THE CROSSING MATRIX IN THE OCTET MODEL

J. J. de Swart *)
CERN, Geneva

ABSTRACT

A general expression is given for the elements of the crossing matrix in the octet model in terms of the Clebsch-Gordan coefficients for SU(3). Several useful relations about the crossing matrices are derived. For two specific cases the crossing matrix is given.

*) On leave from the University of Nijmegen, Nijmegen, The Netherlands.

7021 / TH. 363
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In the octet model \(^1\)-\(^3\)) particles \(a_1\) with similar properties are assigned to the same irreducible representation \(\{\lambda_i\}\) (I.R.) of the group SU(3). Inside a specific I.R. the different particles \(a_1\) are distinguished by the quantum numbers \(\nu_1 = (I, I_z, Y)\). The antiparticle \(\bar{a}_1\) belongs to the conjugate representation \(\{\lambda_i^*\}\) and has the other quantum numbers \(-\nu_1 = - (I, -I_z, -Y)\). The parts of the wave function in "unitary spin space" of the particle \(\phi(\nu)\) and antiparticle \(\phi(-\nu)\) are connected by (8.2) of \(A:\)

\[
\phi(-\nu) = (-\nu) \phi(\nu^*)
\]

(1)

where \(\tilde{J} = I_z + \frac{Y}{2}\).

We will consider the reactions

\[a_1 + a_2 \to a_3 + a_4\]  I

\[a_1 + \bar{a}_3 \to \bar{a}_2 + a_4\]  II

\[a_1 + \bar{a}_4 \to a_3 + \bar{a}_2\]  III

We note that charge conservation in these reactions is expressed by

\[\nu_1 + \bar{\nu}_2 = \bar{\nu}_3 + \bar{\nu}_4\]

(2)

Suppressing the space-spin properties of the amplitudes, we can write for the matrix elements of the transition amplitude in channel I

7021
\[
\left( \begin{array}{c}
\mu_3 \mu_4 \\
\nu_3 \nu_4
\end{array} \right) = \sum_{\mu, \nu, \gamma} \left( \begin{array}{c}
\mu_3 \mu_4 \\
\nu_3 \nu_4
\end{array} \right) F^{I}_{\mu, \nu, \gamma} \left( \begin{array}{c}
\mu_1 \mu_2 \mu_5 \\
\nu_1 \nu_2 \nu_4
\end{array} \right)
\]  

(3a)

Here

\[
\left( \begin{array}{c}
\mu_1 \mu_2 \mu_5 \\
\nu_1 \nu_2 \nu_4
\end{array} \right)
\]

is a Clebsch-Gordan coefficient (C.G. coefficient) of SU(3). We have made here the assumption that the transition amplitude is invariant under all transformations belonging to SU(3). The unitary spin amplitudes \( F_{\mu, \nu, \gamma} \) are then only dependent on the eigenvalues \( \mu \) of the Casimir operators \( F^2 \) and \( G^2 \) and on the eigenvalues \( \beta \) and \( \gamma \) of the operator \( \Gamma \).

Analogously, we can write for the amplitudes in the channels II and III,

\[
\left( \begin{array}{c}
\mu_2 \mu_4 \\
\nu_2 \nu_4
\end{array} \right) = \sum_{\mu, \nu, \beta, \gamma} \left( \begin{array}{c}
\mu_2 \mu_4 \\
\nu_2 \nu_4
\end{array} \right) F^{II}_{\mu, \nu, \beta, \gamma} \left( \begin{array}{c}
\mu_3 \mu_5 \\
\nu_3 \nu_5
\end{array} \right)
\]  

(3b)

\[
\left( \begin{array}{c}
\mu_3 \mu_2 \\
\nu_3 \nu_2
\end{array} \right) = \sum_{\mu, \nu, \beta, \gamma} \left( \begin{array}{c}
\mu_3 \mu_2 \\
\nu_3 \nu_2
\end{array} \right) F^{III}_{\mu, \nu, \beta, \gamma} \left( \begin{array}{c}
\mu_4 \mu_5 \\
\nu_4 \nu_5
\end{array} \right)
\]  

(3c)

