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ARCHIVES

ON THE BEHAVIOUR OF THE PARTIAL WAVE AMPLITUDES
FOR LARGE ANGULAR MOMENTA IN POTENTIAL SCATTERING

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I. INTRODUCTION

Since Regge has published his original work on complex angular momenta in potential scattering ¹⁾, there has been a big effort to present simpler derivations of his important results on the asymptotic behaviour of the partial wave amplitudes for large complex angular momenta ^{2),3),4),5)}. We present here a new derivation which, at least to the author, looks simple and in which the difficulties of the problem are well separated.

II. THE CASE OF REAL ℓ

It is perhaps worthwhile first to recall shortly, for pedagogical reasons, a method which enables us to find in a straightforward way the asymptotic behaviour of the scattering amplitude for large, real ℓ . In the Schrödinger equation

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + k^2 - V(r) \right] u_\ell(r) = 0 \quad (1)$$

we change the scale, using the variable $\rho = kr$:

$$\left[\frac{d^2}{d\rho^2} - \frac{\ell(\ell+1)}{\rho^2} + 1 - V_k(\rho) \right] u_\ell(\rho) = 0 \quad (2)$$

with

$$V_k(\rho) = \frac{1}{k^2} V(\rho/k)$$

Eq. (2) can be replaced by the integral equation

2.

$$\begin{aligned}
 u_\ell(\rho) = & \sqrt{\frac{\pi\rho}{2}} J_{\ell+1/2}(\rho) \\
 & + \int_0^\rho \sqrt{\frac{\pi\rho'}{2}} J_{\ell+1/2}(\rho') H_{\ell+1/2}^{(1)}(\rho) V_k(\rho') u_\ell(\rho') d\rho' \\
 & + \int_\rho^\infty \sqrt{\frac{\pi\rho'}{2}} J_{\ell+1/2}(\rho) H_{\ell+1/2}^{(1)}(\rho') V_k(\rho') u_\ell(\rho') d\rho'
 \end{aligned} \quad (3)$$

then the scattering amplitude is given by

$$f_\ell = e^{i\delta_\ell} \sin \delta_\ell = - \int_0^\infty \sqrt{\frac{\pi\rho}{2}} J_{\ell+1/2}(\rho) V_k(\rho) u_\ell(\rho) d\rho \quad (4)$$

Now let us show that, under very general conditions f_ℓ goes to zero as $\ell \rightarrow \infty$ (ℓ real, not necessarily integer). We make two remarks whose proof can be found in a paper on complex potentials ⁶⁾:

$$\begin{aligned}
 \text{i)} \quad & \left| \sqrt{\frac{\pi\rho}{2}} J_{\ell+1/2}(\rho) \right| < \frac{\rho}{\sqrt{2\ell+1}} \\
 \text{ii)} \quad & \left| J_{\ell+1/2}(\rho) H_{\ell+1/2}^{(1)}(\rho') \right| < \frac{1}{\sqrt{\ell+1/2}} \sqrt{\frac{2}{\pi}} \quad \text{for } \rho' > \rho
 \end{aligned} \quad (5)$$

[these inequalities were proved for integral ρ but the proof can be extended to real non-integral ℓ 7]. Using Eq. (5) one can put an upper bound on the inhomogeneous form of (3), if necessary, and majorize the kernel of equation (3) by a separable kernel :

$$|u_\ell(\rho)| \leq \sqrt{\frac{\pi\rho}{2}} |J_{\ell+1/2}(\rho)| + \sqrt{\frac{\pi}{2\ell+1}} \int_0^\infty \sqrt{\rho\rho'} |V_k(\rho')| |u_\ell(\rho')| d\rho' \quad (6)$$

From Eq. (6) we easily obtain

$$\int_0^\infty \rho |V_k(\rho)| |u_\ell(\rho)| d\rho < \frac{\int \sqrt{\frac{\pi}{2}} \rho |J_{\ell+1/2}(\rho)| |V(\rho)| d\rho}{1 - \sqrt{\frac{\pi}{2\ell+1}} \int_0^\infty \rho |V_k(\rho)| d\rho} \quad (7)$$

for $2\ell+1 > \pi^2 \left[\int_0^\infty \rho |V_k(\rho)| d\rho \right]$.

Then it is straightforward to get an upper bound of $u_\ell(\rho)$ and of the scattering amplitude :

$$\begin{aligned} & \left| \int u_\ell(\rho) \sqrt{\frac{\pi}{2}} J_{\ell+1/2}(\rho) V_k(\rho) d\rho \right| < \\ & \int \frac{\pi \rho}{2} [J_{\ell+1/2}(\rho)]^2 |V_k(\rho)| d\rho \\ & + \sqrt{\frac{\pi}{2\ell+1}} \frac{\left[\int \sqrt{\frac{\pi}{2}} \rho^{1/2} |J_{\ell+1/2}(\rho)| |V_k(\rho)| d\rho \right]^2}{1 - \sqrt{\frac{\pi}{2\ell+1}} \int_0^\infty \rho |V_k(\rho)| d\rho} \quad (8) \\ & < \frac{\int \frac{\pi \rho}{2} (J_{\ell+1/2}(\rho))^2 |V_k(\rho)| d\rho}{1 - \sqrt{\frac{\pi}{2\ell+1}} \int_0^\infty \rho |V_k(\rho)| d\rho} \end{aligned}$$

