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THE CROSSING MATRIX IN THE OCTET MODEL

J. J. de Swart *)

CERN, Geneva

A B S T R A C T

A general expression is given for the elements of the crossing matrix in the octet model in terms of the Clebsch-Gordan coefficients for $SU(3)$. Several useful relations about the crossing matrices are derived. For two specific cases the crossing matrix is given.

*) On leave from the University of Nijmegen, Nijmegen, The Netherlands.

In the octet model ¹⁾⁻³⁾ particles a_i with similar properties are assigned to the same irreducible representation $\{\mu_i\}$ (I.R.) of the group $SU(3)$. Inside a specific I.R. the different particles a_i are distinguished by the quantum numbers $\nu_i = (I, I_z, Y)$. The antiparticle $\bar{a}_i$ belongs to the conjugate representation $\{\mu_i^*\}$ and has the other quantum numbers $-\nu_i = (I, -I_z, -Y)$. The parts of the wave function in "unitary spin space" of the particle $\phi_{\nu}^{(\mu)}$ and antiparticle $\phi_{-\nu}^{(\mu^*)}$ are connected by (8.2) of A :

$$\phi_{\nu}^{(\mu)*} = (-)^{\bar{\nu}} \phi_{-\nu}^{(\mu^*)} \quad (1)$$

where $\bar{\nu} = I_z + \frac{Y}{2}$.

We will consider the reactions

$$a_1 + a_2 \rightarrow a_3 + a_4 \quad \text{I}$$

$$a_1 + \bar{a}_3 \rightarrow \bar{a}_2 + a_4 \quad \text{II}$$

$$a_1 + \bar{a}_4 \rightarrow a_3 + \bar{a}_2 \quad \text{III}$$

We note that charge conservation in these reactions is expressed by

$$\bar{\nu}_1 + \bar{\nu}_2 = \bar{\nu}_3 + \bar{\nu}_4 \quad (2)$$

Suppressing the space-spin properties of the amplitudes, we can write for the matrix elements of the transition amplitude in channel I

2.

$$\left(\begin{matrix} \mu_3 \mu_4 \\ \gamma_3 \gamma_4 \end{matrix} \middle| F \middle| \begin{matrix} \mu_1 \mu_2 \\ \gamma_1 \gamma_2 \end{matrix} \right) = \sum_{\substack{\mu \nu \\ \beta \gamma}} \left(\begin{matrix} \mu_3 \mu_4 \mu \\ \gamma_3 \gamma_4 \gamma \end{matrix} \right) F_{\mu \beta \gamma}^I \left(\begin{matrix} \mu_1 \mu_2 \mu \\ \gamma_1 \gamma_2 \gamma \end{matrix} \right) \quad (3a)$$

Here

$$\left(\begin{matrix} \mu_1 \mu_2 \mu \\ \gamma_1 \gamma_2 \gamma \end{matrix} \right)$$

is a Clebsch-Gordan coefficient ³⁾ (C.G. coefficient) of SU(3). We have made here the assumption that the transition amplitude is invariant under all transformations belonging to SU(3). The unitary spin amplitudes $F_{\mu \beta \gamma}$ are then only dependent on the eigenvalues μ of the Casimir operators F^2 and G^3 and on the eigenvalues β and γ of the operator Γ^3 .

Analogously, we can write for the amplitudes in the channels II and III,

$$\left(\begin{matrix} \mu_2^* \mu_4^* \\ -\gamma_2 \gamma_4 \end{matrix} \middle| F \middle| \begin{matrix} \mu_1 \mu_3^* \\ \gamma_1 -\gamma_3 \end{matrix} \right) = \sum_{\substack{\mu \nu \\ \beta \gamma}} \left(\begin{matrix} \mu_2^* \mu_4 \mu \\ -\gamma_2 \gamma_4 \gamma \beta \end{matrix} \right) F_{\mu \beta \gamma}^{II} \left(\begin{matrix} \mu_1 \mu_3^* \mu \\ \gamma_1 -\gamma_3 \gamma \gamma \end{matrix} \right) \quad (3b)$$

$$\left(\begin{matrix} \mu_3 \mu_2^* \\ \gamma_3 -\gamma_2 \end{matrix} \middle| F \middle| \begin{matrix} \mu_1 \mu_4^* \\ \gamma_1 -\gamma_4 \end{matrix} \right) = \sum_{\substack{\mu \nu \\ \beta \gamma}} \left(\begin{matrix} \mu_3 \mu_2^* \mu \\ \gamma_3 -\gamma_2 \gamma \beta \end{matrix} \right) F_{\mu \beta \gamma}^{III} \left(\begin{matrix} \mu_1 \mu_4^* \mu \\ \gamma_1 -\gamma_4 \gamma \gamma \end{matrix} \right) \quad (3c)$$

