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LEGENDRE TRANSFORMS OF SCATTERING AMPLITUDESAND A CROSSING SYMMETRIC REPRESENTATION

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ABSTRACT

Legendre transforms differ from partial wave amplitudes by the use of variables of integration that are more appropriate for crossing symmetry. It is shown that their holomorphy properties are similar to those of partial wave amplitudes. Using Legendre transforms the Watson - Sommerfeld transformation can be applied in a crossing symmetric form, and the asymptotic terms can be separated from the background integral in all three energy channels simultaneously. The background integral can be expressed with an exponential factor which decreases if the contour can be moved to the left in the complex ℓ plane. The background integral satisfies the Mandelstam representation without subtractions, and each asymptotic term has the corresponding cut-plane analyticity with correct thresholds.

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1. INTRODUCTION

Comparatively little attention has been given to the general problem of expressing a scattering amplitude as an eigenfunction expansion or an integral transform, although a number of special cases have been studied in detail. The latter include : the partial wave expansion in a particular channel and the related Watson-Sommerfeld transform associated with the work of Regge ¹⁾; Hilbert transforms or single variable dispersion relations; the double Hilbert transform introduced by Mandelstam ²⁾. Sometimes an eigenfunction expansion has been combined with an integral transform as with partial wave dispersion relations.

The objectives in the use of these special expansions and transforms have often been very ambitious, such as seeking a closed system of coupled equations from analyticity and unitarity. It now appears that this problem is much harder than was expected. It is therefore desirable to make a general study of expansions and transforms of scattering amplitudes bearing in mind that it is unlikely that all required objectives can be achieved simultaneously. One should therefore seek to classify and combine expansions and transforms to achieve different selections of possible objectives, and to define the range of problems to which they may be applicable. For example, the partial wave expansion provides a simple expression of unitarity in one energy channel (as opposed to channels for crossed processes) and gives a very useful method for studying resonances, but in crossed channels unitarity is very complicated, and the expansion is unsuitable for displaying crossing symmetry.

It is desirable to have the following properties either satisfied by each term of an expansion, or explicitly satisfied by the integrand of an integral transform of a scattering amplitude :

- (1) analyticity in the cut energy plane of one channel;
- (2) correct thresholds in energy, (a) for the right-hand cut,
(b) left hand;
- (3) correct threshold dependence on energy;
- (4) cut plane analyticity of the Mandelstam type, (a) in 2 variables,
(b) in 3 variables;
- (5) explicit crossing symmetry;
- (6) elastic unitarity in a simple form, (a) in 1 channel,
(b) in 3 channels;
- (7) inelastic unitarity;
- (8) useful decoupling of equations, (a) in 1 channel,
(b) in 3 channels;
- (9) minimum of mutually cancelling terms or parts of terms;
- (10) explicit appearance of resonance terms, (a) in 1 channel,
(b) in 3 channels;
- (11) explicit Regge asymptotic terms, (a) in 1 channel,
(b) in 3 channels;
- (12) replacement of the background integral of the Regge
representation, (a) in 1 channel, (b) in 3 channels.

Some, but not all, of these properties are simply achieved by existing representations. No representation achieves either (6b) or (8b), although at one time it was hoped that the Mandelstam representation coupled with a partial wave expansion might do so. No representation throws any real light on the problem of inelastic unitarity (7). Partial wave amplitudes and their dispersion relations are particularly convenient for single channel problems but are inconvenient for crossing symmetry (5), and for asymptotic behaviour (11) and (12). The Regge representation improves on the partial wave series by giving the asymptotic behaviour in 1 channel (11a), but the individual terms have the wrong thresholds (2). Khuri has modified the Regge representation to give the correct energy threshold

(2a) and at the same time removes the background integral in 1 channel (11a) under certain assumptions (Khuri I ³). In a different approach (Khuri II ⁴) he has introduced a crossing symmetric form (5) of the Regge representation satisfying (10b) and (11b), but this form is not suitable for replacing the background integral. This representation (Khuri II ⁴), incorporates the Mandelstam representation, as a background term without subtractions, with Regge asymptotic behaviour in all three (crossed) channels, but unitarity is not simple in any channel.

The object of this paper is to describe a representation based on Legendre transforms (defined in Section 2), which satisfies a set of the above objectives that differs from those of previous representations. Like in Khuri II unitarity is not simple but analyticity, crossing and asymptotic behaviour are all explicit. There appear to be fewer mutually cancelling parts of terms than in Khuri II, and under similar assumptions to those of Khuri I it would be possible to replace the background integral by a sum. Alternatively without these assumptions the method provides a derivation of Regge asymptotic behaviour in crossing symmetric form plus a Mandelstam representation without subtractions.