The latter expression has been obtained by use of the Schwartz inequality. We notice that its numerator is just the Born approximation for the potential $|V|$. So we see that under the condition

$$\int \rho |V_k(\rho)| d\rho \quad \text{finite} \quad (9)$$

the scattering amplitude decreases as fast as the Born approximation for the potential. In particular if V is such that

$$\int \rho^2 |V_k(\rho)| d\rho$$

is finite, the amplitude decreases faster than $1/\ell$. If V can be bounded by a Yukawa potential, the Born approximation decreases exponentially with ℓ . More precisely we get

4.

$$f_e < \frac{C Q_e (1 + \frac{\mu^2}{2k^2})}{1 - \frac{C}{\mu} \sqrt{\frac{\pi}{2e+1}}}$$

for $|V| < C \frac{e^{-\mu r}}{r}$.

Analogous results hold for potentials bounded by $e^{-\mu r}/r^n$ provided $\int r |V(r)| dr$ is finite.

Before leaving the real l case, let us make one more remark. We could have solved the integral equation in the Volterra form, i.e.,

$$\tilde{u}_l(p) = \sqrt{\frac{\pi p}{2}} J_{l+1/2}(p) + \int_0^p K(p, p') V(p') \tilde{u}_l(p') dp'$$

We would have obtained the same kind of results. Then one could obtain the scattering amplitude as given by

$$f_e = e^{i\delta_e} \sin \delta_e = \frac{- \int \sqrt{\frac{\pi p}{2}} J_{l+1/2}(p) V(p) \tilde{u}_l(p) dp}{1 + \int \sqrt{\frac{\pi p}{2}} H_{l+1/2}^{(1)}(p) V(p) \tilde{u}_l(p) dp}$$

Then we can express $\tilde{u}$ in terms of u

$$\tilde{u} = \frac{u}{1 - \int \sqrt{\frac{\pi p}{2}} H_{l+1/2}^{(1)}(p) V(p) u(p) dp}$$

and find separate bounds for the numerator and the denominator of the new expression of $e^{i\delta} \sin \delta$.

III. THE CASE OF $\ell = i\lambda - \frac{1}{2}$, k REAL (SIMPLE CASE)

Equation (3), or maybe preferably the Volterra form of this equation, can still be used for $\ell = i\lambda - \frac{1}{2}$. What we want to do is to find the same circumstances as in the case of real ℓ : find a bound of ten inhomogeneous terms decreasing as $\lambda \rightarrow \infty$, find a separable bound of the kernel decreasing as $\lambda \rightarrow \infty$. We know that this is impossible if we stick to ρ real. However, if we now restrict ourselves to superpositions of Yukawa potentials,

$$V(r) = \int_{\mu}^{\infty} \rho(\alpha) \frac{e^{-\alpha r}}{r} d\alpha \quad (10)$$

We know that $V(r)$ and hence $V_k(\rho)$ for k real can be continued in $-\frac{\pi}{2} \leq \text{Arg } \rho \leq +\frac{\pi}{2}$.

We shall try to find a complex path along which we get the required properties. It is easy to see that this path should start from the origin along the positive imaginary axis: from the series expansion of Bessel functions one easily proves that

$$\left| H_{i\lambda}^{(1)}(ix) \right|, \left| H_{i\lambda}^{(2)}(ix) \right| < \frac{c}{\sqrt{\lambda}} \quad \text{for } x < \sqrt{\lambda}$$

For large modulus of the argument, we easily see that we make both $H_{i\lambda}^{(1)}(z)$ and $H_{i\lambda}^{(2)}(z)$ small by taking $\text{Im } z = \frac{\lambda\pi}{2}$. This is obvious on the asymptotic forms

$$H_{i\lambda}^{(1),(2)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{\pm \left[i \left(z - \frac{\pi}{4} \right) + \lambda \frac{\pi}{2} \right]}$$

That such a path, connecting $z = 0$ and $z = \infty + i \frac{\lambda\pi}{2}$, exists can be proved by studying the Sommerfeld representation of Bessel functions. This is probably the most tedious part of the present work and is contained in the Appendix. The result is as follows:

along the path

$$\Gamma(\lambda) \begin{cases} z = i\gamma & 0 \leq \gamma \leq \lambda, \text{ and} \\ (\operatorname{Re} z)^2 = \lambda^2 \left(\frac{\pi}{2} - t \right) \left[\cotan t + t - \frac{\pi}{2} \right] & (\operatorname{Re} z > 0) \\ (\operatorname{Im} z)^2 = \lambda^2 \left(\frac{\pi}{2} - t \right) \left[\tan t - t + \frac{\pi}{2} \right] & (\operatorname{Im} z > 0) \\ \left(\frac{\pi}{2} \geq t \geq 0 \right) \end{cases}$$

indicated in Fig. 1

Fig. 1

The curve $\Gamma(\lambda)$ in the z plane

we have

$$\left| H_{i\lambda}^{(1),(2)}(z) \right| < \frac{C \sqrt{\frac{2}{\pi}}}{|\lambda^2 + z^2|^{1/4}} \quad (11)$$

and for $|z - \lambda| < \lambda^{1/3}$, along this path,

$$\left| H_{i\lambda}^{(1),(2)}(z) \right| < \frac{C \sqrt{\frac{2}{\pi}}}{\lambda^{1/3}} \quad (12)$$

