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ARCHIVES

ON PARASTATISTICS

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INTRODUCTION

Some attention has been recently paid to a possible deviation from ordinary Bose-Einstein and Fermi-Dirac statistics in the realm of elementary particles ¹⁾. From a practical point of view, the so-called parastatistics of rank 2 are the most appealing ones, on account of their (comparative) simplicity.

It is known ²⁾, however, that the existence of para-assemblies is incompatible with that of a complete set of commuting observables, provided that the particles are indistinguishable. What is meant here by indistinguishability is that

- I) the Hilbert space needed to describe para-assemblies of n identical particles be a subspace of $\tilde{\mathcal{H}}^n$ (defined as $\tilde{\mathcal{H}}_1 \otimes \dots \otimes \tilde{\mathcal{H}}_1$, n times) invariant under π_n (permutation group of order n !).
 - II) the observables must commute with all the permutations.
- We remark that II) is meaningless unless I) is true.

One of the authors ³⁾ has derived an explicit irreducible representation (with vacuum) of the commutation rules for the conventional F_2 and B_2 (parafermions and parabosons of multiplicity two), which clearly shows that I) is violated ⁴⁾. In that paper a new framework was then proposed to circumvent this difficulty. The essential feature of the method consists in using a sequence of central idempotents ⁵⁾, $\{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, \dots\}$ satisfying the following conditions ³⁾:

$$1) \quad \varepsilon_n \in Z(\sigma_n) \quad \text{for every } n, \quad \text{where } Z(\sigma_n) \quad (1)$$

is the centre of the group algebra σ_n of π_n

$$2) \quad \varepsilon_n = \varepsilon_n^+, \quad \varepsilon_n^2 = \varepsilon_n. \quad (2)$$

3) ξ_n has vanishing projections on the Young diagrams having more than r columns (rows), while it actually contains some Young diagrams with r columns (rows). This last condition ensures that the parastatistics thereby described will actually be $F_r(B_r)$; r is called the rank of the parastatistics.

4) If α_k is the unsymmetrized destruction defined by

$$\alpha_k |i_1 \dots i_n\rangle \equiv x_n \delta_{ki_n} |i_1 \dots i_{n-1}\rangle$$

and α_k^+ its Hermitian conjugate, then

$$\varepsilon_{n-1} \alpha_k \varepsilon_n = \alpha_k \varepsilon_n \quad (3)$$

and therefore

$$\varepsilon_n \alpha_k^+ \varepsilon_{n-1} = \varepsilon_n \alpha_k^+ \quad (4)$$

5) If we set $\xi \equiv \bigoplus_{n=0}^{\infty} \varepsilon_n$ and define the parafield operators as

$$a_k = \xi \alpha_k \xi \quad (5)$$

$$a_l^+ = \xi \alpha_l^+ \xi \quad (6)$$

then the algebra $\{a_k, a_k^+\} (k, \ell = 1, 2, \dots)$ must be irreducible in $\mathcal{E} \tilde{h}$, where $\tilde{h} \equiv \bigoplus_{n=0}^{\infty} \tilde{h}^n$.

One easily checks that all these conditions are satisfied by ordinary B_1, F_1 (Bose or Fermi) statistics. Conventional F_2 and B_2 satisfy ³⁾ all but condition 1). The subsequent analysis on the structure of $\mathcal{E}$ implies that we can narrow our conditions - namely that 5) is a consequence of 1) through 4). The problem remained open whether or not the above conditions allow physically interesting "paramodels". As natural criteria in this approach we have required

- A) the existence of commutation relations of finite order which enable us to write in an "almost normal" form any state vector obtained by applying to the vacuum a certain number of creation and destruction operators ^{*)}.
- B) the existence of an expression

$$N_k = P(L_k^{(j)}, M_k^{(i)}), \text{ with } L_k^{(j)} \equiv (a_k)^j (a_k^+)^j, M_k^{(i)} \equiv (a_k^+)^i (a_k)^i \quad (7)$$

for the operator counting the number of particles in the state $|k\rangle$, P being a polynomial.

^{*)} Since one has to expect that no commutation rules of order 2 exist in the general case, the "almost normal" form referred to herein means one in which only a fixed (depending on the parafield) finite number of successive creation operators stand to the right of one destruction operator.

We note that this requirement is much weaker than that of Green ¹⁾.

