CM-P00056905

EXACT SOLUTIONS OF THE BETHE-SALPETER EQUATION FOR FERMIONS

W. Kummer

CERN - Geneva

A B S T R A C T

The properties of the Bethe-Salpeter equation (B.-S. equation) for two fermions with equal masses mediated by a massless vector particle (photon or γ γ μ model as specified in the text) are investigated in the ladder approximation. For vanishing total energy of the bound state, sets of eigen-solutions of the homogeneous B.-S. equation are obtained. Arguments are given that one of these belonging to discrete eigen-values of the coupling constant can be called a bound state, whereas the other continuous sets lack in some properties, attributed to bound states of the usual sense.

INTRODUCTION

The investigation of the scattering amplitude in the complex angular momentum plane has caused renewed interest in problems connected with ladder-like approximations in the field theoretical formalism. All these considerations finally reduce to treatments of B.-S. equations ¹⁾ in the ladder approximation. The known exact solutions ²⁾ here unfortunately restrict themselves to problems where scalar bosons are coupled by bosons. In the case of two fermions it has been even pointed out sometimes that there may be no solution at all in this approximation ³⁾ or at least only very pathological ones ⁴⁾.

The subject of this paper is to consider the properties of a two-fermion system. Fermions with equal masses are coupled by a massless vector particle by means of a γ_μ or $i\gamma_5\gamma_\mu$ interaction ^{*)}. The limit which will be necessary for a solution is that the total energy of the bound state $2E = 0$. Fortunately this is just the region where the properties of the B.-S. amplitude are most interesting with respect to the high energy behaviour at small momentum transfer in the crossed channel. Nevertheless, we do not want to discuss these consequences here, but merely try to give some rigorous solutions of the equations.

Starting with a decomposition of the two-fermion B.-S. amplitude into Dirac matrices in Section 1 we arrive for $E = 0$ at a system of integral equations in which only few quantities are coupled respectively in three different sets of equations. We treat in Section 2 one of these which consists only of one integral equation and one algebraic relation for two quantities. One other set, being one simple integral equation only, is dealt with in Appendix 2. In Chapter 3 the behaviour of the solutions is examined in co-ordinate space, whereas in 4 the transition from the B.-S. amplitudes to mass shell scattering amplitudes is made.

*) The first case is electrodynamics, the second one is to be understood as a model with the propagator $g_{\mu\nu}(k^2+ic)^{-1}$ of electrodynamics but with the vertex γ_μ replaced by $i\gamma_5\gamma_\mu$. It is therefore not the (unrenormalizable) axial vector theory.

2.

1. DECOMPOSITION OF THE B.-S. EQUATION FOR TWO FERMIONS

Picking out ladder contributions one gets for the fermion-fermion amplitude between a bound state b and the vacuum

$$\chi^{--}(x_1, x_2) = \langle 0 | T \{ \psi(x_1) \psi(x_2) \} | b \rangle \quad (1.1)$$

for equal masses in the case of an intermediate photon (1^-) the B.-S. equation of lowest order

$$(i\not{D}_1 - m)(i\not{D}_2 - m)\chi^{--} = -e^2 D_{\mu\nu} \gamma_1^\mu \gamma_2^\nu \chi^{--} \quad *) \quad (1.2)$$

where ∇_1 and ∇_2 are acting on different variables and spinor components and

$$D_{\mu\nu} = -g_{\mu\nu} D$$

$$D = i(2\pi)^{-4} \int d^4k (k^2 + i\epsilon)^{-1} e^{-ikx} = -(4\pi^2 x^2)^{-1}$$

Similarly for the fermion-antifermion amplitude

$$\chi^{+-}(x_1, x_2) = \langle 0 | T \{ \psi(x_1) \bar{\psi}(x_2) \} | b \rangle \quad (1.3)$$

*) $g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $ab = a_0 b_0 - \vec{a} \cdot \vec{b}$; $\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$

$$\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = -i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \not{A} = a_0 \gamma_0 - \vec{a} \cdot \vec{\gamma}$$

the equation reads

$$(i\not{D}_1 - m)\chi^{+-}(-i\not{D}_2 - m) = e^2 D_{\mu\nu} \gamma^\mu \chi^{+-} \gamma^\nu \quad (1.4)$$

In c.m. co-ordinates with a c.m. energy $2E$ and a relative distance $x = x_1 - x_2$ we have

$$\chi = e^{-i(t_1 + t_2)E} f(x) \quad (1.5)$$

and thus

$$(i\not{D} + \gamma_0 E - m) f^{--} (-i\not{D}^T + \gamma_0^T E - m) = e^2 D \gamma^\mu f^{--} \gamma_\mu^T \quad (1.2')$$

$$(i\not{D} + \gamma_0 E - m) f^{+-} (i\not{D} - \gamma_0 E - m) = -e^2 D \gamma^\mu f^{+-} \gamma_\mu \quad (1.4')$$

The difference of both being charge conjugation only, we arrive therefore - splitting off γ_5 - with

$$F = f^{--} C^{-1} \gamma_5 = f^{+-} \gamma_5 \quad (1.6)$$

where C is defined through

$$-\gamma^T = C^{-1} \gamma C$$

4.

at

$$(i\not{D} + \gamma_0 E - m) F (-i\not{D} + \gamma_0 E - m) = e^2 D \gamma^\mu F \gamma_\mu \quad (1.7)$$

In momentum space, we may formulate the homogeneous integral equation necessary for a bound state in terms of $\phi(p)$ defined by

$$F = \int d^4 p \phi(p) e^{-ipx} \quad (1.8)$$

namely

$$(\not{p} + \gamma_0 E - m) \phi(p) (-\not{p} + \gamma_0 E - m) = \frac{e^2 i}{(2\pi)^4} \int \frac{d^4 k}{(p-k)^2} \gamma^\mu \phi(k) \gamma_\mu \quad (1.9)$$

We now make the assumption that we are allowed to turn the integration over k_0 into the imaginary axis of the k_0 plane taking at the same time also p_0 imaginary. This can be done in the propagator, but for $\phi(k, k_0)$ we have to assume that there must not be any singularities in the first and third quadrant of the complex k_0 plane including infinity and in some region around $k_0 = 0$ ⁵⁾. Obviously we shall have to try afterwards if our solutions really satisfy these conditions. In real four-dimensional co-ordinates

$$P = (P_0, \vec{P}) = (ip_0, \vec{P})$$

$$\eta = (\eta, 0) = (iE, 0)$$

$$\tilde{\gamma} = (\tilde{\gamma}_0, \vec{\tilde{\gamma}}) = (i\gamma_0, \vec{\gamma})$$

$$\tilde{\not{P}} = \tilde{\gamma}_0 P_0 + \vec{\tilde{\gamma}} \cdot \vec{P}$$

$$P_\pm = P \pm \eta$$

(1.9) reads then

$$(\tilde{\not{P}}_+ + m) \phi(\tilde{\not{P}}_- - m) = \tilde{\gamma}_\mu I \phi \tilde{\gamma}_\mu \quad (1.10)$$

We have abbreviated with I the integral operation ($\lambda = \alpha \cdot (4\pi)^{-1}$)

$$I \phi = \frac{\lambda}{\pi^2} \int \frac{d^4 P'}{(P' - P)^2} \phi(P') \quad (1.11)$$

ϕ is an Hermitian matrix for real E, P_0 since from

$$\tilde{\not{P}}_\pm^* = -\tilde{\not{P}}_\mp^T$$

follows

$$(-\tilde{\not{P}}_+ - m) \phi^+(-\tilde{\not{P}}_- + m) = \tilde{\gamma}_\mu I \phi^+ \tilde{\gamma}_\mu$$

and comparing with (1.10)

$$\phi = \phi^+ \quad (1.12)$$

In order to simplify the notation we omit from now on the symbol $\sim$ from the $\tilde{\gamma}_\mu$ and write the decomposition of ϕ as

$$\phi = S + \gamma_5 \mathcal{P} + \gamma_5 V_\varrho + \gamma_5 \gamma_\varrho A_\varrho + \sigma_{\varrho\varrho'} T_{\sigma\varrho} \quad (1.13)$$

Because of (1.12) S and $T_{\alpha\beta}$ are real, V_ρ , A_ρ , and $\mathcal{P}$ are purely imaginary quantities. The right-hand side of (1.10) gives

$$\gamma^\mu \phi \gamma^\mu = -4S + 2V_\rho \gamma_\rho + 4\mathcal{P} \gamma_5 - 2A_\rho \gamma_5 \gamma_\rho$$

and is independent of the tensor part. This typical feature of the vector particle interaction will be essential for our solution. If the photon coupling is replaced by the $i\gamma_5 \gamma_\mu$ model, in the interaction energy appears $i\gamma_5 \gamma_\mu$ instead of γ_μ . Hence the right-hand side of (1.2') is to be replaced by

$$-e^2 D \gamma_5 \gamma^\mu f^{--} \gamma_\mu^\tau \gamma_5^\tau$$

With (1.6) and

$$\gamma_5^\tau = C^{-1} \gamma_5 C$$

$e^2 D \gamma^\mu F \gamma_\mu$ in (1.7) must be changed into

$$-e^2 D \gamma_5 \gamma^\mu F \gamma_\mu \gamma_5$$

Thus, after decomposing with (1.13) the only difference appears in the signs of the V_ρ and the A_ρ under the integral.