In Section 2 of this paper Legendre transforms are defined and are shown to have an analytic continuation from integer values under conditions similar to those for partial wave amplitudes. Relations are established between the location of poles of Legendre transforms and those of partial wave amplitudes using the power series coefficients from Khuri II as an intermediate step.

In Section 3 the asymptotic terms that arise from Regge poles moving into the right half angular momentum plane are separated out in two channels. This leaves a residual background term which can be expressed as a double Watson-Sommerfeld transform, and which satisfies the Mandelstam representation without subtractions. The asymptotic terms also separately satisfy the Mandelstam representation and have branch cuts correctly located.

In Section 4 the full problem of a 3-channel process is considered. It is shown that the asymptotic terms can be separated out for all three channels simultaneously, leaving a background term that satisfies the Mandelstam representation without subtractions.

2. LEGENDRE TRANSFORMS: DEFINITIONS, MEROMORPHY AND ASYMPTOTIC BEHAVIOUR

In this Section we limit our discussion to a scattering amplitude that, after a finite number of subtractions, satisfies the Mandelstam representation with a spectral function that is non-zero in one region only (the s, t region). This will be generalized to give a fully crossing symmetric representation in Section 4. We assume also that there are no bound states. Taking particles of equal mass M , define the variables x, y, z in terms of the squares of the invariant energies s, t, u by

$$s = 2M^2(1+x), \quad t = 2M^2(1+y), \quad u = 2M^2(1+z), \quad (2.1)$$

with $x+y+z = -1$.

The amplitude $A(x, y)$ is regular below the first normal threshold in each channel since there are no bound states. In this Section we have only s and t channels, and

$$A(x, y) = F(x, y) + \frac{1}{\pi^2} \int_1^\infty dx' \int_1^\infty dy' \frac{\rho(x', y')}{(x'-x)(y'-y)} \quad (2.2)$$

$$= F_1(x, y) + \frac{1}{\pi} \int_1^\infty dx' \frac{A_1(x', y)}{(x'-x)} \quad (2.3)$$

$$= F_2(x, y) + \frac{1}{\pi} \int_1^\infty dy' \frac{A_2(x, y')}{(y'-y)} \quad (2.4)$$

where F, F_1 and F_2 denote subtraction terms. The functions F_1 and F_2 are polynomials in x and y respectively.

Since $A(x, y)$ is regular in $\{x < 1, y < 1\}$, it can be expanded in a series of Legendre polynomials, or in a double series,

$$A(x, y) = \sum_{\ell=0}^{\infty} (2\ell+1) B(\ell, y) P_{\ell}(x), \quad (2.5)$$

$$= \sum_{m=0}^{\infty} (2m+1) B(x, m) P_m(y), \quad (2.6)$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} (2\ell+1)(2m+1) C(\ell, m) P_{\ell}(x) P_m(y), \quad (2.7)$$

where the Legendre transforms B and C are defined by

$$B(\ell, y) = \frac{1}{2} \int_{-1}^1 dx P_{\ell}(x) A(x, y) \quad (2.8)$$

$$B(x, m) = \frac{1}{2} \int_{-1}^1 dy P_m(y) A(x, y) \quad (2.9)$$

$$C(\ell, m) = \frac{1}{4} \int_{-1}^1 dx \int_{-1}^1 dy P_{\ell}(x) P_m(y) A(x, y) \quad (2.10)$$

For sufficiently large values of ℓ , the polynomial P_{ℓ} makes no contribution to the integral in Eq. (2.8), and hence

$$B(\ell, y) = \int_1^{\infty} dx' Q_{\ell}(x') A_1(x', y). \quad (2.11)$$

Since, by assumption, the integral in Eq. (2.3) is convergent, the equation (2.11) defines a unique analytic continuation of $B(\ell, y)$ to complex values of ℓ in $R(\ell) > \ell_0$, and the continuation is bounded by a polynomial in ℓ in this region, [see Carlson's theorem ⁵⁾]. A similar continuation holds for $B(x, m)$ in $R(m) > m_0$, where m_0 is a fixed real number. The function $F(x, y)$ in Eq. (2.2) consists of a polynomial in x (whose coefficients depend on y and may have branch cuts in $R(y) > 1$), plus a similar polynomial in y . Therefore for $R(\ell) > \ell_0$ and $R(m) > m_0$,

$$C(\ell, m) = \int_1^\infty dx' \int_1^\infty dy' Q_\ell(x') Q_m(y') \rho(x', y'). \quad (2.12)$$

This defines an analytic continuation of the double Legendre transform $C(\ell, m)$.