Now, the purpose of this note is to prove that no paramodel exists satisfying 1) to 4) and either A) or B).

1. STRUCTURE OF $\mathcal{E}_n$

Theorem 1 : Conditions 1) to 4) imply that $\mathcal{E}_n$ is the sum of all the units of σ_n associated with Young diagrams belonging to partitions lesser than or equal to a fixed sequence of positive integers, $\{\Lambda_n\}_1^\infty$ ($\Lambda_1 \geq \Lambda_2 \geq \dots$; $\Lambda_1 = r$). When partitions refer to rows (resp. : columns) one obtains F_r (resp. : B_r).

We first prove the following :

Lemma 1 : If $E^{(\lambda)}$ is the unit given by the partition $[\lambda] \equiv [\lambda_1, \dots, \lambda_m]$, $\sum \lambda_i = s$, then $\alpha_k^{E^{(\lambda)}}$ has non-vanishing projection on each (allowable) partition

$$[\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_m], i = 1, \dots, m. \quad (8)$$

Proof : Take $|\psi\rangle = |k_1\rangle |k_2\rangle \dots |k_{s-1}\rangle |k\rangle$, with $k_j \neq k$. Then the only permutations in $E^{(\lambda)}$ which will contribute to $\alpha_k^{E^{(\lambda)}} |\psi\rangle$ will be those leaving the s -th place unchanged. Now, $E^{(\lambda)}$ is the sum of all primitive idempotents associated to the standard tableaux belonging to $[\lambda]$. We know that the sum of all permutations in those

6.

cannot contain

$$[n'_1, \dots, n'_k, n'_{k+1} + 1, n'_{k+2}, \dots]$$

so that, by iterating this process, $\mathcal{E}_{n+\Lambda_{k+1} - n'_{k+1}}$ cannot contain

$$[\Lambda_1, \dots, \Lambda_k, \Lambda_{k+1}, n'_{k+2}, \dots].$$

Repeating the argument for n'_{k+2} , then for n'_{k+3} , and so on, we arrive at the conclusion that some $\mathcal{E}_N$ cannot contain the maximal partition $[N]$ compatible with $[N] \leq [\Lambda]$. Hence by reductio ad absurdum, the theorem follows.

2. COMMUTATION RELATIONS

Consider parafields of rank r given by (5), (6), where $\mathcal{E}_n$ satisfies requirements 1) to 4). Given a product of $r+1$ creation operators, $a_{k_1}^+, \dots, a_{k_{r+1}}^+$, (6) shows that $a_{k_1}^+ \dots a_{k_{r+1}}^+ = \mathcal{E} \alpha_{k_1}^+ \dots \alpha_{k_{r+1}}^+$.

Now, since $\mathcal{E} \alpha_{k_1}^+ \dots \alpha_{k_{r+1}}^+ |\psi_n\rangle$ will necessarily involve at least two indices of the $k_1, \dots, k_{r+1}$ in the same column (resp. : row) it is clear that

$$S\{a_{k_1}^+ \dots a_{k_{r+1}}^+\} = 0 \quad (\text{resp. : } A\{a_{k_1}^+ \dots a_{k_{r+1}}^+\} = 0) \quad (9)$$

in the F_r (resp. : B_r) case, and where S (resp. : A) is the symmetrizer (resp. : antisymmetrizer) of $(r+1)$ elements.

Similar relations are obtained for destruction operators just by taking Hermitian conjugates in (9).

Concerning relations involving both creation and annihilation operators, we will simply exhibit typical forms of them which should be true according to A) and next we will prove that they cannot hold. Let us consider first the case F_r . Then there should exist a relation of the form

$$a_k (a_k^+)^r + \lambda_1 a_k^+ a_k (a_k^+)^{r-1} + \dots + \mu (a_k^+)^{r-1} = 0 \quad (10)$$

and hence

$$(a_k)^r a_k^+ + \bar{\lambda}_1 (a_k)^{r-1} a_k^+ a_k + \dots + \bar{\mu} (a_k)^{r-1} = 0. \quad (11)$$

Take a state $|\psi\rangle \equiv \mathcal{E} |k \dots k \ell\rangle$ (($r-1$) k 's, $\ell \neq k$). Since here $\mathcal{E}_r = 1$, $a_k |\psi\rangle = 0$ and thus from (11)

$$(a_k)^r a_k^+ |\psi\rangle = 0$$

which is clearly impossible.