For the left-hand side we use the formulae of Appendix I yielding finally the sixteen tensor equations

$$-(P_+ P_-) S - 2i P_{+\mu} P_{-\nu} T_{\mu\nu} - m(P_{-\mu} - P_{+\mu}) V_\mu - m^2 S = -4IS \quad (1.14)$$

$$(P_+ P_-) \mathcal{P} + \varepsilon_{\alpha\beta\mu\nu} P_{+\alpha} P_{-\beta} T_{\mu\nu} - m(P_{-\mu} + P_{+\mu}) A_\mu - m^2 \mathcal{P} = 4I\mathcal{P} \quad (1.15)$$

$$(P_+ P_-) V_\varphi - P_{+\varphi} P_{-\mu} V_\mu - P_{-\varphi} P_{+\mu} V_\mu + i P_{+\mu} P_{-\nu} \varepsilon_{\mu\nu\varphi\sigma} A_\sigma -$$

$$-2im(P_{+\mu} + P_{-\mu}) T_{\mu\varphi} - m(P_{+\varphi} - P_{-\varphi}) S - m^2 V_\varphi = \pm 2I V_\varphi \quad (1.16)$$

$$-(P_+ P_-) A_\varphi + (P_{+\varphi} P_{-\mu} + P_{-\varphi} P_{+\mu}) A_\mu - i P_{+\mu} P_{-\nu} \varepsilon_{\mu\nu\varphi\sigma} V_\sigma -$$

$$-m(P_{+\alpha} - P_{-\alpha}) \varepsilon_{\mu\nu\varphi\alpha} T_{\mu\nu} + m(P_{+\varphi} + P_{-\varphi}) S - m^2 A_\varphi = \mp 2I A_\varphi \quad (1.17)$$

$$\begin{aligned} & -(P_+ P_-) T_{\sigma\varphi} + [(P_{+\varphi} P_{-\beta} - P_{-\varphi} P_{+\beta}) T_{\sigma\beta} - (P_{+\sigma} P_{-\beta} + P_{-\sigma} P_{+\beta}) T_{\varphi\beta}] + \\ & + \frac{i}{2} (P_{+\sigma} P_{-\varphi} - P_{+\varphi} P_{-\sigma}) S + \frac{1}{2} \varepsilon_{\mu\nu\sigma\varphi} P_{+\mu} P_{-\nu} S - \frac{im}{2} [(P_{+\sigma} + P_{-\sigma}) V_\varphi - \\ & - (P_{+\varphi} + P_{-\varphi}) V_\sigma] - \frac{m}{2} (P_{+\alpha} - P_{-\alpha}) A_\mu \varepsilon_{\alpha\mu\sigma\varphi} - m^2 T_{\sigma\varphi} = 0 \quad (1.18) \end{aligned}$$

In (1.16) and (1.17) the upper and lower signs of the right-hand side correspond to γ_μ and $i\gamma_5 \gamma_\mu$ interaction respectively. We shall be mainly interested in strongly bound systems ($\eta = 0$, $P_+ = P_- = P$) where these equations simplify greatly :

$$(P^2 + m^2) S = 4IS \quad (1.19)$$

$$(P^2 - m^2) V_\varphi - 2P_\varphi (PV) - 4im P_\mu T_{\mu\varphi} = \pm 2I V_\varphi \quad (1.20)$$

$$-(P^2 + m^2) T_{\sigma\varphi} + 2[P_\varphi P_\beta T_{\sigma\beta} - P_\sigma P_\beta T_{\varphi\beta}] - im(P_\sigma V_\varphi - P_\varphi V_\sigma) = 0 \quad (1.21)$$

$$(P^2 - m^2) \mathcal{P} - 2m(PA) = 4I\mathcal{P} \quad (1.22)$$

$$-(P^2 + m^2) A_\ell + 2P_\ell(PA) + 2m P_\ell \mathcal{P} = \mp 2I A_\ell \quad (1.23)$$

In this paper only (1.19), (1.20), and (1.21) are treated, since (1.22) and (1.23) are very hard to attack. Hence, the eigensolutions we find there are not all possible ones.

2. SOLUTIONS OF THE "VECTOR-TENSOR" EQUATIONS

The continuous spectrum of (1.19) is only the generalization of Goldstein's result ⁴⁾; it is treated in Appendix 2. We concentrate here on (1.20) and (1.21) which are more promising than (1.22) and (1.23) because of the algebraic character of (1.21). From the latter equation we have with $s = P^2$

$$P_\sigma T_{\sigma\ell} = \frac{-im}{s-m^2} [P_\ell(PV) - sV_\ell] \quad (2.1)$$

and

$$T_{\sigma\ell} = \frac{-im}{s-m^2} (P_\ell V_\sigma - P_\sigma V_\ell) \quad (2.2)$$

Using (2.1) in (1.20) we get

$$(s+m^2) V_\rho - 2 P_\rho P_\sigma V_\rho = \pm \frac{s-m^2}{s+m^2} 2I V_\rho \quad (2.3)$$

We are then dealing with a mathematical problem, similar to the one in classical electrodynamics : we have to solve a coupled system of equations for a "vector potential" V_ρ . In the case of electrodynamics the decoupling is performed making use of the freedom in gauge. Nevertheless we may divide also here V_ρ into a longitudinal and transversal part

$$V_\rho = V_\rho'' + V_\rho^\perp \quad (2.4)$$

with

$$V_\rho^\perp P_\rho = 0 \quad (2.5)$$

The integral operation in (2.3) mixes V_ρ'' and $V_\rho^\perp$. Hence an independent treatment is possible only if either

$$a) \quad V_\rho^\perp \equiv 0$$

or

$$b) \quad V_\rho'' \equiv 0$$

2a. LONGITUDINAL SOLUTION

From (2.3) follows that V_g obeys ($m = 1$)

$$(s+1) V_g = \mp 2I V_g \quad (2.6)$$

This is the same equation as (1.19), except that both signs of the coupling appear. It is shown in Appendix 2 that there is no solution (except the trivial one) for the coupling with a photon. The solution with a $i\gamma_5 \gamma_\mu$ coupling between fermions is given for each component of V by Eq. (A.2.22) of Appendix II with $\beta' = -2\lambda = -\frac{\lambda'}{2} < 0$, $\gamma = [(n+1)^2 - 2\lambda']^{1/2}$. It is therefore connected with the continuous spectrum $\lambda' > 0$. Thus, it is a solution of the homogeneous B.-S. equation which, nevertheless, lacks in the property of discreteness of a "real" bound state. Now the integration constants have to be adjusted in such a way that the parallelity condition is really satisfied. From (A.2.22) (π_n^e are defined by (A.2.5); $\vartheta_1, \vartheta_2, \vartheta_3$ are polar angles in the R_4 , $t = (s+1)^{-1}$)

$$\begin{aligned} (V_{\lambda, m, e})_g &= -\frac{t^2}{1-t} [a_- \omega_{n+}(t) - a_+ \omega_{n-}(t)] \pi_n^e(\omega \vartheta_1) (v_n(\vartheta_2, \vartheta_3))_g \\ &= S^{-\frac{n+3}{2}} (s+1)^{\frac{n-1}{2}} \left[a_- (s+1)^{-\frac{n}{2}} F_+((s+1)^{-1}) - a_+ (s+1)^{\frac{n}{2}} F_-((s+1)^{-1}) \right] \pi_n^e(v_n)_g \quad (2.7) \end{aligned}$$

where $F_{\pm}$ are the hypergeometric functions in (A.2.18) and

$$(v_n)_g = \sum_{e=0}^n \sum_{m=-e}^{+e} (b_{n, m, e})_g P_e^m(\omega \vartheta_2) e^{im\vartheta_3} \quad (2.8)$$

should be parallel to P_g .

As can be seen easily this is only possible for $n = 1$ since, e.g. in co-ordinate space, the "parallelity condition" means that

$$V_p = \partial_p W \quad (2.9)$$

where W is some "potential" depending on the four-dimensional distance only. There are no other possibilities because in a more general W the gradient can never be expressed by a vector (2.8). In momentum space we remain then with

$$V_p = P_p \cdot P^{-1}$$

and n replaced in the "radial factor" of (2.7) by one.