We consider next the relation between poles of the usual partial wave amplitude in the s channel $A(x, m)$ and the poles of the Legendre transform $B(x, m)$. It is convenient to use Khuri's power series coefficients [Khuri II ⁴⁾] as a link between these amplitudes. Writing $\omega = \cos\theta$, we have

$$y = -\frac{1}{2} \{ 1 + x + \omega(1-x) \}. \quad (2.13)$$

We consider the amplitude for fixed y (that is fixed s). In the real region $x < 1$, $y < 1$, the full scattering amplitude $A(x, y)$ can be expanded in a power series in y or a power series in ω ,

$$A(x, y) = \sum_{m=0}^{\infty} f(x, m) y^m, \quad (2.14)$$

$$= \sum_{m=0}^{\infty} g(x, m) \omega^m. \quad (2.15)$$

It can also be expressed as the usual partial wave series

$$A(x, y) = \sum_{m=0}^{\infty} (2m+1) A(x, m) P_m(\omega). \quad (2.16)$$

It should be noted that the relation between the partial wave series Eq. (2.16) and the power series (2.15) is similar to the relation of the Legendre transform series (2.6) and the power series (2.14). This will enable us to make a direct application of the following two theorems from Khuri II ⁴⁾ :

Theorem 1

If $A(x, m)$ satisfies the following two conditions :

- i) it has a unique analytic continuation that is meromorphic in $R(m) > -\frac{1}{2}$, with poles at $m = \beta$, $R(\beta) > -\frac{1}{2}$,
- ii) as $|m| \rightarrow \infty$ in $R(m) > -\frac{1}{2}$, $A(x, m) \sim C m^{\frac{1}{2}} \exp(-m\xi)$ where ξ is real and positive,

then there exists a unique analytic continuation of $g(x, m)$, [see Eq. (2.15)], which is meromorphic in $R(m) > -\frac{1}{2}$. The poles occur at $\beta, \beta-2, \dots, \beta-2n$, where $R(\beta-2n) > -\frac{1}{2}$. As $R(m) \rightarrow \infty$, $g(x, m) \sim (\cosh \xi)^{-m}$, and $g(x, m)$ vanishes as $|Im(m)| \rightarrow \infty$ in $R(m) > -\frac{1}{2}$.

Theorem 2

This is the converse of Theorem 1. If $g(x, m)$ satisfies the conditions derived in Theorem 1 with a pole at $m = \beta$, then $A(x, m)$ satisfies conditions i) and ii) with poles at $\beta, \beta-2, \dots, \beta-2n$, with $R(\beta-2n) > -\frac{1}{2}$.

It is clearly immaterial which variable is used for the expansion of $A(x, y)$ so we deduce at once that Theorems 1 and 2 apply also to the properties of the Legendre transform $B(x, m)$ and the power series coefficient $f(x, m)$. It is therefore necessary only to relate the poles and asymptotic behaviour of the functions f and g in order to complete our derivation of the analogous relation between $A(x, m)$ and $B(x, m)$.

By expanding the denominator of the integral in Eq. (2.4) and comparing with Eq. (2.14), we see that for $R(m)$ sufficiently large,

$$f(x, m) = \frac{1}{\pi} \int_1^{\infty} dy A_2(x, y) y^{-m-1}. \quad (2.17)$$

If $g(x, m)$ has a pole in the m plane at $m = \beta$, the amplitude $A(x, y)$ contains a term proportional to $(-\omega)^\beta$. This follows from applying the Sommerfeld-Watson transform to the integral in Eq. (2.15). Writing this term as a dispersion integral,

$$\frac{(-\omega)^\beta}{\sin \pi \beta} = -\frac{1}{\pi} \int_0^{\infty} d\omega' \frac{(\omega')^\beta}{\omega' - \omega} \quad (2.18)$$

we see that the discontinuity $A_2(x, y)$ in Eq. (2.17) contains

$$(\omega')^\beta = (a + by)^\beta \quad (2.19)$$

where a and b are constants determined from Eq. (2.13). Substituting the right-hand side of Eq. (2.19) for A_2 in Eq. (2.17) and integrating, we see that $f(x, m)$ has poles at $m = \beta, \beta - 1, \beta - 2, \dots$ in $R(m) > -\frac{1}{2}$. It is clear that for these pole terms $f(x, m) \rightarrow 0$ as $|\text{Im}(m)| \rightarrow \infty$.