Due to (1) we have

$$\begin{pmatrix} \mu_3 \mu_4 \\ \gamma_3 \gamma_4 \end{pmatrix} | F | \begin{pmatrix} \mu_1 \mu_2 \\ \gamma_1 \gamma_2 \end{pmatrix} = (-)^{\bar{\gamma}_3 - \bar{\gamma}_2} \begin{pmatrix} \mu_2^* \mu_4 \\ -\gamma_2 \gamma_4 \end{pmatrix} | F | \begin{pmatrix} \mu_1 \mu_3^* \\ \gamma_1 - \gamma_3 \end{pmatrix} \quad (4a)$$

$$\begin{pmatrix} \mu_3 \mu_4 \\ \gamma_3 \gamma_4 \end{pmatrix} | F | \begin{pmatrix} \mu_1 \mu_2 \\ \gamma_1 \gamma_2 \end{pmatrix} = (-)^{\bar{\gamma}_4 - \bar{\gamma}_2} \begin{pmatrix} \mu_3 \mu_2^* \\ \gamma_3 - \gamma_2 \end{pmatrix} | F | \begin{pmatrix} \mu_1 \mu_4^* \\ \gamma_1 - \gamma_4 \end{pmatrix} \quad (4b)$$

The relations (3) and (4) allow us to express the unitary spin amplitudes F^I in terms of the unitary spin amplitudes F^{II} or F^{III} .

$$F_{\mu\beta\gamma}^I = \sum_{\mu'\beta'\gamma'} \langle \mu\beta\gamma | \beta_{II}(\mu_1, \mu_2, \mu_3, \mu_4) | \mu'\beta'\gamma' \rangle F_{\mu'\beta'\gamma'}^{II} \quad (5a)$$

$$F_{\mu\beta\gamma}^I = \sum_{\mu'\beta'\gamma'} \langle \mu\beta\gamma | \beta_{III}(\mu_1, \mu_2, \mu_3, \mu_4) | \mu'\beta'\gamma' \rangle F_{\mu'\beta'\gamma'}^{III} \quad (5b)$$

These equations define the crossing matrices β_{II} and β_{III} , which can easily be deduced from (3) and (4) using the orthogonality relations for the C.G. coefficients (Eq. (11.2) of A). We obtain

$$\begin{aligned} & \langle \mu\beta\gamma | \beta_{II}(\mu_1, \mu_2, \mu_3, \mu_4) | \mu'\beta'\gamma' \rangle = \\ & \sum_{\substack{\gamma_1 \gamma_2 \gamma_3 \gamma_4 \\ \gamma'}} (-)^{\bar{\gamma}_2 - \bar{\gamma}_3} \begin{pmatrix} \mu_1 \mu_2 \mu \\ \gamma_1 \gamma_2 \gamma \end{pmatrix} \begin{pmatrix} \mu_3 \mu_4 \mu \\ \gamma_3 \gamma_4 \gamma \end{pmatrix} \begin{pmatrix} \mu_1 \mu_3^* \mu' \\ \gamma_1 - \gamma_3 \gamma' \end{pmatrix} \begin{pmatrix} \mu_2^* \mu_4 \mu' \\ -\gamma_2 \gamma_4 \gamma' \end{pmatrix} \end{aligned} \quad (6a)$$

4.

and

$$\begin{aligned}
 & (\mu\beta\gamma | \beta_{\text{III}} (\mu_1, \mu_2, \mu_3, \mu_4) | \mu'\beta'\gamma') = \\
 & \sum_{\gamma_1, \gamma_2, \gamma_3, \gamma_4} (-)^{\bar{\gamma}_4 - \bar{\gamma}_2} \begin{pmatrix} \mu_1 & \mu_2 & \mu \\ \gamma_1 & \gamma_2 & \gamma \end{pmatrix} \begin{pmatrix} \mu_3 & \mu_4 & \mu \\ \gamma_3 & \gamma_4 & \gamma \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_4^* & \mu' \\ \gamma_1 & -\gamma_4 & \gamma' \end{pmatrix} \begin{pmatrix} \mu_3 & \mu_2^* & \mu' \\ \gamma_3 & -\gamma_2 & \gamma' \end{pmatrix} \\
 & \hspace{15em} (6b)
 \end{aligned}$$