2b. TRANSVERSAL SOLUTION

If we take the second possibility b) in (2.3) and call $V_p^\perp$ simply V_p , we have then besides an integral equation ($m = 1$)

$$\frac{(s+1)^2}{s-1} V_p = \pm 2I V_p \quad (2.10)$$

a "Lorentz convention"

$$P_p V_p = 0 \quad (2.11)$$

Using the formula for the Green functions of the potential equation in four dimensions

$$\square_p \frac{1}{p^2} = -4\pi^2 \delta^4(P) \quad (2.12)$$

we obtain for each component X of V_ξ a differential equation

$$\square \left[\frac{(s+1)^2}{s-1} X \right] = 2 \lambda' X \quad (2.13)$$

$\lambda' = 4\lambda$ is to be taken bigger than zero for $i\gamma_5 \gamma_\mu$ coupling between fermions, smaller than zero for γ_μ coupling. The separation is possible in four-dimensional central co-ordinates $\sqrt{s}, \vartheta_1, \vartheta_2, \vartheta_3$

$$X = R_n(s) \Pi_n^e(\omega \vartheta_1) P_e^m(\omega \vartheta_2) e^{im \vartheta_3} \quad (2.14)$$

where Π_n^e is given by (A.2.5); the "radial equation" reads (cf. similar steps leading to this equation and the following ones in Appendix II)

$$\left\{ \frac{d}{ds} \left[s^2 \frac{d}{ds} \right] - \frac{n}{2} \left(\frac{n}{2} + 1 \right) \right\} \frac{(s+1)^2}{s-1} R_n(s) = \frac{\lambda' s}{2} R_n(s) \quad (2.15)$$

Reinsertion of (2.15) into (2.10) leads to the boundary conditions (which are the same as in the equation of Appendix II)

$$\begin{aligned} \lim_{s \rightarrow \infty} \left\{ s^{-\frac{n}{2}} \left[s \frac{d}{ds} + \left(\frac{n}{2} + 1 \right) \right] \frac{(s+1)^2}{s-1} R \right\} &= 0 \\ \lim_{s \rightarrow 0} \left\{ s^{\frac{n}{2}+1} \left[s \frac{d}{ds} - \frac{n}{2} \right] \frac{(s+1)^2}{s-1} R \right\} &= 0 \end{aligned} \quad (2.16)$$

It is useful to map the infinite interval onto $\overline{[0,1]}$ by

$$s = \frac{1}{t} - 1 \quad (2.17)$$

and to introduce a function

$$w = \frac{1-t}{t^2(2t-1)} R \quad (2.18)$$

The differential equation becomes

$$(1-t)^2 t^2 w'' + 2t(1-t^2) w' - \left[\frac{n}{2}(\frac{n}{2}+1) + \lambda'(1-t)(\frac{1}{2}-t) \right] w = 0 \quad (2.19)$$

and the boundary conditions (cf. (A.2.15), (A.2.16))

$$\lim_{t \rightarrow 0} \left\{ t^{\frac{n}{2}+1} (t w' - \frac{n}{2} w) \right\} = 0 \quad (2.20)$$

$$\lim_{t \rightarrow 1} \left\{ (1-t)^{\frac{n}{2}} \left[(1-t) w' + \left(\frac{n}{2} + 1 \right) w \right] \right\} = 0 \quad (2.21)$$

(2.19) is a differential equation of Fuchsian type with regular points at $0, 1, \infty$. Splitting off a factor $t^\alpha (1-t)^\beta$ we must therefore arrive at the hypergeometric equation. This is achieved by

$$w = t^{-\frac{1}{2}+\frac{n}{4}} (1-t)^{-\frac{n}{4}} F(t) \quad (2.22)$$

with

$$\mu = [2\lambda' + (n+1)^2]^{1/2} \quad (2.23)$$

and F an appropriate solution of an hypergeometric equation with parameters

$$\begin{aligned} a &= \frac{1}{2}(\mu - \kappa - n) \\ b &= \frac{1}{2}(\mu + \kappa - n) \\ c &= 1 + \mu \end{aligned} \quad (2.24)$$

The appearance of $\kappa = (1+4\lambda')^{1/2}$ as in the S equation (cf. Appendix 2) reflects the specific role of the point $\lambda' = -\frac{1}{4}$ ^{*)}. Taking the two independent solutions of the hypergeometric equation for $a, b, c, c-a, c-b$ non-integer, $F(a, b, c; t), t^{1-c} F(a+1-c, b+1-c, 2-c; t)$, the solutions of (2.19) are

$$\omega_{\pm} = t^{\frac{\pm\mu-1}{2}} (1-t)^{-\frac{n}{2}} F\left(\frac{\pm\mu-\kappa-n}{2}, \frac{\pm\mu+\kappa-n}{2}; 1\pm\mu; t\right) \quad (2.25)$$

Photon coupling ($\lambda' < 0$) may be considered first. From (2.23) then $\mu < n+1$ is real or purely imaginary for $2|\lambda'| > (n+1)^2$. Thus both ω_+ and ω_- of (2.25) obey the boundary condition (2.20). Since the hypergeometric function behaves as a constant in $t = 1$ [cf. ⁶⁾, p. 110, Eq. (12)]

$$\begin{aligned} F &= \Gamma(1\pm\mu)n! [\Gamma(\frac{\pm\mu-\kappa+n}{2}+1) \Gamma(\frac{\pm\mu+\kappa+n}{2}+1)]^{-1} + O(1-t) = \\ &= C_{\pm} \cdot n! + O(1-t) \end{aligned} \quad (2.26)$$

^{*)} This is similar in the boson case ²⁾.

Eq. (2.21) is fulfilled by the linear combination

$$\omega_n = c_- \omega_+(t) - c_+ \omega_-(t) \quad (2.27)$$

yielding a continuous spectrum.

For $if_5 \gamma_\mu$ coupling ($\lambda > 0$) from (2.23) $\mu > n+1$. Therefore ω_+ fulfils (2.20), whereas this limit applied to ω_- gives a divergent result. In $t = 1$ the hypergeometric function in ω_+ behaves as [cf. ⁶⁾, p. 110, Eq. (12)]

$$F \sim \text{const} + O(1-t) + (1-t)^{n+1} \log(1-t) [1 + O(1-t)]$$

so that (2.21) gives something different from zero.

One has to take different solutions if the hypergeometric functions show not only in $t = 1$, but also in $t = 0$ a logarithmic behaviour ^{*)}. Then (the trivial case $\lambda = 0$ is excluded)

$$c = \mu + 1 = \tilde{q} + 3 + n \quad \tilde{q} = 0, 1, 2, \dots$$

Besides ω_+ which obeys (2.20) but not (2.21), as has been shown already, the second solution is now [Ref. ⁶⁾, p. 75, Eqs. (5), (8)]

$$\omega_- = t^{\frac{n+\tilde{q}+1}{2}} (1-t)^{\frac{n}{2}+1} F\left(n+2+\frac{\tilde{q}-k}{2}, n+2+\frac{\tilde{q}+k}{2}; n+2; 1-t\right)$$

which fulfils (2.21), but the hypergeometric function behaves as $t^{-\tilde{q}-2-n}$ in $t = 0$ [Ref. ⁶⁾, p. 110, Eq. (14)] which makes (2.20) impossible. There remain only two possible degenerate cases for the solutions, because if b,

*) Logarithmic behaviour in infinity does not change the considerations in the general case.

also $c-a$ is an integer; a and $c-b$ are coupled in the same way. In the first of these cases

$$\mu + \kappa = n + 2\tilde{q} + 2 \quad \tilde{q} = 0, 1, 2, \dots$$

the first solution is again ω_+ from which we already know that it does not obey (2.21) whereas the second one which contains a polynomial [Ref. 6), p.71, Eq. (6)]

$$\omega_- = t^{\frac{\mu-1}{2}} (1-t)^{\frac{n}{2}+1} F(-\tilde{q}, -\tilde{q}+\kappa; -2\tilde{q}+\kappa-n-1; t)$$

does not behave properly in $t = 0$.