In addition to the terms in $A_2(x, y)$ that correspond to poles in the m plane, there will be a term A_2^0 where $A_2^0(x, y) \sim y^{-\frac{1}{2}}$ as $y \rightarrow \infty$. This term corresponds to the usual background integral of the Regge representation. When substituted into Eq. (2.17) it is clear that for this term also $f(x, m) \rightarrow 0$ as $|\text{Im}(m)| \rightarrow \infty$ in $R(m) > -\frac{1}{2}$.

This completes the derivation of our result that the functions $A(x, m)$, $g(x, m)$, $f(x, m)$, $B(x, m)$, are all related by theorems similar to Khuri's Theorems 1 and 2 that are quoted above.

3. WATSON-SOMMERFELD TRANSFORMS AND DOUBLE TRANSFORMS

When the partial wave amplitude satisfies the conditions of Theorem 1, the Legendre transform satisfies similar conditions and can be used as the basis of a W-S transform of the scattering amplitude,

$$A(x,y) = \frac{i}{2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dm \frac{(2m+1)B(x,m)P_m(-y)}{\sin \pi m} - \pi \sum_{\beta} \frac{(2\beta+1)b(\beta)P_{\beta}(-y)}{\sin \pi \beta} \quad (3.1)$$

Using the other channel,

$$A(x,y) = \frac{i}{2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \frac{(2\ell+1)B(\ell,y)P_{\ell}(-x)}{\sin \pi \ell} - \pi \sum_{\alpha} \frac{(2\alpha+1)b(\alpha)P_{\alpha}(-x)}{\sin \pi \alpha} \quad (3.2)$$

The sums in (3.1) and (3.2) are over the poles of $B(x,m)$ at $m = \beta(x)$ in the domain $R(\beta) > -\frac{1}{2}$, and the poles of $B(\ell,y)$ at $\ell = \alpha(y)$ in $R(\alpha) > -\frac{1}{2}$, respectively.

The full amplitude $A(x,y)$ has branch cuts along the real axis in $R(x) > 1$ and $R(y) > 1$. The branch cut in the y plane is correctly located by the branch cuts of $P_m(-y)$ and $P_{\beta}(-y)$ in Eq. (3.1), so no branch cut cancellation is required (in contrast with the usual Regge representation in which the branch cuts of $P_m(-\omega)$ are not the same as those of $A(x,y)$). Similarly Eq. (3.2) gives the correct branch cuts in the x plane.

We consider next the problem of forming a double W-S transform of the scattering amplitude derived from Eq. (2.7). In forming the single W-S transform in Eq. (3.1) we have separated out the dominant asymptotic terms in y for each particular value of x that determines which poles $\beta(x)$ lie in $R(m) > -\frac{1}{2}$. In order to construct a double

W-S transform with background integrals along $R(\ell) = -\frac{1}{2}$ and $R(m) = -\frac{1}{2}$ we need to separate $C(\ell, m)$ into two parts

$$C(\ell, m) = C_0(\ell, m) + C_1(\ell, m), \quad (3.3)$$

where $C_0(\ell, m)$ is regular in $\{R(\ell) > -\frac{1}{2}\} \times \{R(m) > -\frac{1}{2}\}$. This requires the removal from the background integrals in Eq. (3.1) and (3.2) of all terms that arise from any pole of $B(x, m)$ or $B(\ell, y)$ that moves into the right half ℓ or m planes for any value of x and y on the physical sheet.

We can therefore redefine the sums in Eq. (3.1) and (3.2) to include all such poles. However, since for some values of x , any pole $\beta(x)$ can be found in the left half m plane, it is apparent that $P_\beta(-y)$ for these values of x represents a false asymptotic behaviour that is in fact cancelled by the background integral. It is therefore necessary to modify the asymptotic terms and the background integral in Eq. (3.1) so that when β moves into the left half plane the separated term no longer dominates in the asymptotic region as $y \rightarrow \infty$. A similar modification is required for Eq. (3.2). One method of achieving this result has been described by the author in another paper ⁶⁾ and a similar method is given in Khuri II. However, we shall use a different method here which more closely resembles the spirit of Khuri I in that the modified background integral contains a damping factor which reduces it, if it can be moved further to the left.