Due to (1) we have
\[
\left( \nu_3 \nu_4 \left| F \right| \nu_1 \nu_2 \right) = (-)^{\nu_3 - \nu_2} \left( \nu_3^* \nu_4^* \left| F \right| \nu_1^* \nu_2^* \right) \tag{4a}
\]

\[
\left( \nu_3 \nu_4 \left| F \right| \nu_1 \nu_2 \right) = (-)^{\nu_4 - \nu_2} \left( \nu_3^* \nu_2^* \left| F \right| \nu_1^* \nu_4^* \right) \tag{4b}
\]

The relations (3) and (4) allow us to express the unitary spin amplitudes \( F^I \) in terms of the unitary spin amplitudes \( F^{II} \) or \( F^{III} \).

\[
F^I_{\mu' \beta' \gamma'} = \sum_{\mu' \beta' \gamma'} \left( \mu \beta \gamma \left| \beta^{II} (\nu_1, \nu_2, \nu_3, \nu_4) \right| \mu' \beta' \gamma' \right) F^{II}_{\mu \beta \gamma} \tag{5a}
\]

\[
F^I_{\mu' \beta' \gamma'} = \sum_{\mu' \beta' \gamma'} \left( \mu \beta \gamma \left| \beta^{III} (\nu_1, \nu_2, \nu_3, \nu_4) \right| \mu' \beta' \gamma' \right) F^{III}_{\mu \beta \gamma} \tag{5b}
\]

These equations define the crossing matrices \( \beta^{II} \) and \( \beta^{III} \), which can easily be deduced from (3) and (4) using the orthogonality relations for the C.G. coefficients (Eq. (11.2) of A). We obtain

\[
\left( \mu \beta \gamma \left| \beta^{II} (\nu_1, \nu_2, \nu_3, \nu_4) \right| \mu' \beta' \gamma' \right) =
\sum_{\nu_1, \nu_3, \gamma, \nu_4} (-)^{\nu_1 - \nu_3} \left( \nu_1 \nu_2 \nu_3 \nu_4 \left| \nu_1 \nu_2 \nu_3 \nu_4 \beta \right| \nu_1 \nu_2 \nu_3 \nu_4 \beta' \right) \left( \nu_3 \nu_4 \nu_1 \nu_2 \left| \nu_3 \nu_4 \nu_1 \nu_2 \beta' \gamma' \right| \nu_3 \nu_4 \nu_1 \nu_2 \beta \right) \left( \nu_2 \nu_4 \nu_1 \nu_3 \left| \nu_2 \nu_4 \nu_1 \nu_3 \beta \right| \nu_2 \nu_4 \nu_1 \nu_3 \beta' \right) \tag{6a}
\]
and
\[ (\mu \beta \gamma | \beta_{\mu}' (\nu_1, \nu_2, \nu_3, \nu_4) | \mu' \beta' \gamma') = \]
\[ \sum (-)^{\nu_1 - \nu_2} (\nu_1, \nu_2, \nu_3, \nu_4) (\nu_3, \nu_4, \nu_1, \nu_2) (\nu_1, \nu_2, \nu_3, \nu_4) (\nu_3, \nu_4, \nu_1, \nu_2) \]

(6b)

These expressions (6a) and (6b) could be used to evaluate the elements of the crossing matrix. For \( \gamma \) we can choose the quantum numbers of any state belonging to the I.R. \( \{ \mu \} \). A judicious choice of \( \gamma \) can save a lot of labour in the evaluation of these matrix elements. This arbitrariness of \( \gamma \) can be used to give a more symmetric expression for the crossing matrix.

\[ (\mu \beta \gamma | \beta_{\mu}' (\nu_1, \nu_2, \nu_3, \nu_4) | \mu' \beta' \gamma') = \]
\[ \frac{1}{N_{\mu}} \sum (-)^{\nu_1 - \nu_2} (\nu_1, \nu_2, \nu_3, \nu_4) (\nu_3, \nu_4, \nu_1, \nu_2) (\nu_1, \nu_2, \nu_3, \nu_4) (\nu_3, \nu_4, \nu_1, \nu_2) \]

(7)

where \( N_{\mu} \) is the number of states in the I.R. \( \{ \mu \} \).