Eqs. (11) and (12) provide us the required properties : Eq. (3) takes the form

$$\begin{aligned}
U_{\lambda-1/2}(z) &= \sqrt{\frac{\eta z}{2}} J_{\lambda}(z) \\
&+ \int_0^z \sqrt{\frac{\eta}{2}} \sqrt{zz'} J_{\lambda}(z') H_{\lambda}^{(1)}(z) V_K(z') U_{\lambda-1/2}(z') dz' \\
&+ \int_z^{\infty + i\frac{\lambda\eta}{2}} \sqrt{\frac{\eta}{2}} \sqrt{zz'} J_{\lambda}(z) H_{\lambda}^{(1)}(z') V_K(z') U_{\lambda-1/2}(z') dz'
\end{aligned} \tag{13}$$

Then (13) can be bounded by

$$|U_{\lambda-1/2}(z)| < \frac{C \sqrt{|z|}}{|\lambda^2 + z^2|^{1/4}} \left[1 + C \int_0^{\infty + i\frac{\lambda\eta}{2}} \frac{\sqrt{|z'|}}{|\lambda^2 + z'^2|^{1/4}} |V_K(z')| |U_{\lambda-1/2}(z')| |dz'| \right] \tag{14}$$

where the integral and the point z lie along the integration path $\Gamma(\lambda)$. Then we get

$$\left[\int_{\Gamma} \frac{|U_{\lambda-1/2}(z)| \sqrt{|z|}}{\sqrt{|\lambda^2 + z^2|}} |V_K(z)| |dz| \right] \left[1 - C^2 \int_{\Gamma} \frac{|z| |V_K(z)| |dz|}{\sqrt{|\lambda^2 + z^2|}} \right] < C \int_{\Gamma} \frac{|z| |V_K(z)| |dz|}{\sqrt{|\lambda^2 + z^2|}} \tag{15}$$

The whole question is then to know if

$$I(\lambda) = \int_{\Gamma(\lambda)} \frac{|z| |V_K(z)| |dz|}{\sqrt{|\lambda^2 + z^2|}} \tag{16}$$

goes to zero with $\lambda \rightarrow \infty$. A simple case is

$$|V_K(z)| < \frac{C'}{|z|^{\alpha}} \quad 1 < \alpha < 2 \tag{17}$$

Now if we notice that the path $\Gamma(\lambda)$ has a fixed shape, the scale being fixed by λ , we get

8.

$$I(\lambda) < \frac{\text{const}}{\lambda^{\alpha-1}} \quad (18)$$

$I(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Hence for $1 < \alpha < 2$ we get, for λ big enough,

$$\int_{\Gamma} |U_{\lambda-1/2}(z)| \frac{\sqrt{|z|}}{\sqrt{|z^2+\lambda^2|}} |V_K(z)| |dz| < \frac{\text{const } \lambda^{1-\alpha}}{1 - \text{const } \lambda^{1-\alpha}}$$

Then, if we are allowed to take as a definition of the scattering amplitude

$$f_{\lambda-1/2} = \int_0^{\infty + i\frac{\lambda\eta}{2}} \sqrt{\frac{\eta z}{2}} J_{\lambda}(z) V_K(z) U_{\lambda-1/2}(z) dz \quad (19)$$

we get

$$|f_{\lambda-1/2}| < \frac{C'' \lambda^{1-\alpha}}{1 - C'' \lambda^{1-\alpha}} \quad (20)$$

It is indeed permitted to take (19) as a definition of the scattering amplitude because the existence of a solution of the equation along Γ , ensured by (18), enables us to prolongate this solution outside Γ without special difficulties and due to the exponential decrease of the potential for $\text{Re } z \rightarrow \infty$

$$\int_0^{\infty + i\frac{\lambda\eta}{2}} \sqrt{\frac{\eta z}{2}} J_{\lambda}(z) V_K(z) U_{\lambda-1/2}(z) dz = 0$$

This proof works for a potential like

$$\frac{e^{-\alpha r} - e^{-\beta r}}{r^2}$$

or any reasonable continuous superposition of Yukawa, but not for a pure Yukawa.

For a pure Yukawa, $V(r) = G^2 e^{-\mu r}/r$

$$I(\lambda) = \frac{G^2}{k} \int_0^{i\lambda} \frac{|dz|}{\sqrt{|z^2 + \lambda^2|}} + \frac{G^2}{k} \int_{i\lambda}^{i\frac{\pi}{2} + \infty} \frac{e^{-\frac{\mu}{k} \operatorname{Re} z} |dz|}{\sqrt{|z^2 + \lambda^2|}} < \frac{G^2}{k} \frac{\pi}{2} + \frac{CG^2}{\sqrt{\lambda k}}$$

so, for k^2 big enough, λ large, $C^2 I(\lambda) < 1$, and one can easily prove the $|f_{i\lambda - \frac{1}{2}}|$ is bounded when $\lambda \rightarrow \infty$. However, if k^2 is too small (or G^2 too large) $C^2 I(\lambda)$ has an upper bound larger than unity and the simple treatment breaks down.