We get discrete eigensolutions in the last remaining case, if a becomes an integer number

$$\mu - \kappa = -2q + n \quad (2.28)$$

Since $\kappa+n > \mu$, q defined as in (2.28) assumes only non-negative integer values. Two solutions are now [Ref. 6), p. 71, Eq. (5)]

$$\omega_+ = t^{-\frac{1}{2}+\frac{\mu}{2}} (1-t)^{-\frac{n}{2}} F(-q, -q+\kappa; -2q+\kappa+n+1; t) \quad (2.29)$$

$$\omega_- = t^{-\frac{1}{2}-\frac{\mu}{2}} (1-t)^{\frac{n}{2}+1} F(q+1, q+1-\kappa; 1+2q-\kappa-n; t) \quad (2.30)$$

(2.30) has again the wrong behaviour in $t = 0$ but the solution (2.29) is a useful one not only in $t = 0$. With

$$F(a, b; c; z) = (1-z)^{-a} F(a, c-b, c; z(1-z)^{-1}) \quad (2.31)$$

the hypergeometric function in (2.29) can be transformed into another polynomial

$$F = (1-t)^q F\left(-q, -q+n+1; -2q+\kappa+n+1; \frac{t}{1-t}\right) \quad (2.32)$$

If $q \leq n$ this behaves as a constant in $t = 1$ - in contradiction to (2.21).

If, however,

$$q \geq n+1 \quad (2.33)$$

then $F \sim O(1-t)^{n+1}$ or $\omega_+ \sim O(1-t)^{\frac{n}{2}+1}$ and thus (2.20) is fulfilled. (2.29) with (2.28) and (2.33) give eigenfunctions and eigenvalues. From (2.28) we have more explicitly

$$\begin{aligned} \lambda_{1/2} &= 3q(q-n) + n\left(n+\frac{1}{2}\right) + \sqrt{D} \\ D &= 8q^2(q-n)^2 + q(q-n)(6n^2+4n+1) + \\ &\quad + \left[n\left(n+\frac{1}{2}\right)\right]^2 \end{aligned} \quad (2.34)$$

which is a two parametric manifold in q and n (see Fig. 1) with

$$q-1 \geq n \geq 0$$

Eigenvalues and eigenfunctions are most simple for the lowest $q = n+1$

$$\omega_{n+1,n} = t^{\frac{3n}{2}+2} (1-t)^{\frac{n}{2}+1} \quad (2.35)$$

$$\lambda^1 = (2n+3)(2n+4) \quad (2.36)$$

Orthogonality relations for fixed n and different q can be stated easily with the aid of the differential equation (2.19) because $t^2(\omega_{q2} \omega'_{q1} - \omega_{q1} \omega'_{q2})$ vanishes at the boundaries of $[0,1]$. This is, however, a more or less accidental effect, since the boundary conditions (2.19) and (2.21) are obviously much weaker. Thus the problem as formulated by (2.19), (2.20), (2.21) is not a Sturm-Liouville problem, but it, nevertheless, possesses Sturm-Liouville type eigenfunctions.

One has with a norm N_q^2

$$\int_0^1 \frac{\left(\frac{1}{2}-t\right)}{1-t} \omega_{q1} \omega_{q2} dt = \delta_{q1q2} N_{q1}^2 \quad (2.37)$$

There is no difficulty in carrying out the integration over a product of polynomials which result from using (2.29) together with (2.32). Compatibility with the "Lorentz convention" (2.11) can be achieved by putting $V_0 = 0$ and choosing such a

$$\vec{v}_e = \sum_{m=-e}^{+e} b_{m,e} y_{e,m}(\vartheta_1, \vartheta_3) \quad (2.38)$$

that

$$\begin{aligned} \vec{V}_{q,n,e} &= - \frac{t^2(2t-1)}{1-t} \omega_{q,n}(t) \pi_n^e(\cos \vartheta_1) \vec{v}_e = \\ &= (s-1) s^{q-\frac{n}{2}-1} (s+1)^{-\frac{n-n+3}{2}-q} F\left(-q, -q+n+1, -2q+n+1; \frac{1}{s}\right) \times (2.39) \\ &\quad \times \pi_n^e \vec{v}_e \end{aligned}$$

[cf. (2.14), (2.17), (2.18) and (2.29)] obeys (2.11).

Using, for convenience, spherical functions

$$M_{\alpha\beta\gamma} = |\vec{r}|^{\ell+1} \partial_1^\alpha \partial_2^\beta \partial_3^\gamma |\vec{r}|^{-1}$$

with $\alpha+\beta+\gamma = \ell$ instead of spherical harmonics, one can prove for every $\ell \geq 1$ that there is a linear combination which fulfils (2.5). For lowest ℓ one has (with free parameters a, b, c and unit vector components e_i)

$$\vec{V}_1 = (a e_2 + b e_3, -a e_1 + c e_3, -c e_2 - b e_1)$$

$$\vec{V}_2 = (a e_2 e_3, b e_1 e_3, -(a+b) e_1 e_2)$$

$$\vec{V}_3 = (a M_{13} + b M_{32}, -a M_{13} + c M_{31}, -c M_{23} - b M_{12})$$

$$M_{ij} = e_i - 5 e_j^2 e_i$$

⋮

We may try to find also eigenstates of "total angular momentum". Well-known formulae can be used in co-ordinate space for the "solenoidal gauge" of multipole fields [cf. e.g. Rose ⁷⁾] which just correspond to solutions of the form (2.39) obeying our "Lorentz condition" (2.11) in co-ordinate space ($V_0 = 0$)

$$\vec{\partial} \vec{V} = 0 \quad (2.40)$$

By analogy between $\vec{V}$ and the vector potential of electrodynamics one finds easily that only the "magnetic multipole potential" with $\ell = L$ (total angular momentum) and parity $(-)^L$

$$\vec{V}_{L,q,m} = \xi_{L,q,m}(x_0, |\vec{r}|) \vec{Y}_{L,L}^m(\vartheta_1, \vartheta_2)$$

is consistent with (2.40). The "electrical multipole part" has to be excluded here because our radial function ξ has not the simple properties of spherical Bessel functions.

Besides, (2.39) considered as a function of p_0 has in fact the necessary properties for turning the integration path of p_0 in the transformation of (1.9). There are branch cuts or poles only at $p_0 = \pm \sqrt{\vec{p}^2}$, $\pm \sqrt{1+\vec{p}^2}$ and for $p_0 \gg 0$ the solution goes as $p_0^{-(\mu+1)}$ (similar results hold for the continuous sets).

In principle one could start now the development of a perturbation calculation for energies $E \neq 0$ since (1.14) - (1.18) seem to be too difficult for an exact solution. We will only sketch the way. By inspection of (1.16) and (1.18) one finds that corrections to V, T can be only of the order η^2 because S, P, V are for our solution of the order η . The first order corrections in $S = \eta S^{(1)} + O(\eta^2)$, T etc. obey integral equations which are inhomogeneous in the zero order V, T . With $((q, n) = \beta)$

$$\begin{aligned}\lambda_\beta &= \lambda_\beta^{(0)} + \eta^2 \lambda_\beta^{(2)} + \dots \\ V_{\beta, \mu} &= V_{\beta, \mu}^{(0)} + \eta^2 V_{\beta, \mu}^{(2)} + \dots \\ \Omega_{\mu\nu} &= \frac{(s+1)^2}{s-1} \delta_{\mu\nu} - \frac{2(s+1)}{s-1} P_\mu P_\nu \\ I &= \lambda \int\end{aligned}$$

one obtains in lowest order the equations

$$\begin{aligned}(\Omega - 2\lambda_\beta^{(0)} \int) V_\beta^{(0)} &= 0 \\ (\Omega - 2\lambda_\beta^{(0)} \int) V_\beta^{(2)} &= \Lambda_\beta + 2\lambda_\beta^{(2)} \int V_\beta^{(0)}\end{aligned}$$

where Λ depends on $V^{(0)}$, $S^{(1)}$, $P^{(1)}$ and $A^{(1)}$. The completeness property of the orthogonal set $V_{n,q}^{(0)}$ allows the definition of coefficients $a_{\beta,\beta'}$, $b_{\beta,\beta'}$:

$$V_{\beta}^{(2)} = \sum_{\beta'} a_{\beta\beta'} V_{\beta'}^{(0)}, \quad \Lambda_{\beta} = \sum_{\beta'} b_{\beta\beta'} V_{\beta'}^0 = \sum_{\beta'} \frac{b_{\beta\beta'}}{2\lambda_{\beta'}} \Omega V_{\beta'}^{(0)}$$

connected by simple algebraic relations

$$2(\lambda_{\beta'}^{(0)} - \lambda_{\beta}^{(0)}) a_{\beta\beta'} = b_{\beta\beta'} + 2\lambda_{\beta}^{(2)} \delta_{\beta\beta'}$$

which have the solutions

$$\lambda_{\beta}^{(2)} = - \frac{b_{\beta\beta'}}{2}$$

$$a_{\beta\beta'} = \frac{b_{\beta\beta'}}{2(\lambda_{\beta'}^{(0)} - \lambda_{\beta}^{(0)})}$$

There are, however, no simple statements possible concerning the $b_{\beta,\beta'}$: since the integral equations for the first order corrections have to be solved with big λ an iteration procedure seems to be rather hopeless. Hence we do not intend to write down the complicated equations here in detail.