Make the following substitution in the background integral in Eq. (3.2), (see Khuri I),

$$\frac{2^{1/2} \pi (2\ell+1) P_\ell(-x)}{\sin \pi \ell} = - \int_{-\infty}^0 \frac{d\xi \sinh \xi \exp(\ell + \frac{1}{2})\xi}{(\cosh \xi - x)^{3/2}} - \int_0^{\infty} \frac{d\xi \sinh \xi \exp(\ell + \frac{1}{2})\xi}{(\cosh \xi - x)^{3/2}} \quad (3.4)$$

Define $B'(\xi, y)$ by,

$$B'(\xi, y) = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell B(\ell, y) \exp(\ell + \frac{1}{2})\xi \quad (3.5)$$

Eq. (3.2) can be rewritten to give,

$$A(x, y) = \frac{1}{2^{1/2}} \int_{-\infty}^0 \frac{d\xi \sinh \xi B'(\xi, y)}{(\cosh \xi - x)^{3/2}} + \frac{1}{2^{1/2}} \int_0^{\infty} \frac{d\xi \sinh \xi B'(\xi, y)}{(\cosh \xi - x)^{3/2}} - \pi \sum_{\alpha} \frac{(2\alpha+1)b(\alpha)P_{\alpha}(-x)}{\sin \pi \alpha} \quad (3.6)$$

For convergence of the integral in Eq. (3.5) we require $B(\ell, y) \rightarrow 0$ as $|\operatorname{Im}(\ell)| \rightarrow \infty$, and we assume this to be satisfied.

In the first integral in Eq. (3.6), $\xi < 0$, so the contour in Eq. (3.5) for $B'(\xi, y)$ can be closed to the right in the ℓ plane. Using meromorphy and boundedness of $B(\ell, y)$ we collect only the residues giving,

$$B'(\xi, y) = -\sum_{\alpha} b(\alpha) \exp(2\alpha+1)\xi, \quad \text{for } \xi < 0. \quad (3.7)$$

Substituting from Eq. (3.7) for B' in the first integral of Eq. (3.6), and from Eq. (3.4) for $P_{\alpha}(-x)$ in Eq. (3.6), we obtain

$$A(x, y) = \frac{1}{2^{1/2}} \int_0^{\infty} \frac{d\xi \sinh \xi B'(\xi, y)}{(\cosh \xi - x)^{3/2}} - \sum_{\alpha} b(\alpha) R_{\alpha}(-x), \quad (3.8)$$

where

$$R_{\alpha}(-x) = - \int_0^{\infty} \frac{d\xi \sinh \xi \exp(\alpha + \frac{1}{2})\xi}{2^{1/2} (\cosh \xi - x)^{3/2}} \quad (3.9)$$

From Eq. (3.4), when $R(\alpha + \frac{1}{2}) > 0$,

$$R_{\alpha}(-x) \sim \frac{\pi (2\alpha+1) P_{\alpha}(-x)}{\sin \pi \alpha}, \quad \text{as } x \rightarrow \infty. \quad (3.10)$$

When the real part of $(\alpha + \frac{1}{2})$ is negative $R_{\alpha}(-x)$ is always dominated by the background integral in Eq. (3.8) as $x \rightarrow \infty$.

We now note three properties of the representation in Eq. (3.8). Firstly, if we distort the contour defining B' , Eq. (3.5), to the left across a pole of $B(\ell, y)$ we will collect the residue at that pole, which has exactly the same form as the terms in the sum in Eq. (3.8). We can therefore redefine B' by,

$$B'(\xi, y) = \frac{1}{2\pi i} \int_C d\ell B(\ell, y) \exp(\ell + \frac{1}{2})\xi, \quad (3.11)$$

where the contour C is defined so that for every value of y it lies to the left of any pole $\alpha(y)$ of $B(\ell, y)$, which for some value of y moves into the right half-plane $R(\alpha + \frac{1}{2}) > 0$. The sum in Eq. (3.8) is then defined to include all such poles α , so for a given value of y some of these poles will lie to the left of $R(\alpha) = \frac{1}{2}$, but all will lie to the right of the contour C . This procedure separates from the background integral all terms which dominate over it, when x is large, for some value of y . We assume that the number of these poles is finite.