Using the symmetry properties of the C.G. coefficients we will deduce useful relations between the different elements of the crossing matrix. As a specific example we will apply these relations to the case where all the particles belong to the I.R. \( \{ 3 \} \).
I. Performing the crossing of the particles 1 and 3 or 2 and 4 twice, gives us back the original situation. Therefore,

\[ \sum_{\mu' \rho' \gamma'} (\mu' \rho' \gamma' | \beta_\Pi (\mu_1, \mu_2, \mu_3, \mu_4) | \mu' \rho' \gamma') (\mu' \rho' \gamma' | \beta_\Pi (\mu_1, \mu_2, \mu_3, \mu_4) | \mu' \rho' \gamma') = \delta_{\mu' \mu''} \delta_{\rho' \rho''} \delta_{\gamma' \gamma''} \]

\[ \sum_{\mu' \rho' \gamma'} (\mu' \rho' \gamma' | \beta_\Pi (\mu_1, \mu_2, \mu_3, \mu_4) | \mu' \rho' \gamma') (\mu' \rho' \gamma' | \beta_\Pi (\mu_1, \mu_2, \mu_3, \mu_4) | \mu' \rho' \gamma') = \delta_{\mu' \mu''} \delta_{\rho' \rho''} \delta_{\gamma' \gamma''} \]

For the special case \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = 8 \) this implies for the matrices

\[ \beta_\Pi^2 = 1 \quad \text{and} \quad \beta_\Pi^2 = 1 \]

II. Rearranging the terms in the summation in (\( \gamma \)) gives

\[ N_\mu (\mu' \rho \gamma | \beta_\Pi (\mu_1, \mu_2, \mu_3, \mu_4) | \mu' \rho \gamma') = N_\mu (\mu' \rho \gamma | \beta_\Pi (\mu_1, \mu_2, \mu_3, \mu_4) | \mu' \rho \gamma) \]

\[ N_\mu (\mu' \rho \gamma | \beta_\Pi (\mu_1, \mu_2, \mu_3, \mu_4) | \mu' \rho \gamma') = N_\mu (\mu' \rho \gamma | \beta_\Pi (\mu_1, \mu_2, \mu_3, \mu_4) | \mu' \rho \gamma) \]

For the case \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = 8 \) this implies for the matrices \( 5 \)

\[ N \beta_\Pi = \beta_\Pi^T N \quad \text{and} \quad N \beta_\Pi = \beta_\Pi^T N \]

7021
Here \( N \) is a diagonal matrix with the matrix elements

\[
(\mu \beta \gamma \mid N \mid \mu' \beta' \gamma') = N_{\mu' \mu} \delta_{\beta \beta'} \delta_{\gamma \gamma'}
\]

III. Using the property (A, 14.12)

\[
\begin{pmatrix}
\mu_1 & \mu_2 & \mu_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{pmatrix} = \bar{\mathcal{F}}_3 \begin{pmatrix}
\mu_1 & \mu_2 & \mu_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{pmatrix} \begin{pmatrix}
\mu_1^* & \mu_2^* & \mu_3^* \\
\gamma_1^* & \gamma_2^* & \gamma_3^*
\end{pmatrix}
\]

we obtain

\[
(\mu \beta \gamma \mid \beta_\Pi \mid \mu' \beta' \gamma') = \bar{\mathcal{F}}_3(\mu_1, \mu_2, \mu_3) \bar{\mathcal{F}}_3(\mu_3, \mu_4, \mu_5) \bar{\mathcal{F}}_3(\mu_5, \mu_6, \mu_7)
\]

\[
\bar{\mathcal{F}}_3(\mu_1, \mu_2, \mu_3) \bar{\mathcal{F}}_3(\mu_2, \mu_4, \mu_5) (\mu_3 \beta \gamma \mid \beta_\Pi \mid \mu_4 \beta' \gamma')
\]

This relation gives for the case \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = 8 \)

\[
(\mathcal{F}_1 \mid \beta_\Pi \mid \mu) = (\mathcal{F}_2 \mid \beta_\Pi \mid \mu) = (\mathcal{F}_3 \mid \beta_\Pi \mid \mu) = (1 \mid \beta_\Pi \mid \mu) = 0
\]

for \( \mu = 8_{12} \) and \( 8_{21} \),

\[
(10 \mid \beta_\Pi \mid \mu) = -(10^* \mid \beta_\Pi \mid \mu) \quad \text{for} \quad \mu = 8_{12} \text{ and } 8_{21} \text{, and}
\]

\[
(10 \mid \beta_\Pi \mid \mu) = (10^* \mid \beta_\Pi \mid \mu) \quad \text{for} \quad \mu = 27, 10, 10^*, 8_{11}, 8_{22}, \text{and } 1.
\]
IV. The symmetry property (8) and (A, 14.1)