Application of (A.4.4) leads for the asymptotic case of large space-time distances r to

$$\lim_{n \rightarrow \infty} V_n = 2(-1)^{q-k-n} \Gamma\left(\frac{n}{2}\right)^{q-1+\frac{n-n}{2}} e^{-r} \quad (3.5)$$

If $r = |X|$ real (space-like distances) we thus arrive at a familiar result : the solution is an exponential decreasing function times some positive power of r , which here is determined also by the quantum numbers q, n . Due to the independence of k every term in the polynomial F of (3.2) has the same limit. For time-like distances (r purely imaginary) the wave function is oscillating and diverging. Nevertheless, the norm is finite. This is obvious because it is possible to turn the time integration in the imaginary axis and to perform then the integration in terms of (2.39) in momentum space.

The investigation of the behaviour in co-ordinate space is more difficult for the continuous sets (2.7) and (2.27) because there is an infinite sum of terms (3.3). Nevertheless, the asymptotic behaviour for big r can be obtained in both cases. From the first term in (2.7) one gets for the radial part ($n = 1$)

$$R(s) = s^{-2} (1+s)^{-\frac{q}{2}} F_+((s+1)^{-1})$$

using

$$F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z)$$

the integral

$$V(r) = 2n^{\frac{q}{2}} \int_0^{\infty} \frac{d\rho \rho^2}{(\rho^2 + r^2)^{\frac{q}{2} + \frac{q}{2}}} F\left(\frac{q}{2} + 1, \frac{q}{2} + 2; q + 1; \frac{r^2}{r^2 + \rho^2}\right) d\rho \quad (3.6)$$

or with (3.4)

$$V(r) = 2r^{\gamma} \sum_{\nu=0}^{\infty} a_{\nu} \left(\frac{r}{2}\right)^{2\nu} T_{3,2,\gamma/2+\nu+1}$$

Here a_{ν} are the coefficients of a hypergeometric series

$$F\left(\gamma/2+1, \gamma/2+2; \gamma+1; r\right) = \sum_{\nu=0}^{\infty} a_{\nu} r^{\nu} \quad (3.7)$$

We know the asymptotic behaviour of each term of the infinite sum, namely (A.4.4). If one inserts the latter equation the series can be summed up again yielding as a factor the hypergeometric function (3.7) whose asymptotic behaviour is known to be

$$\lim_{r \rightarrow \infty} F \propto r^{-1-\frac{\gamma}{2}}$$

and thus

$$\lim_{r \rightarrow \infty} V(r) \propto r^{-3/2} e^{-r} \quad (3.8)$$

Similarly for the solution (2.27), the first term of the "radial part" gives

$$V(r) = \sum_{k=0}^{\infty} b_k r^{2k+\mu+1} \left[T_{3-n, n+1, \frac{\mu-n+1}{2}+k} - r^2 T_{1-n, n+1, \frac{\mu-n+1}{2}+k} \right]$$

where the b_k are the constant coefficients of the hypergeometric series given by the parameters a, b, c as written in (2.25). Insertion of (A.4.4) and

summing up the hypergeometric series again leads with $(a \leq b)$

$$\lim_{z \rightarrow \infty} F(a, b, c; z) \propto z^{-a}$$

to

$$\lim_{r \rightarrow \infty} V \propto r^{\frac{\kappa}{2}+1} e^{-r} \quad (3.9)$$

As it is impossible to use a similar procedure in the case of small space-time distances r , one could try to compute the solution directly in co-ordinate space and choose appropriate linear combinations, so that the asymptotic behaviour for the two continuous sets is (3.8) or (3.9) respectively. This has been done by Goldstein ⁴⁾ for the solution (2.7) for $n = 0$. Since the solutions in the general case $n \neq 0$ are modified Bessel functions of index $\pm \gamma$ times r^{-1} , the combination known as modified Bessel function of second kind has the behaviour (3.8). Thus one arrives at a singular behaviour at the origin

$$\lim_{r \rightarrow \infty} V = r^{-\gamma-1} \quad (3.10)$$

A similar procedure unfortunately leads for (2.10) in co-ordinate space to a fourth order differential equation with four solutions which behave at $r = 0$ as $r^{-1 \pm \nu}$, r^{n+2} , r^{-n} .

Mandelstam ⁹⁾ has shown that solutions which are more singular than r^{-1} in the origin must be excluded because of the requirement of normalizability of the B.-S. wave functions. Whereas this condition leads to a rejection of the continuous "longitudinal" solution, it cannot be used directly for the continuous solution of (2.10), because it seems to be very difficult to find

the proper linear combination of the four solutions which behave asymptotically as required by (3.9). Nevertheless, one may expect that at least one of the singular solutions is contained in this linear combination. Then this solution would have to be rejected on similar grounds as the first continuous set. An argument which is much simpler will be given, however, in the next chapter.

4. MASS SHELL SCATTERING AMPLITUDES

The relation between the B.-S. amplitude considered so far and the matrix element of the decay process

bound state $\rightarrow$ fermion, antifermion

with momenta p_1, p_2 can be found from the reduction formula

$$\langle \text{pair} | b \rangle = \int d^4x d^4x' \bar{f}^{p_1}(x) (i \not{\partial} - m) \langle 0 | T [\psi(x), \bar{\psi}(x')] | b \rangle \times (-i \not{\partial}' - m) \times g^{p_2}(x') \quad (4.1)$$

f and g contain the spinors u and v of the fermion and antifermion respectively ($\bar{u}u = 2m$)

$$f^{p_1}(x) = [(2\pi)^3 2\omega_1]^{-\frac{1}{2}} e^{-ip_1 x} u_{p_1}, \quad g^{p_2}(x) = [(2\pi)^3 2\omega_2]^{-\frac{1}{2}} e^{ip_2 x} v_{p_2}$$

Remembering (1.1), (1.6) and (1.3) we arrive at

$$\langle \text{pair} | t \rangle = - \frac{i(2\pi)^4 d^4(p_1 + p_2 - 2E) T}{\sqrt{16\omega_1 \omega_2 E^2}} \quad (4.2)$$

$$\begin{aligned} T/E = & 4\pi i \bar{u}(\vec{p}_1 E) \left[(-\vec{\not{p}} + \gamma_0 E - m) \phi_u(\vec{p}_1, p_0) + \right. \\ & \left. \times (-\vec{\not{p}} - \gamma_0 E + m) \right] \Big|_{\substack{\vec{p}^2 = E^2 - m^2 \\ p_0 = 0}} \gamma_5 v(-\vec{p}_1 E) \end{aligned} \quad (4.3)$$

Between the spinors we find essentially the left-hand side of (1.10). We introduce our solutions ($E=0$) in (4.3) and then investigate the limit $p_0 = 0$, $\vec{p}^2 = -m^2 = -1$.

We have to deal with something like a "scattering" situation; from the formulation of the problem in terms of a homogeneous B.-S. equation there is no incoming but only an "outgoing" state (with momentum purely imaginary). Hence, the scattering amplitude which is essentially the quotient of the corresponding wave functions, can be expected to diverge for a "real" bound state. The discrete transversal solution (2.39) may be considered first. With an amplitude, which contains V_p obeying (2.5) and the tensor part (2.2), the expression in square brackets of Eq. (4.3) becomes ($m=1$)

$$N = (s+1)^{\frac{1}{2}} (s-1)^{-\frac{1}{2}} = s^{q-\frac{n}{2}-1} (s+1)^{-\frac{n-n-1}{2}-q} F\left(\frac{1}{s}\right) \Pi_n^e \quad (4.4)$$

Here F is the degenerated hypergeometric function of (2.39). For $p_0 = 0$, $\vec{p}^2 = -1$ because of

$$\Pi_n^e(z) = \begin{cases} \begin{pmatrix} -(e+1) \\ \frac{n-e}{2} \end{pmatrix} & n-e \text{ even} \\ q(z) & n-e \text{ odd} \end{cases}$$

we have

$$\lim_{p_0 \rightarrow 0, \vec{p}^2 = -1} N \propto p_0^{-(\mu - n - 1 + 2q)} \quad (4.5)$$

if $n - \ell$ even

$$\lim_{p_0 \rightarrow 0, \vec{p}^2 = -1} N \propto p_0^{-(\mu - n - 2 + 2q)}$$

if $n - \ell$ odd. At general quantum numbers q, n μ [cf. (2.23)] is non-integer. The matrix element diverges therefore as a non-integer negative power of p_0 ; if $q = n+1$ ($\mu = 3n+5$), as an integer power. In the case of the continuous solution (2.7) [the tensor part vanishes because of (2.2)] simple Dirac algebra gives

$$N = (\vec{p}^2 + m^2) \vec{\not{p}} \omega(t) \quad (4.6)$$

where

$$\omega(t) = -t^2(1-t) [a_- \omega_+(t) - a_+ \omega_-(t)] \quad (4.7)$$

Since for $m = 1$

$$S+1 = \frac{1}{t} = p_0^2 \rightarrow 0 \quad (4.8)$$

the matrix element is determined [cf. (4.6) and (4.8)] by the limit of $t^{-1}\omega$ for t going to infinity.

