw \sqrt{1-w^{(1/2-F)^{-1}}} 
\frac{A w^2 + 2 \text{sgn}(m) w^{(1-F)(1/2-F)^{-1}} + A^{-1}}{A^{-1} w^2 + 2 \text{sgn}(m) w^{F(1/2-F)^{-1}} + A}, & 0 < F < \frac{1}{2} \\
0, & F = \frac{1}{2} \\
\frac{1}{\pi} \int_0^1 dw \sqrt{1-w^{F-\frac{1}{2}}} 
\frac{A w^2 + 2 \text{sgn}(m) w^{F(1/2-F)^{-1}} + A}{A^{-1} w^2 + 2 \text{sgn}(m) w^{F(1/2-F)^{-1}} + A}, & \frac{1}{2} < F < 1 
\end{cases}$$

(29)

The following relations also hold:

$$\frac{\partial}{\partial \Theta} [\text{sgn}(m)S_1(F, \Theta) + S_2(F, \Theta)] = \lim_{v \rightarrow \infty} \frac{1}{\pi \cos \Theta} \frac{\sqrt{v - 1}}{A v^F + 2 \text{sgn}(m) + A^{-1} v^{1-F}} =
\begin{cases} 
0, & F \neq \frac{1}{2} \\
\frac{1}{2\pi}, & F = \frac{1}{2} 
\end{cases}$$

(30)

$$\frac{\partial}{\partial F} [\text{sgn}(m)S_1(F, \Theta) + S_2(F, \Theta)] = \lim_{v \rightarrow \infty} \frac{1}{\pi} \frac{\sqrt{v - 1} \left( \pi \text{ctg} \pi + \ln \frac{v}{4} \right)}{A v^F + 2 \text{sgn}(m) + A^{-1} v^{1-F}} =
\begin{cases} 
0, & F \neq \frac{1}{2} \\
0, & \Theta = \left( \pm \frac{\pi}{2} \right) \mod 2\pi, F = \frac{1}{2} \\
\infty, & \Theta \neq \left( \pm \frac{\pi}{2} \right) \mod 2\pi, F = \frac{1}{2} 
\end{cases}$$

(31)
In the case of $F = \frac{1}{2}$ we can obtain (compare with (1))

$$Q^{(I)} = -\frac{1}{2\pi} \arctg \left\{ \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} (1 - \text{sgn}(m)) \right) \right\},$$  \hspace{1cm} (32)

$$\Phi^{(I)} = -\frac{e^2}{8\pi^2 m} \arctg \left\{ \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} (1 - \text{sgn}(m)) \right) \right\}. \hspace{1cm} (33)$$

4. The vacuum magnetic flux considered as a function of $\Theta$ has a discontinuity at $\Theta = \Theta_0$, where

$$\Theta_0 = \left\{ \frac{3\pi}{2} - 2\text{sgn}(m)\arctg \left[ \frac{2^{2F-1} \Gamma(F)}{\Gamma(1-F)} \right] \right\} \mod 2\pi; \hspace{1cm} (34)$$

At this value of $\Theta$, the energy of the bound state vanishes (see (17)). The vacuum charge, as a function of $\Theta$, has discontinuities not only at $\Theta = \Theta_0$ but also at $\Theta = \Theta_-'$ for $0 < F < \frac{1}{2}$ and at $\Theta = \Theta_+$ for $\frac{1}{2} < F < 1$, where

$$\Theta_\pm = \left( \mp \frac{\pi}{2} \right) \mod 2\pi; \hspace{1cm} (35)$$

By virtue of relation (30), the vacuum charge is a constant for $F \neq \frac{1}{2}$. It should be emphasized that the vacuum quantum numbers are indeterminate at $\Theta = \Theta_0$ given by (34).

For $\Theta$ values from the region determined by (14), the vacuum quantum numbers as functions of $F$ have discontinuities at the $F$ value satisfying relation (17). The values of the vacuum quantum numbers are indeterminate at this point discontinuity. In addition, the vacuum charge as a function of $F$ has a discontinuity at $F = \frac{1}{2}$, provided that $\Theta \neq (\pm \frac{\pi}{2}) \mod 2\pi$:

$$Q^{(I)} \mid_{F = \frac{1}{2}} = Q^{(I)} \mid_{F = \frac{1}{2}} + \frac{1}{\pi} \arctg \left[ \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) \right]; \hspace{1cm} (36)$$

$$Q^{(I)} \mid_{F \searrow \frac{1}{2}} = Q^{(I)} \mid_{F = \frac{1}{2}} + \frac{1}{\pi} \arctg \left[ \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) \right] - \frac{1}{2} \text{sgn} \left[ \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) \right]; \hspace{1cm} (37)$$

When $\Theta = \left\{ \frac{\pi}{2} [1 + \text{sgn}(m)] \right\} \mod 2\pi$, the two discontinuities coincide, and the value of the vacuum charge is indeterminate at this point.
Taking into account (31), (36) and (37), we find that, for \( F \neq \frac{1}{2} \), the vacuum charge can be represented as

\[
Q^{(I)} = \begin{cases} 
\frac{1}{2} \text{sgn}(m)(1 - F), & -1 < \text{sgn}(m)A < \infty \\
-\frac{1}{2} \text{sgn}(m)(1 + F), & -\infty < \text{sgn}(m)A < -1 \\
-\frac{1}{2} \text{sgn}(m)F, & A^{-1} = 0 
\end{cases}, \quad 0 < F < \frac{1}{2};
\]

\[
Q^{(I)} = \begin{cases} 
-\frac{1}{2} \text{sgn}(m)F, & -1 < \text{sgn}(m)A^{-1} < \infty \\
\frac{1}{2} \text{sgn}(m)(2 - F), & -\infty < \text{sgn}(m)A^{-1} < -1 \\
\frac{1}{2} \text{sgn}(m)(1 - F), & A = 0 
\end{cases}, \quad \frac{1}{2} < F < 1.
\]

(38)

The vacuum quantum numbers are displayed in Figs. 1 and 2 for \( m > 0 \) and in Figs. 3 and 4 for \( m < 0 \). As can be seen from (14), a bound state exists for \( \frac{\pi}{2} < \Theta < \frac{3\pi}{2} \) in the former case (\(|E| < m\)) and for \( -\frac{\pi}{2} < \Theta < \frac{\pi}{2} \) in the latter case (\(|E| < -m\)).