In the case $C^2 I(\lambda) \rightarrow 0$ for $\lambda \rightarrow \infty$ it is not difficult to extend the results to $|\ell| \rightarrow \infty$, $|\operatorname{Arg} \ell| < \frac{\pi}{2}$. Indeed

$$f_\ell = e^{i\delta_\ell} \tan \delta_\ell = \frac{S(\ell) - 1}{2i}$$

Now we know

i) $S(\ell) S^*(\ell^*) = 1$ so $S(-i\lambda - \frac{1}{2}) \rightarrow 1$ as $\lambda \rightarrow \infty$;

ii) $S(\ell)$ has no poles for $\operatorname{Re} \ell > -\frac{1}{2}$, $\operatorname{Im} \ell < 0$ and is bounded by $e^{\operatorname{Im} \ell \pi}$ in this region as it has been shown under very general assumptions by Regge.

Then the function

$$\frac{S(\ell) - 1}{2i} e^{\alpha \ell}$$

where α is related to the range of the potential :

$$\alpha = \log \left[1 + \frac{\mu^2}{2k^2} + \sqrt{\left(1 + \frac{\mu^2}{2k^2} \right)^2 - 1} \right] - \varepsilon$$

$$|A(\ell)| < \text{const exp}(-\alpha \text{Re} \ell)$$

$$|B(\ell)| < \text{const exp}(-\alpha \text{Re} \ell)$$

(25)

for $|\ell| > L$. In this way we prove that $f(\ell) = e^{i\int_0^\ell \sin \delta_\ell}$ goes to zero for $|\ell| \rightarrow \infty$, $|\text{Arg } \ell| < \frac{\pi}{2} - \varepsilon$, and also that $|f(\ell)| < C$ for $\text{Im } \ell \rightarrow \pm \infty$, $\text{Re } \ell > L_0$; L_0 being chosen large enough.

For $I(\lambda) \sim \lambda$ as $\lambda \rightarrow \infty$, which is the case for the exponential potential $e^{-\mu r}$,

$$I(\lambda) = \int_0^{i\lambda} \frac{|z| |dz|}{\sqrt{|z^2 + \lambda^2|}} + \int_{i\lambda}^{\frac{i\lambda}{2} + \infty} \frac{|z| e^{-\mu \text{Re } z} |dz|}{\sqrt{|\lambda^2 + z^2|}} \approx \lambda$$

we get $|f(\ell)| \rightarrow 0$ for $|\ell| \rightarrow \infty$, $|\text{Arg } \ell| < \frac{\pi}{2} - \varepsilon$.

However, it is not satisfactory that we cannot get $f(i\lambda - \frac{1}{2}) \rightarrow 0$ for $\lambda \rightarrow \infty$ in the case $I(\lambda) \rightarrow C$ for $\lambda \rightarrow \infty$.

This can be obtained at the expense of a less simple treatment: we define

$$W = \hat{u} - \sqrt{\frac{\pi z}{2}} J_{i\lambda}(z)$$

W satisfies the integral equation

$$\begin{aligned} W(z) = & \int_0^z \frac{\pi}{2} \sqrt{2z'} \left[J_{i\lambda}(z) H_{i\lambda}^{(1)}(z') - J_{i\lambda}(z') H_{i\lambda}^{(1)}(z) \right] V_K(z') \sqrt{\frac{\pi z'}{2}} J_{i\lambda}(z') dz' \\ & + \int_0^z \frac{\pi}{2} \sqrt{2z'} \left[J_{i\lambda}(z) H_{i\lambda}^{(1)}(z') - J_{i\lambda}(z') H_{i\lambda}^{(1)}(z) \right] V_K(z') W(z') dz' \quad (21') \end{aligned}$$

The kernel of (21') is the same as (21). However, the inhomogeneous term is changed, and we shall see that one can put a much better bound on the inhomogeneous term of (21') than on the inhomogeneous term of (21). More specifically, for $V(r) = G(e^{-\mu r})/r$ one can prove the following for $0 \leq y \leq i\lambda$:

$$\left\{ \begin{aligned} & \left| \int_0^{iy} \frac{\pi}{z} z' (J_{i\lambda}(z'))^2 V_K(z') dz' \right| \\ & \left| \int_0^{iy} \frac{\pi}{z} z' J_{i\lambda}(z') H_{i\lambda}^{(1)}(z') V_K(z') dz' \right| \end{aligned} \right\} < \frac{C'}{\lambda^{1/2}}$$