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 \end{pmatrix} = z_1(\mu_1, \mu_2, \mu_3, \mu_4) \begin{pmatrix} \mu_2^* & \mu_3^* & \mu_4^* \end{pmatrix}$$

(9)

gives

$$\begin{pmatrix} \kappa \beta \gamma \end{pmatrix} = z_2(\kappa, \kappa^*, \gamma, \gamma^*) z_3(\kappa, \kappa^*, \gamma, \gamma^*)$$

Application to the case $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$ gives

$$\begin{pmatrix} \beta \end{pmatrix} = \begin{pmatrix} \beta \end{pmatrix}$$

and

$$\begin{pmatrix} \beta \end{pmatrix} = \begin{pmatrix} \beta \end{pmatrix}$$

V. We have the explicit form for the special C.G. coefficient

$$\begin{pmatrix} \mu & \mu^* & 1 \end{pmatrix} = \frac{\overline{\gamma}_+ \overline{\gamma}_-}{\sqrt{N}} \overline{\overline{\gamma}_-}$$
where \( \gamma \) are the quantum numbers of the highest eigenstate of the I.R. \( \{ \mu \}^3 \). This gives

\[
(m_1 m_2 m_3 | \beta'' (m_1', m_2', m_3') | 1 ) = \frac{1}{N_1 N_2} \sqrt{\gamma_{1m} + \gamma_{2m}} \delta_{\beta \gamma}
\]

Here \( \gamma_{1m} \) is the highest eigenvalue and \( N_1 \) the number of states of the I.R. \( \{ \mu_i \} \).

For the case \( m_1 = m_2 = m_3 = m_4 = 8 \) this means

\[
(m \beta \gamma | \beta'' | 1 ) = \frac{1}{8} \delta_{\beta \gamma}
\]

VI. Using the property (9) of the C.G. coefficients we get

\[
(m \rho \gamma | \beta'' (m_1', m_2', m_3') | m' \rho' \gamma') = \delta_i \left( \sum_{\gamma} (m_1 \rho \gamma \beta | m_2 \rho m_3 \gamma) \right) \delta_i \left( m' \rho' \gamma' | m_2 \rho m_3 \gamma \right)
\]

For the case \( m_1 = m_2 = m_3 = m_4 = 8 \) this means that the matrices \( \beta'' \) and \( \beta''' \) are simply related by

\[
(m \rho \gamma | \beta'' | m' \rho' \gamma') = \delta_i \left( s_8 m_1 \rho \gamma \beta | m_2 \rho m_3 \gamma \right) \delta_i \left( s_8 m_1 \rho \gamma | m_2 \rho m_3 \gamma \right)
\]
VII. Explicit evaluation of (6) gives for the case \( \mu_1 = \mu_2 = \mu_3 = \mu_4 = 8 \) the matrix elements \( \langle \mu' \beta' / \beta_3 / \mu_3' \beta' \rangle \) as given in Table 1. The matrix elements of \( \beta_3 \) can be derived very easily using this Table and Table 1 of A. We note that this matrix agrees up to trivial sign conventions with the crossing matrices as given by others (6)-(8).

We have also evaluated the crossing matrices for the case \( \mu_1 = \mu_3 = 8, \mu_2 = \mu_4 = 10 \). The elements of \( \langle \mu / \beta_2 / \mu' \beta' \rangle \) are given in Table 2 and of \( \langle \mu / \beta_3 / \mu' \rangle \) in Table 3.

We noticed that for the two cases evaluated

\[
\sum_{\mu} N_\mu \langle \mu \beta / \beta_2 (\mu', \nu', \mu', \mu', \nu') | \mu' \beta' \rangle = \sqrt{N_1 N_2} \delta(\mu', 1)
\]

but we were, however, not able to prove this result in general.
REFERENCES


2) Y. Ne'eman, Nuclear Physics 26, 222 (1961).


4) We will use the same notation and definitions as in A. Especially the phase convention for the operators $I_\pm$ and $K_\pm$ are essential for the validity of (1).

5) $\beta^T$ is the transpose of the matrix $\beta$.

6) R.E. Cutkosky (to be published).

7) D.E. Neville (to be published).