From (A.2.18) for $n = 1$ and Ref. ⁶⁾, p. 109, Eq. (7) we find

$$\lim_{t \rightarrow \infty} \omega_{\pm}(t) = \frac{\Gamma(1 \pm \gamma) e^{-\frac{i\pi}{2}(1 \mp \gamma)}}{\Gamma(\pm \frac{\gamma}{2}) \Gamma(2 \pm \frac{\gamma}{2})}$$

and after manipulations with the relation between Γ functions

$$\Gamma(z) \Gamma(1-z) = \pi \sin^{-1} \pi z$$

finally

$$\lim_{t \rightarrow \infty} N = \frac{4}{\pi i} \cancel{\gamma} \sin \frac{\pi \gamma}{2} \times (1 - \frac{\gamma^2}{4})^{-1} \quad (4.9)$$

Obviously, the finite result (4.9) is in strict contradiction to what has been stated above for a bound state of usual type. If in the first part of (4.4) the continuous "transversal" set (2.27) with (2.14), (2.18) is introduced one finds that the matrix element behaves as

$$\lim_{s \rightarrow -1} (s+1)^2 (s-1) V = \lim_{p_0 \rightarrow 0} p_0^{1-\kappa} = 0$$

Thus there is again no divergent behaviour.

SUMMARY

The B.-S. equation for two fermions can be split into the three mutually independent systems of equations (1.19), (1.20) and (1.21), (1.22) and (1.23) if the total energy of the bound state is zero. The first two of these terms have been treated in the case of an γ_μ or $i\gamma_5 \gamma_\mu$ interaction with zero mass propagator in the ladder approximation.

The general solution of (1.19) has been found for both couplings to be admitting continuous values of the coupling constant. It is the generalization of an angular independent solution which has been discussed by Goldstein⁴⁾ some years ago.

In (1.20), (1.21) we looked for solutions which obey certain additional conditions so that the system of equations can be decoupled. The "longitudinal" solution is for vector coupling only the trivial one, for $i\gamma_5 \gamma_\mu$ coupling it is essentially the same as the one of (1.19). The "transversal" solution turns out to be again a continuous one if the fermions are coupled by a photon, a discrete set, however, is obtained in the $i\gamma_5 \gamma_\mu$ model.

A common feature of all continuous solutions is their highly singular behaviour at the origin of the co-ordinate space and the fact that their mass shell limits are finite or zero instead of diverging as one would expect for a "real" bound state. The discrete set of solutions is regular at the origin of co-ordinate space and leads to a divergent matrix element on the mass shell. But the model in which we obtained this discrete set, unfortunately does not represent directly a physical problem. However, it seems close enough to reality so that there is now a strong argument that discrete solutions of the B.-S. equation exist also in the case of quantum electrodynamics. In fact, our solutions are not all possible ones of the two fermion problem with vector interaction, since the

solutions of (1.22), (1.23) have not yet been investigated. There the discrete solutions of the problem with photon coupling may be contained. Moreover, even in (1.20), (1.21) only solutions restricted by additional conditions have been considered.

Among the physical consequences of our solution we only want to point out that simply their existence is another argument that there can be really a ladder mechanism to form a "vector meson" as a strongly coupled two fermion bound state ¹⁰⁾. The old idea, to consider bosons as two fermion bound states gains therefore some new "practical" support.

The author wishes to thank Dr. G. Furlan for very useful discussions and reading the manuscript. He is indebted to Professor S. Fubini for suggesting this field of investigation.

APPENDIX 1

Some relations are given which have been used in deriving (1.14) to (1.18). Because of

$$[\gamma_\mu, \gamma_\nu] = -2\delta_{\mu\nu} \quad \sigma_{\sigma\rho} = \frac{1}{2i} (\gamma_\sigma \gamma_\rho - \gamma_\rho \gamma_\sigma)$$

one has

$$\not{P}_+ \not{P}_- = -P_+ P_- + i P_{+\sigma} P_{-\rho} \sigma_{\sigma\rho} \quad (\text{A.1.1})$$

$$\not{P}_+ \gamma_5 \not{P}_- = P_+ P_- \gamma_5 + \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} P_{+\mu} P_{-\nu} \sigma_{\sigma\rho} \quad (\text{A.1.2})$$

$$\begin{aligned} \not{P}_+ \gamma_\mu \not{P}_- &= [-P_{+\rho} P_{-\mu} + P_+ P_- \delta_{\mu\rho} - P_{+\mu} P_{-\rho}] \gamma_\rho - \\ &\quad - i P_{+\sigma} P_{-\nu} \epsilon_{\sigma\nu\rho\mu} \gamma_5 \gamma_\rho \end{aligned} \quad (\text{A.1.3})$$

$$\begin{aligned} \not{P}_+ \gamma_5 \gamma_\mu \not{P}_- &= [-P_{+\rho} P_{-\mu} + P_+ P_- \delta_{\mu\rho} - P_{+\mu} P_{-\rho}] \gamma_5 \gamma_\rho + \\ &\quad + i P_{+\sigma} P_{-\nu} \epsilon_{\sigma\nu\rho\mu} \gamma_\rho \end{aligned} \quad (\text{A.1.4})$$

$$\begin{aligned} T_{\mu\nu} \not{P}_+ \sigma_{\mu\nu} \not{P}_- &= \left\{ -i (P_{+\mu} P_{-\nu} - P_{+\nu} P_{-\mu}) + \epsilon_{\alpha\beta\mu\nu} P_{+\alpha} P_{-\beta} \gamma_5 + \right. \\ &\quad \left. + [-P_+ P_- \delta_{\rho\nu} \delta_{\sigma\mu} + \delta_{\sigma\mu} (P_{+\rho} P_{-\nu} + P_{-\rho} P_{+\nu}) - \delta_{\rho\mu} (P_{+\sigma} P_{-\nu} + P_{-\sigma} P_{+\nu})] \right\} T_{\mu\nu} \quad (\text{A.1.5}) \end{aligned}$$

$$\not{P}_+ - \not{P}_- = (P_{+p} - P_{-p}) \gamma_p \quad (\text{A.1.6})$$

$$(\not{P}_+ + \not{P}_-) \gamma_5 = -(P_{+p} + P_{-p}) \gamma_5 \gamma_p \quad (\text{A.1.7})$$

$$\not{P}_+ \gamma_\mu - \gamma_\mu \not{P}_- = (P_{-\mu} - P_{+\mu}) + i(P_{+5} + P_{-5}) \gamma_5 \sigma_\mu \quad (\text{A.1.8})$$

$$\not{P}_+ \gamma_5 \gamma_\mu - \gamma_5 \gamma_\mu \not{P}_- = (P_{+\mu} + P_{-\mu}) \gamma_5 + \frac{1}{2} \epsilon_{\alpha\mu\sigma\rho} (P_{+\alpha} - P_{-\alpha}) \gamma_5 \sigma_\rho \quad (\text{A.1.9})$$

$$\begin{aligned} \not{P}_+ \sigma_{\mu\nu} - \sigma_{\mu\nu} \not{P}_- &= 2i (P_{+\mu} + P_{-\mu}) \gamma_\nu + (P_{+\alpha} - P_{-\alpha}) \\ &\quad \times \epsilon_{\mu\nu\rho\alpha} \gamma_5 \gamma_\rho \end{aligned} \quad (\text{A.1.10})$$