Then we get for $z = iy$, $0 < y < \lambda$:

$$|w(z)| < \frac{2C'}{\lambda^{1/2}} \frac{\sqrt{|z|}}{\sqrt{\lambda^2 + z^2}} \exp[2C^2 I(\lambda)]$$

and hence

$$|A(\lambda)| \text{ and } |B(\lambda)| < \frac{\text{const}}{\lambda^{1/2}} + \int_{i\lambda}^{i\frac{\lambda}{2} + \infty} \frac{\pi}{z} |J_{i\lambda}(z)|^2 |V_K(z)| |dz|,$$

as $\lambda \rightarrow \infty$ the only important contributions in the integrals are furnished by z close to iy , because the rest of the integral is suppressed by the exponential decrease of the potential as $\text{Re } z$ increases. Using the bounds on $J_{i\lambda}$ and $H_{i\lambda}^{(1)}$ one easily gets $|A(\lambda)|$ and $|B(\lambda)| < \text{const } \lambda^{-\frac{1}{2}}$ and hence $|f(i\lambda - \frac{1}{2})| < \text{const } \lambda^{-\frac{1}{2}}$ for $\lambda \rightarrow \infty$.

V. CONCLUDING REMARKS

This new approach to the complex large angular momentum, as can be seen by comparison of Sections II and III, is essentially an extension of a proof, valid for ℓ real, to the case of $\ell = i) - \frac{1}{2}$. The proof is only really simple if $|V| < 1/(|z|^{1+\epsilon})$ for $\text{Re } z \geq 0$. We are sorry to give a less straightforward derivation for the pure Yukawa case. Let us notice that it is not the first time that this difficulty is encountered : for instance, the arguments of Regge¹⁾ to locate the poles are more involved for $|V| < \frac{C}{|z|}$ than for $|V| < \frac{C}{|z|^2}$.

The connection with the WKB method is this : the path of integration we choose coincides with the Stokes lines of the WKB approximation for the Bessel functions. However, WKB approximation has not been used to majorize the Bessel functions. We think it is safer to use the Sommerfeld integral representation of Bessel functions.

Let us make one more comment. The existence of the path $\Gamma(\lambda)$ might be used for numerical computation, for finite λ , of the scattering amplitude. This might permit to exhaust all possible trajectories crossing the line $\text{Re } \ell = -\frac{1}{2}$.

APPENDIX

I. UPPER BOUNDS AND ASYMPTOTIC EXPRESSIONS OF BESSEL FUNCTIONS

We start from the Sommerfeld integral representations :

$$H_{i\lambda}^{(1)}(z) = \frac{1}{\pi} \int_{-i\infty+\eta}^{i\infty-\eta} e^{iz \cosh t + \lambda(\frac{\pi}{2}-t)} dt$$

$$H_{i\lambda}^{(2)}(z) = \frac{1}{\pi} \int_{-i\infty+\eta}^{i\infty+2\pi-\eta} e^{iz \cosh t + \lambda(\frac{\pi}{2}-t)} dt$$

valid for $-\eta < \text{Arg } z < \pi - \eta$. We have to distinguish two cases : $\text{Im } z < \lambda$,
and $\text{Im } z > \lambda$.

a) $\text{Im } z < \lambda$.

What we want to find is a set of values of z such that there exist integration paths for both $H^{(1)}$ and $H^{(2)}$ along which the modulus of the integrand is less than unity.

The series expansion around $z = 0$ suggests that part of the interesting values of z lie on the imaginary axis. We set $\eta = 0$ and we first study $z = iy$, $0 < y < \lambda$; then one easily gets the following map for the integrand.

[See Fig. 2.] We find two saddle points for

$$\begin{aligned} \text{Re } t_{0,1} &= \frac{\pi}{2} \\ \text{Im } t_{0,1} &= \pm \cosh^{-1}\left(\frac{\lambda}{y}\right) \end{aligned}$$

One can easily convince oneself that the contributions to the integral from points at a distance larger than a finite point are negligible for $\lambda \rightarrow \infty$. The

Fig. 2

The t plane for $z = iy$, $0 < y < \lambda$
 The unshaded areas correspond to an integrand less than unity. The dotted lines represent convenient integration paths for calculating $H_{i\lambda}^{(1)}$ and $H_{i\lambda}^{(2)}$.

contributions of the neighbourhood of the saddle points are given in a reliable way by the steepest descent method as long as the two points are at finite distance in the t plane and as long as they are at non-zero distance from each other, i.e., $\lambda \varepsilon < y < (1 - \varepsilon)\lambda$; then the contributions of the saddle points are, if we write the integrand as $\exp[f(\lambda, z, t)]$,

$$C \frac{\exp[f(\lambda, z, t_{0,1})]}{\sqrt{\frac{d^2 f}{dt^2}(\lambda, z, t_{0,1})}}$$

In this way we get

$$H_{i\lambda}^{(1)}(iy) \approx -i\sqrt{\frac{2}{\pi}} \frac{\sin\left(\frac{\pi}{4} + \lambda \cos^{-1}\frac{\lambda}{y} - \sqrt{\lambda^2 - y^2}\right)}{(\lambda^2 - y^2)^{1/4}}$$

$$H_{i\lambda}^{(2)}(iy) \approx \sqrt{\frac{1}{2\pi}} \frac{e^{+i\left[\frac{\pi}{4} + \lambda \cos^{-1}\frac{\lambda}{y} - \sqrt{\lambda^2 - y^2}\right]}}{(\lambda^2 - y^2)^{1/4}}$$

for $\lambda \varepsilon < y < \lambda(1 - \varepsilon)$.