The second property of the representation (3.8) that we require is that the individual terms and the background integral, taken separately, have the cut plane analyticity of the Mandelstam representation. That is

to say they each satisfy the Mandelstam representation with sufficient subtractions. It is clear from Eq. (3.9) that $R_{\alpha}(-x)$ has cut plane analyticity in x with the cut from $x = 1$ to ∞ , and provided the location of the Regge pole $\alpha(y)$ has the usual properties the whole term will be analytic in the y plane cut from 1 to ∞ . Similarly the background integral in Eq. (3.8) is clearly analytic in the cut x plane and from Eq. (3.5) has the correct branch cut in y . It is important here that the Legendre transform $B(\ell, y)$ has only a right hand branch cut in y (in contrast to the usual partial wave amplitude).

The third property of the representation (3.8) will not be required here but it may be useful in other contexts, for example in practical calculations. If $B(\ell, y)$ is suitably bounded and meromorphic in the left half ℓ plane (as well as the right half), then, since in the background integral $\xi > 0$, we see that the contour C in Eq. (3.11) can be closed on the left of the ℓ plane. We would collect residues at the poles and the amplitude would take the form,

$$A(x, y) = - \sum_{\alpha} b(\alpha) R_{\alpha}(-x), \quad (3.12)$$

where the sum over α now includes all poles of $B(\ell, y)$ in the entire ℓ plane.

The background integral in Eq. (3.8) satisfies the Mandelstam representation. No subtractions are required in the x dispersion relation since the dominant x terms have been separated for all values of y . We can now apply the same procedure to the background integral and express it in the form that removes the dominant y terms,

$$\frac{1}{2^{1/2}} \int_0^{\infty} \frac{d\eta \sinh \eta B''(x, \eta)}{(\cosh \eta - y)^{3/2}} - \sum_{\beta} b(\beta) R_{\beta}(-y), \quad (3.13)$$

in which the sum includes all poles of $B(x, m)$ that enter the right half m plane for some value of x on the physical sheet. $B''(x, \gamma)$ is defined in a similar way to Eq. (3.11) but is based on a Legendre transform derived from the background integral of Eq. (3.8) not from the full amplitude.

Provided the residues $b(\beta)$ satisfy,

$$b\{\beta(x)\} \sim x^{-1/2}, \quad \text{as } x \rightarrow \infty, \quad (3.14)$$

then the separation in formula (3.13) does not introduce any x dependence that dominates over the background integral of Eq. (3.8) for large x . Therefore from (3.8) and (3.13),

$$A(x, y) = A_0(x, y) - \sum_{\alpha} b(\alpha) R_{\alpha}(-x) - \sum_{\beta} b(\beta) R_{\beta}(-y) \quad (3.15)$$

where $A_0(x, y)$ denotes the first term in formula (3.13), and satisfies the Mandelstam representation without subtractions,

$$A_0(x, y) = \frac{1}{\pi^2} \int_1^{\infty} dx' \int_1^{\infty} dy' \frac{\rho_0(x', y')}{(x' - x)(y' - y)} \quad (3.16)$$

The amplitude $A_0(x, y)$ can be expressed in either of the following equivalent forms,

$$A_0(x, y) = \frac{i}{2} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \frac{(2\ell+1) B_0(\ell, y) P_{\ell}(-x)}{\sin \pi \ell} \quad (3.17)$$

$$= \frac{i}{\pi^{1/2}} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell B_0(\ell, y) \int_0^{\infty} \frac{d\xi \sinh \xi \exp(\ell + \frac{1}{2}) \xi}{(\cosh \xi - x)^{3/2}} \quad (3.18)$$

The function $B_0(\ell, y)$ differs from $B(\ell, y)$ only by the absence of those poles of the latter that enter the right half ℓ plane. There are similar formulae using $B_0(x, m)$.

The corresponding double Legendre transform $C_0(\ell, m)$ is regular in $\{R(\ell) > -\frac{1}{2}\} \times \{R(m) > -\frac{1}{2}\}$, and we can express $A_0(x, y)$ as a double W-S transform. This gives from Eq. (3.8),

$$A(x, y) = -\frac{1}{4} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dm \frac{(2\ell+1)(2m+1)C_0(\ell, m)P_\ell(-x)P_m(-y)}{\sin\pi\ell \sin\pi m} \\ - \sum_{\alpha} b(\alpha) R_{\alpha}(-x) - \sum_{\beta} b(\beta) R_{\beta}(-y). \quad (3.19)$$

The integrand satisfies the Mandelstam representation; in fact it is a product of two terms each of which satisfies a single variable dispersion relation. Similarly each term of the two sums satisfies the Mandelstam representation with subtractions, if the usual assumption is made that the positions of the poles $\alpha(y)$ and $\beta(x)$, and the corresponding residues, satisfy dispersion relations.