APPENDIX 2

We consider the integral equation

$$(s+1)X = - \frac{\beta}{\pi^2} \int d^4 p' (p'-p)^{-2} X(p') \quad (\text{A.2.1})$$

which can be transformed immediately into a differential equation with the aid of

$$\square_P \frac{1}{p^2} = -4\pi^2 \delta^4(p) \quad (\text{A.2.2})$$

where $\square_P$ is the Laplace operator in the four-dimensional space of momentum P . More explicitly the latter may be written as ($P^2 = s$)

$$\square = 4 \left[s \frac{d^2}{ds^2} + 2 \frac{d}{ds} \right] + \frac{\mathcal{L}}{s} \quad (\text{A.2.3})$$

$$\begin{aligned} \mathcal{L} = & \frac{\partial^2}{\partial \vartheta_1^2} + 2 \cot \vartheta_1 \vartheta_1 \frac{\partial}{\partial \vartheta_1} + \frac{1}{\sin^2 \vartheta_1} \left\{ \frac{\partial^2}{\partial \vartheta_2^2} + \right. \\ & \left. + \cot \vartheta_2 \vartheta_2 \frac{\partial}{\partial \vartheta_2} + \frac{1}{\sin^2 \vartheta_2} \frac{\partial^2}{\partial \vartheta_3^2} \right\} \end{aligned} \quad (\text{A.2.4})$$

We introduce four-dimensional spherical harmonics (omitting normalization factors)

$$\begin{aligned} Y_{n,e,m} &= \sin^e \vartheta_1 C_{n-e}^{e+1}(\cos \vartheta_1) P_e^m(\cos \vartheta_2) e^{im \vartheta_3} = \\ &= \Pi_n^e(\cos \vartheta_1) P_e^m(\cos \vartheta_2) e^{im \vartheta_3} \quad n \geq e \geq |m| \end{aligned} \quad (\text{A.2.5})$$

which obey

$$\mathcal{L} y_{n,\ell,m} = -n(n+2) y_{n,\ell,m} \quad (\text{A.2.6})$$

Inserting

$$X = R(s) y_{n,\ell,m} \quad (\text{A.2.7})$$

into (A.2.1) after application of (A.2.3) and using (A.2.2), (A.2.6) a radial equation ($\beta' = 4\beta$) results

$$\left\{ \frac{d}{ds} \left[s^2 \frac{d}{ds} \right] - \frac{n}{2} \left(\frac{n}{2} + 1 \right) \right\} (s+1) R = \beta' s R \quad (\text{A.2.8})$$

We arrive at the boundary conditions by reinserting $\beta' \cdot s \cdot R \cdot Y$ with (A.2.7), (A.2.8) into (A.2.1). Integration by parts in $s' = P'^2$ and making use of (A.2.2), (A.2.3) leads to

$$\int d\Omega' y(\Omega') \left\{ s'^2 \frac{d}{ds'} [(s'+1) R(s')] \right\} \frac{1}{(P'-P)^2} - s'^2 (s'+1) R(s') \frac{d}{ds'} \left(\frac{1}{(P'-P)^2} \right) \bigg|_{s'=0}^{s'=\infty} = 0 \quad (\text{A.2.9})$$

Now the propagator may be written (ϑ is the angle between P' and P , $\theta(x)$ the step function)

$$\frac{1}{(P'-P)^2} = \frac{\theta(s'-s)}{s'} \left(1 + \frac{s}{s'} - 2 \sqrt{\frac{s}{s'}} \cos \vartheta \right)^{-1} + \frac{\theta(s-s')}{s} \left(1 + \frac{s'}{s} - 2 \sqrt{\frac{s'}{s}} \cos \vartheta \right)^{-1}$$

Then we use the definition of Gegenbauer polynomials for $x = \sqrt{s/s'}$ or $\sqrt{s'/s}$ in the two terms respectively ($x \leq 1$)

$$(1+x^2-2xy)^{-\nu} = \sum_{\mu=0}^{\infty} C_{\mu}^{\nu}(y) x^{\mu}$$

The addition theorem of Gegenbauer polynomials [Ref. ⁶⁾, p. 177, Eq. (19)] together with the same theorem for Legendre polynomials states that (the coefficients $a_{n,\ell,m}$ are not of interest in this connection)

$$C_n^1(\cos \theta) = \sum_{\ell=0}^n \sum_{m=-\ell}^{+\ell} a_{n,m,\ell} Y_{\ell,m}(\Omega) Y_{n,\ell,m}^*(\Omega') \quad (\text{A.2.10})$$

The integration over angles in (A.2.9) projects out the quantum numbers n, ℓ, m ; thus we are allowed to replace $(P' \cdot P)^{-2}$ simply by

$$\frac{\theta(s'-s)}{s'} \left(\frac{s}{s'}\right)^{\frac{n}{2}} + \frac{\theta(s-s')}{s} \left(\frac{s'}{s}\right)^{\frac{n}{2}}$$

(A.2.9) becomes

$$\begin{aligned} \lim_{s \rightarrow \infty} \left\{ s^{-\frac{n}{2}} \left[s \frac{d}{ds} + \frac{n}{2} + 1 \right] (s+1) R \right\} &= 0 \\ \lim_{s \rightarrow 0} \left\{ s^{\frac{n}{2}+1} \left[s \frac{d}{ds} - \frac{n}{2} \right] (s+1) R \right\} &= 0 \end{aligned} \quad (\text{A.2.11})$$

If the infinite interval is mapped onto $[0,1]$ by

$$s = \frac{1}{t} - 1 \quad (\text{A.2.12})$$

and a new dependent variable ω is defined by

$$R = - \frac{t^2}{1-t} \omega \quad (\text{A.2.13})$$

we arrive at the eigenvalue problem

$$t^2(1-t)^2 \omega'' + (1-t)^2 2t \omega' - [\beta'(1-t) + \frac{n}{2}(n+1)] \omega = 0 \quad (\text{A.2.14})$$

$$\lim_{t \rightarrow 0} \left\{ t^{\frac{n}{2}+1} (t \omega' - \frac{n}{2} \omega) \right\} = 0 \quad (\text{A.2.15})$$

$$\lim_{t \rightarrow 1} \left\{ (1-t)^{\frac{n}{2}} [(1-t) \omega' + (\frac{n}{2}+1) \omega] \right\} = 0 \quad (\text{A.2.16})$$

The differential equation can be reduced to a hypergeometric equation. With

$$\gamma = \sqrt{4\beta' + (n+1)^2} \quad (\text{A.2.17})$$

two solutions are (γ non-integer)

$$\omega_{\pm} = t^{-\frac{1}{2} \pm \frac{\gamma}{2}} (1-t)^{-\frac{n}{2}} F \left(\frac{\pm \gamma - n - 1}{2}, \frac{\pm \gamma - n + 1}{2}; 1 \pm \gamma; t \right) \quad (\text{A.2.18})$$

If $\gamma = 2m+1$ odd integer ($m > \frac{n}{2}$, n odd) instead of ω_- the second solution is [Ref. 6), p. 75, Eq. (8)]

$$\omega_- = (1-t)^{\frac{n}{2}+1} t^m F\left(m + \frac{n}{2} + 1, m + \frac{n}{2} + 2; n+2; 1-t\right) \quad (\text{A.2.19})$$

As third possibility $\gamma = 2m+n+3$ ($m \geq 0$) yields the degenerated case of the hypergeometric function. Here the second solution [Ref. 6), p. 72, Eq. (20)] is a rational function in $\frac{1}{t}$

$$\omega_- = t^{\frac{n}{2}-1} (1-t)^{-\frac{n}{2}} F\left(-m-n-1, m+2; 2; \frac{1}{t}\right) \quad (\text{A.2.20})$$

In the last two "discrete" cases we have assumed $\beta' > 0$.

Now the boundary conditions have to be investigated.

~~γ non-integer, $\beta' > 0$:~~ in this case only ω_+ fulfils (A.2.15). According to Ref. 6) [p. 110, Eq. (12)] F of ω_+ behaves as a constant in $t = 1$

$$F\left(\gamma - \frac{n-1}{2}, \gamma - \frac{n+1}{2}; 1+\gamma; t\right) = \frac{\Gamma(1+\gamma) \Gamma(n+1)}{\Gamma\left(\gamma + \frac{n+1}{2}\right) \Gamma\left(\gamma + \frac{n+3}{2}\right)} + O(1-t)$$

Insertion of ω_+ into (A.2.16) leads therefore to

$$a_+ = \frac{\Gamma(1+\gamma) \Gamma(n+2)}{\Gamma\left(\gamma + \frac{n+1}{2}\right) \Gamma\left(\gamma + \frac{n+3}{2}\right)} \quad (\text{A.2.21})$$

instead of zero. Hence there is no solution at all.

γ non-integer, $\beta \neq 0$: here ω_+ and ω_- fulfil (A.2.15). If we understand a_- as given by (A.2.21) with $-\gamma$ instead of γ

$$X_{n,m,e} = -\frac{t^2}{1-t} [a_{-\omega_+(t)} - a_{+\omega_-(t)}] Y_{n,e,m} \quad (\text{A.2.22})$$

is the eigensolution of the problem.