These asymptotic estimates can be taken as strict bounds because one can work with the modulus of the integrand, bound it conveniently, and integrate along straight lines at angles $\pm 45^\circ$ passing through the saddle points. Then one gets

$$|H_{i\lambda}^{(1),(2)}(iy)| < \frac{C}{(\lambda^2 - y^2)^{1/4}}$$

for $0 \ll y < \lambda$.

For y close to λ one can improve the bound by noticing that to calculate $H^{(1)}$ it is permissible to integrate along a straight line $0 < \operatorname{Re} t = \text{const} < \frac{\pi}{2}$. Adjusting it to be $\operatorname{Re} t = \frac{\pi}{2} - \frac{\lambda^{2/3}}{y}$ we get easily

$$|H_{i\lambda}^{(1)}(iy)| < \frac{C}{\lambda^{1/3}}$$

for $y > \lambda - \lambda^{1/3}$.

For $H^{(2)}$ the same operation is possible: one integrates on $\operatorname{Re} t = \frac{\pi}{2} - \frac{\lambda^{2/3}}{y}$ for $\operatorname{Im} t < 0$, then $\operatorname{Im} t = 0$, $\frac{\pi}{2} - \frac{\lambda^{2/3}}{y} < \operatorname{Re} t < \frac{3\pi}{2} + \varepsilon$, then $\operatorname{Im} t > 0$, $\operatorname{Re} t = \frac{3\pi}{2} + \varepsilon$, and one gets

$$|H_{i\lambda}^{(2)}(iy)| < \frac{C}{\lambda^{1/3}}$$

for $\lambda \gg y > \lambda - \lambda^{1/3}$.

b) $\text{Im } z > \frac{\pi}{2}$.

We know in advance that the region of interest is $\frac{\pi}{2} > \text{Im } z > \frac{\pi}{2}$ since, from the asymptotic form of $H_{i\lambda}^{(1),(2)}$ one sees that these functions will both be small for $\text{Re } z \rightarrow \infty$, $\text{Im } z = \frac{\pi}{2}$.

The problem is to find values of z such that one can simultaneously find a path for $H^{(1)}$ and a path for $H^{(2)}$ along which the integrands are less than unity. So one has to study the sign of

$$\phi(z, t) = \text{Re} [iz \cos t + \lambda(\frac{\pi}{2} - t)]$$

It is easy to see that $\phi(z, t) = -\phi(z, \pi - t)$, i.e., we have an antisymmetry with respect to $t = \frac{\pi}{2}$. Now if there is a path connecting $+i\infty - \eta$ to $-i\infty + \eta$, $0 < \eta < \frac{\pi}{2}$ along which we have $\phi(z, t) \leq -\varepsilon < 0$ we easily see that it will be impossible to find another path connecting $-i\infty + \eta$ to $i\infty + 2\pi - \eta$, due to the antisymmetry of the function $\phi(z, t)$ with respect to $t = \frac{\pi}{2}$. Now

$$\phi(z, t) = -\text{Im } z \cos t_1 \cosh t_2 + \text{Re } z \sin t_1 \sinh t_2 - \lambda t_1 + \lambda \frac{\pi}{2}$$

with $t = t_1 + it_2$. For $-\frac{\pi}{2} < t_1 < +\frac{\pi}{2}$, $\phi(z, t)$ cannot be strictly positive along a line $t_1 = \text{const}$. Indeed if we have $0 < t_1 < \frac{\pi}{2}$ we can always make ϕ negative by taking t_2 negative, large enough; if $-\frac{\pi}{2} < t_1 < 0$ we make ϕ negative by taking t_2 positive large enough. If we had, on the other hand, $\phi(z, t)$ strictly negative on a line $t_1 = \text{const}$, $-\frac{\pi}{2} < t_1 < +\frac{\pi}{2}$ we would have correspondingly, according to the antisymmetry $\phi(z, t)$ strictly positive along the line $\pi - t_1$, $\pi - t_1 < 3\frac{\pi}{2}$, and the integrand of $H^{(2)}$ would become somewhere larger than unity in modulus.

So along a line $t_1 = \text{const}$, $-\frac{\pi}{2} < t_1 < +\frac{\pi}{2}$, $\phi(z, t_1 + it_2)$ must have at least one zero in t_2 . If we put $\cosh t_2 = \frac{1}{2}(x + \frac{1}{x})$, $\phi = 0$ becomes

$$[-\text{Im } z \cos t_1 + \text{Re } z \sin t_1] x^2 + 2\lambda x [\frac{\pi}{2} - t_1] - [\text{Im } z \cos t_1 + \text{Re } z \sin t_1] = 0$$