These expressions (6a) and (6b) could be used to evaluate the elements of the crossing matrix. For γ we can choose the quantum numbers of any state belonging to the I.R. $\{\mu\}$. A judicious choice of γ can save a lot of labour in the evaluation of these matrix elements. This arbitrariness of γ can be used to give a more symmetric expression for the crossing matrix.

$$\begin{aligned}
 & (\mu\beta\gamma | \beta_{\text{II}} (\mu_1, \mu_2, \mu_3, \mu_4) | \mu'\beta'\gamma') = \\
 & \frac{1}{N_\mu} \sum_{\substack{\gamma_1, \gamma_2, \gamma_3, \gamma_4 \\ \gamma, \gamma'}} (-)^{\bar{\gamma}_2 - \bar{\gamma}_3} \begin{pmatrix} \mu_1 & \mu_2 & \mu \\ \gamma_1 & \gamma_2 & \gamma \end{pmatrix} \begin{pmatrix} \mu_3 & \mu_4 & \mu \\ \gamma_3 & \gamma_4 & \gamma \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_3^* & \mu' \\ \gamma_1 & -\gamma_3 & \gamma' \end{pmatrix} \begin{pmatrix} \mu_2 & \mu_4^* & \mu' \\ -\gamma_2 & \gamma_4 & \gamma' \end{pmatrix} \\
 & \hspace{15em} (7)
 \end{aligned}$$

where N_μ is the number of states in the I.R. $\{\mu\}$.

Using the symmetry properties of the C.G. coefficients we will deduce useful relations between the different elements of the crossing matrix. As a specific example we will apply these relations to the case where all the particles belong to the I.R. $\{8\}$.

- I. Performing the crossing of the particles 1 and 3 or 2 and 4 twice, gives us back the original situation. Therefore,

$$\begin{aligned} \sum_{\mu'\beta'\gamma'} (\mu\beta\gamma | \beta_{\text{II}}(\mu_1, \mu_2, \mu_3, \mu_4) | \mu'\beta'\gamma') (\mu'\beta'\gamma' | \beta_{\text{II}}(\mu_1, \mu_3^*, \mu_2^*, \mu_4) | \mu''\beta''\gamma'') = \\ = \delta_{\mu\mu''} \delta_{\beta\beta''} \delta_{\gamma\gamma''} \end{aligned}$$

$$\begin{aligned} \sum_{\mu'\beta'\gamma'} (\mu\beta\gamma | \beta_{\text{III}}(\mu_1, \mu_2, \mu_3, \mu_4) | \mu'\beta'\gamma') (\mu'\beta'\gamma' | \beta_{\text{III}}(\mu_1, \mu_4^*, \mu_3, \mu_2^*) | \mu''\beta''\gamma'') = \\ = \delta_{\mu\mu''} \delta_{\beta\beta''} \delta_{\gamma\gamma''} \end{aligned}$$

For the special case $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 8$ this implies for the matrices

$$\beta_{\text{II}}^2 = 1 \quad \text{and} \quad \beta_{\text{III}}^2 = 1$$

- II. Rearranging the terms in the summation in (7) gives

$$N_{\mu}(\mu\beta\gamma | \beta_{\text{II}}(\mu_1, \mu_2, \mu_3, \mu_4) | \mu'\beta'\gamma') = N_{\mu'}(\mu'\beta'\gamma' | \beta_{\text{II}}(\mu_1, \mu_3^*, \mu_2^*, \mu_4) | \mu\beta\gamma)$$

$$N_{\mu}(\mu\beta\gamma | \beta_{\text{III}}(\mu_1, \mu_2, \mu_3, \mu_4) | \mu'\beta'\gamma') = N_{\mu'}(\mu'\beta'\gamma' | \beta_{\text{III}}(\mu_1, \mu_4^*, \mu_3, \mu_2^*) | \mu\beta\gamma)$$

For the case $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 8$ this implies for the matrices ⁵⁾

$$N\beta_{\text{II}} = \beta_{\text{II}}^T N \quad \text{and} \quad N\beta_{\text{III}} = \beta_{\text{III}}^T N$$

6.