The branch cuts of $A(x, y)$ are correctly located for all terms in Eq. (3.12), and the asymptotic behaviour for large x , or for large y , is given by terms in the second or first sum, respectively.

4. CROSSING SYMMETRY IN THE RELATIVISTIC PROBLEM

We consider the equal mass problem, with neutral spinless particles, and in which no bound states or single particle states occur. We express the amplitude as a function $A(x,y,z)$ depending on the variables x,y,z with $x+y+z = 1$ [see Eq. (2.1)]. Divide the amplitude into 4 parts, consisting of subtraction terms $G(x,y,z)$ and 3 parts $L_{12}(x,y)$, $L_{23}(y,z)$, $L_{31}(z,x)$ that correspond to the three spectral regions of the Mandelstam representation,

$$L_{12}(x,y) = \frac{1}{\pi^2} \int_1^\infty dx' \int_1^\infty dy' \frac{\rho_{12}(x',y')}{(x'-x)(y'-y)} \quad (4.1)$$

It is more convenient to apply the method of the previous Sections separately to L_{12} , L_{23} and L_{31} , rather than to use the method of odd and even amplitudes developed for partial waves. The latter would be necessary if we wished to form the single Legendre transform, $B(\ell, y)$ for example, of the full amplitude. This would involve us in the unsymmetric procedure of forming $B(\ell, y)$ at fixed values of y from $L_{23}(y,z)$ and $L_{31}(z,x)$. There would then be the usual complications of the left hand cut in the x variable that would introduce a left hand integral into the formula in Eq. (2.11). Then odd and even amplitudes would be required so as to give unique analytic continuation in the variable from integer values. These difficulties are avoided by forming separately the transforms of each part of the amplitude.

We use the variables ℓ, m, n to correspond to x, y, z respectively. We form the following transforms

$$\text{from } L_{12}(x,y) \rightarrow B_{12}(\ell, y), B_{12}(x, m), C_{12}(\ell, m), \quad (4.2)$$

$$\text{from } L_{23}(y,z) \rightarrow B_{23}(m, z), B_{23}(y, n), C_{23}(m, n), \quad (4.3)$$

$$\text{from } L_{31}(z,x) \rightarrow B_{31}(n, x), B_{31}(z, \ell), C_{31}(n, \ell). \quad (4.4)$$

Next we remove from the amplitude L_{12} all asymptotic terms arising from poles of $B_{12}(\ell, y)$ that enter the right half ℓ plane. Denote these terms by

$$- \sum_{\alpha} b^{12} \{ \alpha(y) \} R_{\alpha}(-x). \quad (4.5)$$

Similarly we remove the analogous terms coming from poles in $B_{12}(x, m)$. Denote these by

$$- \sum_{\beta'} b^{12} \{ \beta'(x) \} R_{\beta'}(-y). \quad (4.6)$$

The resulting part of the amplitude $L_{12}^{(0)}$ will satisfy the Mandelstam representation without subtractions.

When this procedure is applied also to L_{23} and L_{31} the following crossing symmetric formula is obtained for the scattering amplitude,

$$\begin{aligned} A(x, y, z) = & L_{12}^{(0)}(x, y) + L_{23}^{(0)}(y, z) + L_{31}^{(0)}(z, x) \\ & - \sum_{\alpha} b^{12} \{ \alpha(y) \} R_{\alpha}(-x) - \sum_{\beta'} b^{12} \{ \beta'(x) \} R_{\beta'}(-y) \\ & - \sum_{\beta} b^{23} \{ \beta(z) \} R_{\beta}(-y) - \sum_{\gamma'} b^{23} \{ \gamma'(y) \} R_{\gamma'}(-z) \\ & - \sum_{\gamma} b^{31} \{ \gamma(x) \} R_{\gamma}(-z) - \sum_{\alpha'} b^{31} \{ \alpha'(z) \} R_{\alpha'}(-x). \end{aligned} \quad (4.7)$$