So we get the necessary condition

$$\Delta'(t_1) = \lambda^2 \left(\frac{\eta}{2} - t_1 \right)^2 + (\operatorname{Re} z)^2 \sin^2 t_1 - (\operatorname{Im} z)^2 \cos^2 t_1 \geq 0$$

for $-\frac{\eta}{2} < t_1 < +\frac{\eta}{2}$. The only acceptable roots of the equation are those corresponding to $x > 0$. For $t_1 < \frac{\eta}{2}$ both roots will be negative only if simultaneously :

$$-\operatorname{Im} z \cos t_1 + \operatorname{Re} z \sin t_1 > 0$$

$$\operatorname{Im} z \cos t_1 + \operatorname{Re} z \sin t_1 < 0$$

which implies, for $\operatorname{Im} z > 0$, $\cos t_1 < 0$. So at least one root is acceptable for $-\frac{\eta}{2} < t_1 < +\frac{\eta}{2}$. The same is true for $\frac{\eta}{2} < t_1 < 3\frac{\pi}{2}$ because if a root x is associated with t_1 , a root $\frac{1}{x}$ is associated with $\pi - t_1$.

We conclude that the condition $\Delta' \geq 0$ for $-\frac{\eta}{2} < t_1 < +\frac{\pi}{2}$ ensures for every $-\frac{\eta}{2} < t_1 < \frac{\eta}{2}$ the existence of a point $t_1 + it_2$ where $\phi(z, t_1 + it_2) = 0$.

We know that the path of integration of $H_{\lambda}^{(1)}$ must go at least through one point $T_1 + iT_2$ where $\phi(z, T_1 + iT_2) = 0$. However along this path we must have $\phi(z, t_1 + it_2) \leq 0$. So the point $T_1 + iT_2$ must be a double zero for ϕ . Hence we get

$$\Delta'(T_1) = 0$$

since Δ' must be otherwise positive, we get

$$\begin{cases} \Delta'(T_1) = 0 \\ \frac{d}{dt_1} \Delta'(t_1 = T_1) = 0 \end{cases}$$

i.e.,

$$\lambda^2 \left(\frac{\eta}{2} - T_1 \right)^2 + (\operatorname{Re} z)^2 \sin^2 T_1 - (\operatorname{Im} z)^2 \cos^2 T_1 = 0$$

$$-\lambda^2 \left(\frac{\eta}{2} - T_1 \right) + \left((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 \right) \sin T_1 \cos T_1 = 0$$

for $\text{Im } z < \frac{\lambda\pi}{2}$ the first equation can only have solutions for $0 \leq T_1 \leq \pi$ and due to the symmetry it is enough to investigate $0 \leq T_1 \leq \frac{\pi}{2}$. Hence we get

$$(\text{Re } z)^2 = \lambda^2 \left(\frac{\pi}{2} - T_1 \right) \left[\cotan T_1 + T_1 - \frac{\pi}{2} \right] \quad (\text{Re } z > 0)$$

$$(\text{Im } z)^2 = \lambda^2 \left(\frac{\pi}{2} - T_1 \right) \left[\tan T_1 - T_1 + \frac{\pi}{2} \right] \quad (\text{Im } z > 0)$$

$$0 \leq T_1 \leq \frac{\pi}{2}$$

This is the parametric equation of a curve in the z complex plane. The question is now: if z lies on this curve, can we really find paths for $H_{,\lambda}^{(1)}$ and $H_{,\lambda}^{(2)}$ such that along these paths $\phi(z,t) \leq 0$?

Careful examination of the situation leads to the following picture

[Fig. 3]:

Fig. 3

The shaded areas correspond to $\phi(z,t) > 0$. The non-shaded areas to $\phi(z,t) < 0$; integration paths for $H_{,\lambda}^{(1)}$ $H_{,\lambda}^{(2)}$, are indicated by dotted lines.

It is seen that one can manage to find integration paths for both $H^{(1)}_{i\lambda}$ and $H^{(2)}_{i\lambda}$. The plot of the associated curve in the z plane has been given. Let us indicate that for $T_1 \rightarrow 0$ we get $z = \infty + i \frac{\lambda\pi}{2}$, and for $T_1 \rightarrow \frac{\pi}{2}$, $z = i\lambda$. Hence we connect $z = 0$ to $z = \infty + i \frac{\lambda\pi}{2}$ by a continuous curve.

It remains to find suitable bounds on $H^{(1)}_{i\lambda}$ and $H^{(2)}_{i\lambda}$. The steepest descent method gives

$$|H^{(1),(2)}_{i\lambda}(z)| \approx \frac{C}{|\lambda^2 + z^2|^{1/4}} \quad (|z - i\lambda| > \lambda^{-1/3})$$