Here N is a diagonal matrix with the matrix elements

$$(\mu\beta\gamma|N|\mu'\beta'\gamma') = N_\mu \delta_{\mu\mu'} \delta_{\beta\beta'} \delta_{\gamma\gamma'}$$

III. Using the property (A, 14.12)

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} = \bar{3}_3(\mu_1, \mu_2, \mu_3) \begin{pmatrix} \mu_1^* & \mu_2^* & \mu_3^* \\ -\gamma_1 & -\gamma_2 & -\gamma_3 \end{pmatrix} \quad (8)$$

we obtain

$$\begin{aligned} (\mu\beta\gamma|\beta_{\text{II}}(\mu_1, \mu_2, \mu_3, \mu_4)|\mu'\beta'\gamma') &= \bar{3}_3(\mu_1, \mu_2, \mu_3) \bar{3}_3(\mu_3, \mu_4, \mu_\beta) \\ &\bar{3}_3(\mu_1, \mu_3^*, \mu_\gamma') \bar{3}_3(\mu_2^*, \mu_4, \mu_{\beta'}) (\mu^*\beta\gamma|\beta_{\text{II}}(\mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*)|\mu'^*\beta'\gamma') \end{aligned}$$

This relation gives for the case $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 8$

$$(27|\beta_{\text{II}}|\mu) = (8_{11}|\beta_{\text{II}}|\mu) = (8_{22}|\beta_{\text{II}}|\mu) = (1|\beta_{\text{II}}|\mu) = 0$$

for $\mu = 8_{12}$ and 8_{21} ,

$$(10|\beta_{\text{II}}|\mu) = -(10^*|\beta_{\text{II}}|\mu) \quad \text{for } \mu = 8_{12} \text{ and } 8_{21}, \text{ and}$$

$$(10|\beta_{\text{II}}|\mu) = (10^*|\beta_{\text{II}}|\mu^*) \quad \text{for } \mu = 27, 10, 10^*, 8_{11}, 8_{22}, \text{ and } 1.$$

IV. The symmetry property (8) and (A, 14.1)

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu & \gamma \\ \nu_1 & \nu_2 & \nu & \gamma \end{pmatrix} = \bar{\zeta}_1(\mu_1, \mu_2, \mu, \gamma) \begin{pmatrix} \mu_2 & \mu_1 & \mu & \gamma \\ \nu_2 & \nu_1 & \nu & \gamma \end{pmatrix} \quad (9)$$

gives

$$\begin{aligned} (\mu, \gamma | \beta_{\text{II}} (\mu_1, \mu_2, \mu_3, \mu_4) | \mu' \beta' \gamma') &= \bar{\zeta}_1(\mu, \mu_3^* \mu' \gamma') \bar{\zeta}_3(\mu, \mu_3^* \mu' \gamma') \\ \bar{\zeta}_1(\mu_2^* \mu_4 \mu' \beta') \bar{\zeta}_3(\mu_2^* \mu_4 \mu' \beta') &(\mu, \gamma | \beta_{\text{II}} (\mu_3, \mu_4, \mu_1, \mu_2) | \mu' \beta' \gamma') \end{aligned}$$

Application to the case $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 8$ gives

$$(g_{12} | \beta_{\text{II}} | g_{kl}) = (g_{21} | \beta_{\text{II}} | g_{kl})$$

and

$$(g_{12} | \beta_{\text{II}} | 10) = (g_{21} | \beta_{\text{II}} | 10^*)$$

V. We have the explicit form for the special C.G. coefficient

$$\begin{pmatrix} \mu & \mu^* & 1 \\ \nu & -\nu & 0 \end{pmatrix} = (-1)^{\bar{\nu} + \bar{\nu}_H} \frac{1}{\sqrt{N_\mu}}, \quad ,$$

8.

where ν_H are the quantum numbers of the highest eigenstate of the I.R. $\{\mu\}$ ³⁾.
This gives

$$(\mu\beta\gamma|\beta_{II}(\mu_1, \mu_2, \mu_1, \mu_2)|1) = (-)^{\bar{\nu}_{1H} + \bar{\nu}_{2H}} \frac{1}{\sqrt{N_1 N_2}} \delta_{\beta\gamma}$$

Here ν_{iH} is the highest eigenvalue and N_i the number of states of the I.R. $\{\mu_i\}$.