8) H. Goldberg (to be published).
## Table 1

<table>
<thead>
<tr>
<th>( \mu' \rho' \gamma' )</th>
<th>27</th>
<th>10</th>
<th>10*</th>
<th>8(_{11})</th>
<th>8(_{12})</th>
<th>8(_{21})</th>
<th>8(_{22})</th>
<th>1</th>
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<tbody>
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<td>27</td>
<td>7/40</td>
<td>-1/12</td>
<td>-1/12</td>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>-1/3</td>
<td>1/8</td>
</tr>
<tr>
<td>10</td>
<td>-9/40</td>
<td>1/4</td>
<td>1/4</td>
<td>-2/5</td>
<td>-1/(\sqrt{5})</td>
<td>1/(\sqrt{5})</td>
<td>0</td>
<td>1/8</td>
</tr>
<tr>
<td>10*</td>
<td>-9/40</td>
<td>1/4</td>
<td>1/4</td>
<td>-2/5</td>
<td>1/(\sqrt{5})</td>
<td>-1/(\sqrt{5})</td>
<td>0</td>
<td>1/8</td>
</tr>
<tr>
<td>8(_{11})</td>
<td>27/40</td>
<td>-1/2</td>
<td>-1/2</td>
<td>-3/10</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>1/8</td>
</tr>
<tr>
<td>8(_{12})</td>
<td>0</td>
<td>-(\sqrt{5}/4)</td>
<td>(\sqrt{5}/4)</td>
<td>0</td>
<td>-1/2</td>
<td>-1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8(_{21})</td>
<td>0</td>
<td>(\sqrt{5}/4)</td>
<td>-(\sqrt{5}/4)</td>
<td>0</td>
<td>-1/2</td>
<td>-1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8(_{22})</td>
<td>-9/8</td>
<td>0</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
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<td>1/8</td>
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<tr>
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<td>27/8</td>
<td>5/4</td>
<td>5/4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1/8</td>
</tr>
</tbody>
</table>

The elements \( \mu' \rho' \gamma'/\beta_{II}(\delta,\delta,\delta)/\mu' \rho' \gamma' \) of the crossing matrix.
### Table 2

<table>
<thead>
<tr>
<th>$\mu'$</th>
<th>$\mu$</th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
<th>$\gamma_3$</th>
</tr>
</thead>
<tbody>
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<td>27</td>
<td>$\sqrt{7}/140$</td>
<td>$\sqrt{2}/10$</td>
<td>$-\sqrt{10}/10$</td>
<td>$\sqrt{5}/20$</td>
</tr>
<tr>
<td>27</td>
<td>$-\sqrt{7}/20$</td>
<td>$-3\sqrt{2}/10$</td>
<td>$\sqrt{10}/30$</td>
<td>$\sqrt{5}/20$</td>
</tr>
<tr>
<td>10</td>
<td>$-9\sqrt{7}/20$</td>
<td>$3\sqrt{2}/10$</td>
<td>$\sqrt{10}/10$</td>
<td>$\sqrt{5}/20$</td>
</tr>
<tr>
<td>8</td>
<td>$9\sqrt{7}/20$</td>
<td>$\sqrt{2}/5$</td>
<td>$\sqrt{10}/5$</td>
<td>$\sqrt{5}/20$</td>
</tr>
</tbody>
</table>

The elements $(\mu/\beta_{\Pi}(8,10,8,10)/\mu')$ of the crossing matrix.

### Table 3

<table>
<thead>
<tr>
<th>$\mu'$</th>
<th>$\mu$</th>
<th>$\mu$</th>
<th>$\mu$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
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<td>$1/8$</td>
<td>$9/40$</td>
<td>$1/4$</td>
<td>$2/5$</td>
</tr>
<tr>
<td>$27^*$</td>
<td>$7/24$</td>
<td>$37/40$</td>
<td>$-1/12$</td>
<td>$-2/15$</td>
</tr>
<tr>
<td>$10^*$</td>
<td>$7/8$</td>
<td>$-9/40$</td>
<td>$3/4$</td>
<td>$-2/5$</td>
</tr>
<tr>
<td>$8$</td>
<td>$7/4$</td>
<td>$-9/20$</td>
<td>$-1/2$</td>
<td>$1/5$</td>
</tr>
</tbody>
</table>

The elements $(\mu/\beta_{\Pi}(8,10,8,10)/\mu')$ of the crossing matrix.