The case B_r is more involved, so that we will only sketch the argument for B_2 , since the general case may be treated along the same lines. Suppose then that there existed a relation of the form

$$a_\ell (a_k^+)^{n-1} + \lambda_1 a_k^+ a_\ell (a_k^+)^{n-2} + \dots + \lambda_{n-1} (a_k^+)^{n-1} a_\ell = 0 \quad (12)$$

8.

and therefore

$$(a_k)^{n-1} a_l^+ + \bar{\lambda}_1 (a_k)^{n-2} a_l^+ a_k + \dots + \bar{\lambda}_{n-1} a_l^+ (a_k)^{n-1} = 0 \quad (l \neq k) \quad (13)$$

Take now $|\psi\rangle \equiv \varepsilon |k \dots k l\rangle ((n-1)k\text{'s}, l \neq k)$. One easily verifies that

$$\varepsilon |k k \dots k l\rangle = |k k \dots k l\rangle \quad (14)$$

and hence

$$a_k |\psi\rangle = 0$$

while $(a_k)^{n-1} a_l^+ |\psi\rangle \neq 0$; this completes the proof ^{*)}.

3. NUMBER OPERATOR

In order to avoid inessential complications we will restrict ourselves to considering only the typical cases F_2, B_2 with either $\bigwedge_1 = 2, \bigwedge_2 = \bigwedge_3 = \dots = 1$, or $\bigwedge_1 = \bigwedge_2 = \dots = 2$.

Suppose that the number operator N_k is of the form given by (7). It is clear that it may always be written as

*) It can be explicitly be shown for F_2, B_2 and $\bigwedge_i = 2$ ($i = 1, 2, \dots$) that there exists no order 3 commutation relations of any kind (i.e., not necessarily of the type described above) involving both creation and annihilation operators.

$$N_k = p(L_k^{(j)}) + q(M_k^{(i)}) + r(L_k^{(j)}, M_k^{(i)}) + c\varepsilon \quad (i, j = 1, 2, \dots, s) \quad (15)$$

where p, q, r are polynomials with the property

$$p(0) = q(0) = r(0, y) = r(x, 0) = 0.$$

We intend to prove that no such expression is possible.

3.1

$$\text{Case } \mathbb{F}_2, \Lambda_1 = 2, \Lambda_2 = \dots = \Lambda_n = \dots = 1$$

Take $|\psi\rangle \equiv \varepsilon \{ |k\ell\rangle - |k\ell\rangle \}$; then simple algebraic manipulations show that

$$|\psi\rangle \neq 0, M_k^{(i)} |\psi\rangle = 0, L_k^{(j)} |\psi\rangle = 0.$$

As $N_k |\psi\rangle$ has to be equal to $1 \cdot |\psi\rangle = |\psi\rangle$, it follows $c = 1$. Consider next the subspace $\bar{n}_{(k)} \subset \varepsilon \bar{n}$, consisting of all states with no particle in the state $|k\rangle$. Since $M_k^{(i)}$ annihilates $\bar{n}_{(k)}$, one gets

$$p(L_k^{(j)}) + \varepsilon = 0, \text{ on } \bar{n}_{(k)} \quad (16)$$

or, more explicitly, since $L_k^{(j)} \equiv 0$ ($j \geq 3$, see (9)),

$$p(L_k^{(1)}, L_k^{(2)}) + \varepsilon = 0 \quad \text{on } \tilde{h}_{(k)} \quad (17)$$

Let $|\psi\rangle \in \varepsilon_{n-1} \tilde{h}_{(k)}$; then, as $\varepsilon_{n-1} = \varepsilon'_{n-1} + \varepsilon''_{n-1}$
with

$$\varepsilon'_{n-1} = [1, 1, \dots, 1], \quad \varepsilon''_{n-1} = [2, 1, \dots, 1]$$

it follows

$$L_k^{(1)} |\psi\rangle = x_n^2 \left\{ \varepsilon'_{n-1} + \frac{n-1}{n(n-2)} \varepsilon''_{n-1} \right\} |\psi\rangle \quad (18)$$

$$L_k^{(2)} |\psi\rangle = 2 x_n^2 x_{n+1}^2 \frac{1}{n+1} \varepsilon'_{n-1} |\psi\rangle.$$