For large values of x with y fixed, $x \rightarrow +\infty$

$$A(x, y, z) \sim - \sum_{\alpha} b^{12} \{ \alpha(y) \} R_{\alpha}(-x) - \sum_{\gamma'} b^{23} \{ \gamma'(y) \} R_{\gamma'}(-z) \quad (4.8)$$

$$\sim - \sum_{\alpha} b^{12} \{ \alpha(y) \} (-x)^{\alpha} - \sum_{\gamma'} b^{23} \{ \gamma'(y) \} (-z)^{\gamma'}. \quad (4.9)$$

Since y is fixed, as $x \rightarrow +\infty$, we have $z = 1-x-y \rightarrow -\infty$. This accounts for the presence of a z dependent term in Eqs. (4.8) and (4.9). It also shows that the poles $\alpha'(z)$ and $\gamma(x)$ recede into the left half ℓ and n planes respectively and corresponding functions $R_{\alpha'}(-x)$ and $R_{\gamma}(-z)$ become negligible for large $|x|$ and $|z|$.

The function $L_{12}^{(0)}$ can be expressed either as a Mandelstam integral or as a double Watson-Sommerfeld transform, or as a double transform of the type (3.18),

$$L_{12}^{(0)}(x, y) = \frac{1}{\pi^2} \int_1^\infty dx' \int_1^\infty dy' \frac{\rho_{12}^{(0)}(x', y')}{(x' - x)(y' - y)} \quad (4.10)$$

$$= -\frac{1}{4} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dm \frac{(2\ell+1)(2m+1) C_{12}^{(0)}(\ell, m) P_\ell(-x) P_m(-y)}{\sin \pi \ell \sin \pi m} \quad (4.11)$$

$$= \frac{1}{2\pi^2} \int_0^\infty \frac{d\xi \sinh \frac{\xi}{2}}{(\cosh \frac{\xi}{2} - x)^{3/2}} \int_0^\infty \frac{d\eta \sinh \frac{\eta}{2}}{(\cosh \frac{\eta}{2} - y)^{3/2}} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell \exp(\ell + \frac{1}{2}) \xi \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dm \exp(m + \frac{1}{2}) \eta C_{12}^{(0)}(\ell, m) \quad (4.12)$$

Comparing Eq. (4.10) with Eq. (4.11) we see that,

$$\rho_{12}^{(0)}(x, y) = -\frac{1}{4} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\ell (2\ell+1) \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dm (2m+1) C_{12}^{(0)}(\ell, m) P_\ell(x) P_m(y). \quad (4.13)$$

Similar formulae hold for $L_{23}^{(0)}(y, z)$ and $L_{31}^{(0)}(z, x)$.

5. DISCUSSION

We have established a crossing symmetric form of the Mandelstam-Regge representation, Eq. (4.6). The background integral is of Mandelstam type without subtractions, and the asymptotic terms are of Regge type when their pole location is in the right half plane, but they tend to zero when it is in the left half plane. The asymptotic terms occur in a form that is crossing symmetric, but this depends on the residues at Regge poles tending to zero asymptotically so that they do not dominate over the background integral.

Our representation differs from that of Khuri ⁴⁾ by a somewhat different separation between the background integral and the asymptotic terms. Our choice of asymptotic terms arises from our use of Legendre transforms. The method also provides the possibility of replacing the entire background integral by a sum over residues if the Legendre transform is also meromorphic and suitably bounded in the left half plane. In principle this could be done in a crossing symmetric form but it is doubtful whether the necessary conditions of meromorphy and boundedness in the left half planes hold in the relativistic case ⁷⁾.

The use of Legendre transforms not only permits crossing symmetry to be established but it also simplifies the Regge formula because, (in contrast with the use of partial wave amplitudes), the branch cuts are correctly located. The compensating disadvantage is that unitarity will be rather complicated in all three channels instead of in only two channels as with partial wave amplitudes.

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R E F E R E N C E S

- 1) T. Regge, Nuovo Cimento 14, 951 (1959). See also A. Bottino, A.M. Longoni and T. Regge, ibid. 23, 954 (1962).
- 2) S. Mandelstam, Phys.Rev. 112, 1344 (1958).
- 3) N.N. Khuri, Phys.Rev. 130, 429 (1963), denoted Khuri I in the text.
- 4) N.N. Khuri, Phys.Rev. Letters 10, 420 (1963), and Princeton Institute for Advanced Study preprint, denoted Khuri II in the text.
- 5) Carlson's theorem, see for example E.C. Titchmarsh, Theory of Functions, Oxford (1939).
- 6) R.J. Eden, Phys.Rev. (in press).
- 7) V.N. Gribov, Soviet Physics J.E.T.P. 14, 1395 (1962).