For the case $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 8$ this means

$$(\mu\beta\gamma|\beta_{II}|1) = \frac{1}{8} \delta_{\beta\gamma}$$

VI. Using the property (9) of the C.G. coefficients we get

$$(\mu\beta\gamma|\beta_{II}(\mu_1, \mu_1, \mu_3, \mu_4)|\mu'\beta'\gamma') = \xi_1(\mu_3\mu_4\mu_\beta)\xi_1(\mu_2^*\mu_4\mu'_{\beta'})$$

$$(\mu\beta\gamma|\beta_{III}(\mu_1, \mu_2, \mu_4, \mu_3)|\mu'\beta'\gamma')$$

For the case $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 8$ this means that the matrices β_{II} and β_{III} are simply related by

$$(\mu\beta\gamma|\beta_{II}|\mu'\beta'\gamma') = \xi_1(88\mu_\beta)\xi_1(88\mu'_{\beta'}) (\mu\beta\gamma|\beta_{III}|\mu'\beta'\gamma')$$

VII. Explicit evaluation of (6) gives for the case $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 8$ the matrix elements $(\mu\beta\gamma/\beta_{II}/\mu'\beta'\gamma')$ as given in Table 1. The matrix elements of β_{III} can be derived very easily using this Table and Table 1 of A. We note that this matrix agrees up to trivial sign conventions with the crossing matrices as given by others ⁶⁾⁻⁸⁾.

We have also evaluated the crossing matrices for the case $\mu_1 = \mu_3 = 8$, $\mu_2 = \mu_4 = 10$. The elements of $(\mu/\beta_{II}/\mu'\beta')$ are given in Table 2 and of $(\mu/\beta_{III}/\mu')$ in Table 3.

We noticed that for the two cases evaluated

$$\sum_{\mu\gamma} N_{\mu} (\mu\gamma\gamma/\beta_{II}(\mu_1, \mu_2, \mu_1, \mu_2)/\mu'\beta\beta) = \sqrt{N_1 N_2} \delta(\mu'_\beta, 1)$$

but we were, however, not able to prove this result in general.

R E F E R E N C E S

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- 2) Y. Ne'eman, Nuclear Physics 26, 222 (1961).
- 3) J.J. de Swart, "The Octet Model and its Clebsch-Gordan Coefficients".
Rev. Mod. Phys. 35, October (1963).
- 4) We will use the same notation and definitions as in A. Especially the
phase convention for the operators $I_{\pm}$ and $K_{\pm}$ are essential for the
validity of (1).
- 5) β^T is the transpose of the matrix β .
- 6) R.E. Cutkosky (to be published).
- 7) D.E. Neville (to be published).
- 8) H. Goldberg (to be published).

TABLE 1

		$\mu' \rho' \gamma'$							
		27	10	10*	8_{11}	8_{12}	8_{21}	8_{22}	1
$\mu \rho \gamma$	27	7/40	- 1/12	- 1/12	1/5	0	0	-1/3	1/8
	10	- 9/40	1/4	1/4	-2/5	$-1/\sqrt{5}$	$1/\sqrt{5}$	0	1/8
	10*	- 9/40	1/4	1/4	-2/5	$1/\sqrt{5}$	$-1/\sqrt{5}$	0	1/8
	8_{11}	27/40	- 1/2	- 1/2	-3/10	0	0	1/2	1/8
	8_{12}	0	$-\sqrt{5}/4$	$\sqrt{5}/4$	0	-1/2	-1/2	0	0
	8_{21}	0	$\sqrt{5}/4$	$-\sqrt{5}/4$	0	-1/2	-1/2	0	0
	8_{22}	- 9/8	0	0	1/2	0	0	1/2	1/8
	1	27/8	5/4	5/4	1	0	0	1	1/8

The elements $(\mu \rho \gamma / \beta_{II}(8,8,8,8) / \mu' \rho' \gamma')$ of the crossing matrix.

TABLE 2

		$\mu'_{\gamma'}$			
		27	8_1	8_2	1
μ	35	$9\sqrt{7}/140$	$\sqrt{2}/10$	$-\sqrt{10}/10$	$\sqrt{5}/20$
	27	$-\sqrt{7}/20$	$-3\sqrt{2}/10$	$\sqrt{10}/30$	$\sqrt{5}/20$
	10	$-9\sqrt{7}/20$	$3\sqrt{2}/10$	$\sqrt{10}/10$	$\sqrt{5}/20$
	8	$9\sqrt{7}/20$	$\sqrt{2}/5$	$\sqrt{10}/5$	$\sqrt{5}/20$

The elements $(\mu/\beta_{II}(8,10,8,10)/\mu'_{\gamma'})$ of the crossing matrix.

TABLE 3

		μ'			
		35^*	27	10^*	8
μ	35	$1/8$	$9/40$	$1/4$	$2/5$
	27	$7/24$	$37/40$	$-1/12$	$-2/15$
	10	$7/8$	$-9/40$	$3/4$	$-2/5$
	8	$7/4$	$-9/20$	$-1/2$	$1/5$

The elements $(\mu/\beta_{III}(8,10,8,10)/\mu')$ of the crossing matrix.