γ integer, $\beta \neq 0$: in the two cases cited above the analytic continuation of the solutions can be done easily because the analytic continuation of the hypergeometric function is well known. We merely state the result that there is no solution of this type obeying (A.2.15), (A.2.16).

APPENDIX 3

Plane wave in a Euclidean R_4

The reasoning goes along the same lines as in the R_3 . As the regular solution of the "radial" wave equation

$$\rho^2 R'' + 3\rho R' + [\rho^2 - n(n+2)]R = 0 \quad (\text{A.3.1})$$

is $\rho^{-1} J_{n+1}(\rho)$, the plane wave must be a linear combination, in the axial symmetric case multiplied by the spherical harmonics with $\ell = 0$

$$e^{iPX} = \sum_n a_n \frac{J_{n+1}(\sqrt{s}r)}{\sqrt{s}r} \pi_n^0(\cos\vartheta) \quad (\text{A.3.2})$$

($r = |X|$, $s = P^2$, ϑ is the angle between P and X). Applying

$$\int_0^\pi \pi_n^0 \sin^2\vartheta d\vartheta$$

comparing coefficients of lowest order in r and using

$$\cos^n\vartheta \pi_n^0 = 2^{-n} \sum_{k=0}^n \binom{n}{k} \pi_{2(n-k)} \quad (\text{A.3.3})$$

one gets

$$a_n = 2(n+1) i^n \quad (\text{A.3.4})$$

$\cos \mathcal{S}$ is the internal product of two unit vectors in the X and P space.

Thus we can use the addition theorem of Gegenbauer polynomials [Ref. ⁶⁾, p. 177, Eq. (19)] and spherical functions, so that with (A.2.5)

$$e^{iPX} = \sum_{n=0}^{\infty} \sum_{\ell=0}^n \sum_{m=-\ell}^{\ell} (2n+2)(2\ell+1) i^n \frac{(n-\ell)! (\ell-m)!}{(n+\ell+1)! (\ell+m)!} [(2\ell)!!]^2 \times$$

$$\times \frac{\int_{n+1}(\vec{r})}{r^n} y_{n,\ell,m}^*(\Omega) y_{n,\ell,m}(\Omega') \quad (\text{A.3.5})$$

APPENDIX 4

Asymptotic behaviour of $T_{\alpha\beta\gamma}$ (Eq. (3.4))

Small r is considered first. In our problem we have always

$$\alpha + \beta - 2(\gamma + 1) = -2m < 0$$

therefore if m is non-integer

$$\lim_{r \rightarrow 0} T_{\alpha\beta\gamma} \rightarrow \pi \left[2^{2\gamma+3-2} \Gamma(\gamma+1) \sin \pi \left(\gamma+1 - \frac{\alpha+\beta}{2} \right) \right]^{-1} \left(\frac{r}{2} \right)^{\alpha+\beta-2(\gamma+1)} \quad (\text{A.4.1})$$

One must treat more carefully the case m integer which occurs for μ integer in (3.3). This happens, e.g., for the special solution (2.37), (2.38). The combination (3.4) of generalized hypergeometric functions has quite similar properties as a modified Bessel function of second kind times a power of r . It reduces in fact to such a function if $\alpha = \beta + 2$. As in this special case, the limit $m \rightarrow \text{integer}$ has a meaning also in the general expression (3.4), because the expression in square brackets happens to vanish as well as the argument of the sine in the denominator. Using De l'Hospital's rule in calculations similar to the ones in the usual derivations of Neumann's functions for integer order, one gets

$$\begin{aligned} T_{\alpha\beta\gamma} &= (-1)^m \left[2^{\alpha+\beta-2(m+1)} \Gamma\left(\frac{\alpha+\beta}{2} + m\right) \right]^{-1} r \\ &\times \left\{ -2 \ln \frac{r}{2} {}_1F_2\left(\frac{\alpha+\beta}{2} + m; m+\gamma+1, m+1; \frac{r^2}{4}\right) + \right. \\ &+ \left(\frac{r}{2} \right)^{-2m} \sum_{k=0}^{m-1} \frac{\Gamma\left(\frac{\alpha+\beta}{2} + k\right)}{\Gamma(\gamma+1+k) k!} \left(\frac{r}{2} \right)^{2k} (-1)^{m-k} (m-k-1)! + \\ &+ \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{\alpha+\beta}{2} + k+m\right) \psi(k+1)}{\Gamma(\gamma+k+m+1) (k+m)! \Gamma(k+1)} \left(\frac{r}{2} \right)^{2k} - \frac{\partial}{\partial m} {}_1F_2\left(\dots; \frac{r^2}{4}\right) \end{aligned}$$

Here $\Psi(z)$ is the logarithmic derivative of $\Gamma(z)$. We did not want to perform the derivative in the last term, because it does not influence the behaviour in r . Thus, for small r and integer m (A.4.1) has to be replaced by

$$\lim_{n \rightarrow 0} T_{\alpha\beta\gamma} = \frac{(-)^{\gamma+1-\frac{\alpha+\beta}{2}} \Gamma(\frac{\alpha+\beta}{2}) \Gamma(\gamma+1-\frac{\alpha+\beta}{2})}{2^{2\gamma+3-\alpha} \Gamma(\gamma+1) \Gamma(\beta+1)} \left(\frac{n}{2}\right)^{\alpha+\beta-2(\gamma+1)} \quad (\text{A.4.2})$$

In the region of big r E.M. Wright's treatment¹¹⁾ of the asymptotic expressions for generalized hypergeometric functions can be used. He shows that each generalized hypergeometric function ${}_1\tilde{F}_2$ behaves as

$$\lim_{z \rightarrow 0} {}_1\tilde{F}_2(z) \rightarrow A_0 (z_1^{\mathcal{J}} e^{z_1} + z_2^{\mathcal{J}} e^{z_2}) \quad (\text{A.4.3})$$

where in our case $z_1 = -r$, $z_2 = +r$, and A_0 and $\mathcal{J}$ depend on the parameters of the hypergeometric function. The second term in (A.4.3) will therefore be the dominant one. Nevertheless, inserting Wright's expressions into $T_{\alpha\beta\gamma}$, it turns out that the increasing contributions of the two ${}_1\tilde{F}_2$ just cancel. Therefore we may look first for big purely imaginary r , where clearly both terms in (A.4.3) have the same influence, introduce (A.4.3) into (3.4) and continue to real r after having noticed the cancellation of e^+ . We do not want to reproduce Wright's formulae and the intermediate steps of the straightforward calculation here, but merely write down the result

$$\lim_{n \rightarrow \infty} T_{\alpha\beta\gamma}(n) = \sqrt{\frac{2\pi}{n}} \frac{e^{i\pi(\frac{\alpha-\beta}{2}-1)}}{2^{i(\gamma+1)-\alpha}} \left(\frac{n}{2}\right)^{\alpha-\gamma-2} e^{-n} \quad (\text{A.4.4})$$

The factor $\sin^{-1}\pi(\gamma+1-\frac{\alpha+\beta}{2})$ has dropped out, (A.4.4) is therefore valid also for integer values of m .

Fig.1 THE LOWEST EIGENVALUES OF λ

R E F E R E N C E S

- 1) H.A. Bethe and E.E. Salpeter, Phys. Rev. 84, 1232 (1951);
M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).
- 2) G.C. Wick, Phys. Rev. 96, 1124 (1954);
R.E. Cutkosky, Phys. Rev. 96, 1135 (1954).
C. Ceolin, F. Duimio, R. Stroppolini and S. Fubini, Nuovo Cimento 26,
247 (1962).
- 3) C. Hayashi and Y. Munataka, Progr. Theor. Phys. 7, 481 (1952).
- 4) J.S. Goldstein, Phys. Rev. 91, 1516 (1953).
- 5) G.C. Wick ²⁾ proved this for the "physical" B.-S. amplitude quite generally.
However, formulating the problem as an eigenvalue problem for fixed
energy in the coupling parameter λ one has not in general a
"physical situation". Sufficient big coupling constants may in principle
produce analytic properties of the amplitude quite different from the
ones for the real physical situation. We prefer therefore the more
cautious formulation given in the text.
- 6) A. Erdélyi, Higher transcendental functions (Bateman manuscript project),
Vol. I, McGraw Hill, N.Y. 1953.
- 7) M.E. Rose, Multipole fields, Wiley, N.Y. 1955.
- 8) G.N. Watson, Theory of Bessel functions, University press, Cambridge 1952,
p. 434.
- 9) S. Mandelstam, Proc. R. Soc. A 233, 248 (1955-56).
- 10) K. Baumann and W. Thirring, Nuovo Cimento 18, 357 (1960).
- 11) E.M. Wright, Journ. of London Math. Soc. 10, 386 (1935).