One can check that taking C big enough, this is a bound. So all along $\Gamma(\lambda)$ one has

$$|H^{(1),(2)}_{i\lambda}(z)| < \frac{C}{|\lambda^2 + z^2|^{1/4}}$$

II. ESTIMATES OF THE INTEGRALS

$$\int_0^y y' [J_{i\lambda}(iy')]^2 V_k(iy') dy', \quad \int_0^y y' J_{i\lambda}(iy') H^{(1)}_{i\lambda}(iy') V_k(iy') dy',$$

$$y < \lambda, \quad V_k(z) = \frac{G^2 - \mu^2 z/k}{z}$$

We easily see, using the asymptotic expressions of $H^{(1)}_{i\lambda}$ and $H^{(2)}_{i\lambda}$ that to put bounds on these integrals it is enough to study

$$(A) \int_0^y \frac{e^{-i\frac{\mu}{k}y'}}{\sqrt{\lambda^2 - y'^2}} dy', \quad (B) \int_0^y \frac{e^{-i\left[\frac{\pi}{2} + 2\lambda \operatorname{erf}^{-1} \frac{\lambda}{y'} - 2\sqrt{\lambda^2 - y'^2} + \frac{\mu y'}{k}\right]}}{\sqrt{\lambda^2 - y'^2}} dy'$$

$$(C) \int_0^y \frac{e^{+i\left[\frac{\pi}{2} + 2\lambda \operatorname{erf}^{-1} \frac{\lambda}{y'} - 2\sqrt{\lambda^2 - y'^2} - \frac{\mu y'}{k}\right]}}{\sqrt{\lambda^2 - y'^2}} dy'$$

if we put $x = \frac{y}{\lambda}$ these are integrals of the type

$$\int_0^x \frac{e^{i\lambda \phi(x')} dx'}{\sqrt{1-x'^2}}$$

If $\phi'(x)$ has no zero in the interval of integration, which is the case for (A) and (C), we can just write

$$\int_0^x \frac{e^{i\lambda \phi(x')} dx'}{\sqrt{1-x'^2}} = \frac{1}{\lambda} \left[\frac{e^{i\lambda \phi(x)}}{\phi'(x) \sqrt{1-x^2}} \right]_0^x - \frac{1}{\lambda} \int_0^x e^{i\lambda \phi(x')} \frac{d}{dx'} \left[\frac{1}{\phi'(x') \sqrt{1-x'^2}} \right] dx'$$

and hence

$$\left| \int_0^x \frac{e^{i\lambda \phi(x')} dx'}{\sqrt{1-x'^2}} \right| < \frac{1}{\lambda} \left[\frac{1}{|\phi'(x)| \sqrt{1-x^2}} + \frac{1}{|\phi'(0)|} + \int_0^x \left| \frac{d}{dx'} \left[\frac{1}{\phi'(x') \sqrt{1-x'^2}} \right] \right| dx' \right]$$

so that for $x < 1-\varepsilon$, i.e., $y < \lambda(1-\varepsilon)$ (A) and (C) are less than $\frac{\text{const}}{\lambda}$. In integral (B) $\phi'(x)$ vanishes at one well-defined value $0 < x_0 < 1$; as long as y is less than $\lambda x_0(1-\varepsilon)$ the results are unchanged. For $x > x_0(1-\varepsilon)$ we are led to cut the integral into three pieces:

$$0 < x' \leq x_0 - \frac{1}{\lambda^{1/2}}, \quad x_0 - \frac{1}{\lambda^{1/2}} < x' < x_0 + \frac{1}{\lambda^{1/2}}, \quad x_0 + \frac{1}{\lambda^{1/2}} < x' < x$$

for $x < (1-\varepsilon)$ we get that (B) is less than $C/\lambda^{1/2}$.

For x very close to 1, i.e., y very close to λ , we have to be more careful because the asymptotic expressions of $H_{i,\lambda}^{(1)}$ and $H_{i,\lambda}^{(2)}$ cease to be valid. However, careful examination of the steepest descent method shows that these expressions still hold as long as $z = iy$, $y < \lambda - \lambda^{1/3+\varepsilon}$, ε positive arbitrarily small. So one easily proves that

$$\int_0^y e^{-i\lambda y'} (J_{i,\lambda}(iy'))^2 dy' \quad \text{and} \quad \int_0^y e^{-i\lambda y'} J_{i,\lambda}(iy') H_{i,\lambda}^{(1)}(iy') dy'$$

are still bounded by $c\lambda^{-1/2}$ for $y < \lambda - \lambda^{1/3+\varepsilon}$.

For $y > \lambda - \lambda^{1/3+\varepsilon}$ one can make use of the following remarks :

$$i) \quad \left| \sqrt{y} H_{i\lambda}^{(1),(2)}(iy) \right| < \lambda^{-1/6}, \quad y < \lambda$$

ii) from the differential equation for Bessel functions we get

$$\left| \frac{d^2}{dy^2} \sqrt{y} H_{i\lambda}^{(1),(2)}(iy) \right| < \lambda^{-1/2+\varepsilon}$$

for $\lambda - \lambda^{1/3+\varepsilon} < y \leq \lambda$, and by integration, using

$$\left| \frac{d}{dy} \left[\sqrt{y} H_{i\lambda}^{(1),(2)}(iy) \right]_{y=\lambda} \right| < \lambda^{-1/6}$$

$$iii) \quad \left| \frac{d}{dy} \left[\sqrt{y} H_{i\lambda}^{(1),(2)}(iy) \right] \right| < \lambda^{-1/6+2\varepsilon}, \quad \text{for } \lambda - \lambda^{1/3+\varepsilon} < y \leq \lambda.$$

Then, by integration by parts, it is easy to prove that the contributions of the integrals from $\lambda - \lambda^{1/3+\varepsilon}$ to y are negligible for ε sufficiently small.

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