5. In this study, we consider the vacuum quantum numbers for the most general boundary conditions at the point \( r = 0 \). These conditions can violate C symmetry. Under charge conjugation, we have \( \Phi^{(0)} \rightarrow -\Phi^{(0)} \) and \( \Upsilon \rightarrow -\Upsilon \); this is equivalent to the substitution \( F \rightarrow 1 - F \). In the case of boundary conditions conserving C symmetry, we have \( Q^{(I)} \rightarrow -Q^{(I)} \) and \( \Phi^{(I)} \rightarrow -\Phi^{(I)} \). Several examples of boundary conditions conserving C symmetry are represented in [4]. For one boundary condition of this type, the condition of weaker singularity of wave function at the point \( r = 0 \) (that is, the condition requiring that the divergence for \( r \rightarrow 0 \) not be stronger than \( r^{-p} \), where \( p \leq \frac{1}{2} \)), the vacuum charge was calculated in [7].

It is clear from our consideration (see Figs. 1 and 3) that any boundary condition corresponding to \( -\frac{\pi}{2} \text{sgn}(m) < \Theta < \pi - \frac{\pi}{2} \text{sgn}(m) \) for \( F \neq \frac{1}{2} \) (\( 0 < \text{sgn}(m)A < \infty \))

\(^{1)}\)For the condition of weaker singularity, the vacuum angular momentum was calculated in [8].
and \( \Theta = \frac{\pi}{2}[1 - \text{sgn}(m)] \) for \( F = \frac{1}{2} \) \((\text{sgn}(m)A = 1)\) conserves C symmetry for the vacuum charge. In this case, however, C symmetry is violated for the vacuum magnetic flux (see Figs. 2 and 4). The boundary conditions that conserve C symmetry and periodicity in the magnetic flux of the string both for the vacuum charge and for the magnetic flux are given by

\[
\begin{align*}
\Theta &= \Theta_C \text{mod} 2\pi, & 0 < F < \frac{1}{2} \\
\Theta &= \left\{ \frac{\pi}{2}[1 - \text{sgn}(m)] \right\} \text{mod} 2\pi, & F = \frac{1}{2} \\
\Theta &= (-\Theta_C) \text{mod} 2\pi, & \frac{1}{2} < F < 1
\end{align*}
\]

It is clear that, by choosing a different \( \Theta \) values for different values of the magnetic flux of the string, we can specify boundary conditions violating the periodicity of the vacuum numbers in \( \Phi^{(0)} \). Accordingly, there exist a number of boundary conditions that conserve C symmetry and which violate periodicity in \( \Phi^{(0)} \) (an example of this type is considered in [4]).

Returning to boundary conditions that are periodic in \( \Phi^{(0)} \), we emphasize that, by taking an appropriate value of \( \Theta \) from the interval \(-\frac{\pi}{2}\text{sgn}(m) < \Theta < \pi - \frac{\pi}{2}\text{sgn}(m)\) for each value of \( F \) from the interval \( 0 < F < 1 \), we can annihilate the vacuum magnetic flux for all values of \( \Phi^{(0)} \) (see Figs. 2 and 4). However, it is impossible to achieve this for the vacuum charge (see Figs. 1 and 3). Hence, there is no physically acceptable boundary condition (that is, a boundary condition compatible with the self-adjointness of the Hamiltonian) that ensures the vanishing of the vacuum quantum numbers for all values of the magnetic flux of the string. 2)

In conclusion, we note that the region of the vacuum quantum numbers is restricted by the conditions

\[
-\frac{3}{4} < Q^{(I)} < \frac{3}{4}, \quad -\frac{4}{5} < -\frac{2\pi|m|}{e^2F(1-F)}\Phi^{(I)} < \frac{4}{5}; \tag{41}
\]

In this case, we can therefore speak about the fractional charge in the primary sense of this notion [10, 11]. That the vacuum charge grows indefinitely with increasing magnetic flux of the string, as was stated in [12, 13], is incorrect.

2) The opposite statement in [9] is erroneous.
This work was supported in part by the State Committee for Science, Technologies, and Industrial Policy of Ukraine and by the program Few-Body Physics Network (INTAS-93-337).

References


Fig. 1. $Q^{(l)}$ in the region $-0.5 < \Theta \pi^{-1} < 1.5$, $0 < F < 1$ ($m > 0$).
Fig. 2. \(2\pi|m|\left| e^2 F(1 - F) \right|^{-1}\Phi^{(I)} \) in the region \(-0.5 < \Theta^{-1} < 1.5, \quad 0 < F < 1\).
Fig. 3. \( Q^{(f)} \) in the region \(-0.5 < \Theta \pi^{-1} < 1.5, 0 < F < 1 \) (\( m < 0 \)).
Fig. 4. $2\pi|m|[e^2F(1 - F)]^{-1}\Phi^{(I)}$ in the region $-0.5 < \Theta^{-1} < 1.5$, $0 < F < 1$. 