From (17), and using the fact that $\varepsilon'_{n-1} \varepsilon''_{n-1} = \varepsilon''_{n-1} \varepsilon'_{n-1} = 0$,
 $(\varepsilon'_{n-1})^2 = \varepsilon'_{n-1}$, $(\varepsilon''_{n-1})^2 = \varepsilon''_{n-1}$, we can write

$$p\left(x_n^2 \frac{n-1}{n(n-2)}, 0\right) + 1 = 0 \quad (19)$$

$$p\left(x_n^2, 2 x_n^2 x_{n+1}^2 \frac{1}{n+1}\right) + 1 = 0.$$

The first equation requires that

$$x_n = O(n^{1/2}) \quad \text{as } n \rightarrow \infty$$

while with such a behaviour the second one cannot be fulfilled. We are thus led to contradiction, Q.E.D.

3.2 Case $B_2, \Lambda_1 = 2, \Lambda_2 = \dots = 1$

By restricting N_k to $\tilde{H}_{(k)}$, arguments similar to the above ones lead to

$$p(L_k^{(j)}) + c\varepsilon \equiv 0 \quad \text{on } \varepsilon \tilde{H} \quad (20)$$

i.e.,

$$N_k = q(M_k^{(i)}) + r(L_k^{(j)}, M_k^{(i)}) \quad \text{[REDACTED]} \quad (21)$$

Take now $|\psi\rangle \equiv \varepsilon_2 |k\ell\rangle = |k\ell\rangle$, $k \neq \ell$. Then,

$$M_k^{(i)} |\psi\rangle = 0, \quad i = 1, 2, \dots$$

On the other hand, the same computations as those carried out to prove (14) lead to

$$L_k^{(j)} |\psi\rangle = \text{constant} \times |\psi\rangle. \quad (22)$$

Thus

$$\{q(M_k^{(i)}) + r(L_k^{(j)}, M_k^{(i)})\} |\psi\rangle = 0$$

whilst $N_k |\psi\rangle$ must equal $|\psi\rangle$: we are again led to contradiction.

$$3.3 \quad F_2, B_2 \quad \text{and} \quad \bigwedge_1 = \dots = \bigwedge_n = \dots = 2$$

This case is similar to 3.1 and 3.2 and corresponding arguments apply.

4. CONCLUSIONS

The preceding analysis was aimed at circumventing in a natural way those difficulties inherent to the usual parastatistics framework as regards the indistinguishability of the constituent particles. In view of the negative aspects of this paper we feel tempted to state that either

- i) no para-particles exist in Nature,
- ii) if they exist, and are described by the conventional formalism, then indiscernability in the usual sense is lost. To maintain it would thus require a reformulation of this fundamental concept. Or, finally,
- iii) their description within a mathematical frame preserving indistinguishability and similar to the one given herein would make an intrinsic analysis of the field quantities ⁶⁾ extremely complicated (or likely impossible).

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Irreducibles de Paracampos", (to be published).
- 4) As an illustrative example consider an assembly of 3 parafermions of
rank 2, with state vector $|\psi\rangle = a_k^+ a_k^+ a_\ell^+ |0\rangle$. The commutation
relations $[a_i^+ a_j^+ a_k^+]_+ = 0$ (see Ref. 1) imply that the manifold
 $\mathcal{M}$ of states spanned by all $P|\psi\rangle$, with $P \in \pi_3$, is one-
dimensional. On the other hand, since all three-particle states
have a vanishing projection on the diagram $\square 3$, it follows
clearly that the reduction of π_3 to $\mathcal{M}$ is the antisymmetric
representation of π_3 . But $|\psi\rangle \neq 0$, while $(23)|\psi\rangle = 0$ (!!).
This paradox comes from the fact that the commutation rules are
brought into false relations when a general permutation is applied
to them. One concludes thus that they are incompatible with the
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- 6) A possible deviation from the proposal scheme might be the removal of condition 4), (formulae (3) and (4)), by choosing, e.g., $\mathcal{E}_n = [2, 1, \dots, 1]$. Nevertheless, one can show in this particular case, and in several others as well, that no order 3 commutation relations exist involving both destruction and creation operators neither expressions for N_k as a linear combination of $L_k^{(1)}$ and $M_k^{(1)}$ (Green's condition).