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DERIVATION OF THE GINZBURG-LANDAU THEORY
FOR A LOCAL SUPERCONDUCTOR *)

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ABSTRACT

The microscopic equations for a superconductor developed in an earlier paper are expanded, in a gauge invariant way, under the assumption that the magnetic vector potential is small and slowly varying and that the energy gap is also slowly varying. The equations obtained represent a generalization of the Ginzburg-Landau theory for a local superconductor, which should be valid at temperatures considerably lower than the critical temperature. A comparison with previous work is made and the question of the magnetic field dependence of the energy gap is discussed.

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1. INTRODUCTION

The phenomenological equations given by Ginzburg and Landau ¹⁾ to describe the behaviour of a superconductor in a static magnetic field have been derived by Gor'kov ²⁾ from the microscopic theory. Gor'kov made the assumption that the magnetic potential and the gap are slowly varying functions of position and furthermore he assumed that the temperature of the system is very close to the critical temperature, so that the gap is small. These assumptions correspond closely to the original assumptions made by Ginzburg and Landau; their effective single particle wave function is defined in microscopic terms as being proportional to the local value of the gap.

More recently, Tewordt ³⁾ and Werthamer ⁴⁾ have derived generalizations of the Ginzburg-Landau theory. Their derivations do not assume that the gap is small and therefore their equations should be valid also at lower temperatures. These authors, like Gor'kov, use as a starting point for their derivation the theory of thermal Green's functions. The generalized equations they obtain are not identical, although they agree with each other, and with Gor'kov's equation, near the critical temperature.

In the present paper we give a systematic derivation of the generalized Ginzburg-Landau theory. We use as our starting point the equations for a superconductor in a static magnetic field derived previously ⁵⁾ by means of a generalized canonical transformation and a variational principle for the grand potential. We then perform a gauge invariant expansion of these equations and obtain a result which is in complete agreement with that of Werthamer's ⁴⁾. Actually, this is not too surprising because, as we remark at the end of Section 2, our basic equations can be transformed so as to show a strong similarity to those of the theory of thermal Green's functions. Furthermore, our systematic expansion procedure is very similar to the method used by Werthamer, except for the fact that our expansion is completely gauge invariant, a fact which permits a continuous consistency check during the course of the calculations. On the other hand, the disagreement with Tewordt's result can be traced to the fact that he did not systematically evaluate all terms of a given order, but chose to neglect some. This is explained in Section 4.

It has been shown before ⁶⁾ that the Ginzburg-Landau theory implies the correct quantization of the fluxoid in a multiply connected superconductor. We can make a further simple application of the theory by calculating the dependence of the gap upon the magnetic potential in the case of weak fields. This is done in Section 5, where our results are also compared with the previous calculations of Gupta and Mathur ⁷⁾ and of Nambu and Tuan ⁸⁾.

2. BASIC EQUATIONS

In I we studied the behaviour of a superconductor in a static magnetic field by means of a variational principle. The two unknown matrices

$$F_{ff'} = \langle a_f a_{f'} \rangle \quad (1)$$

and

$$G_{ff'} + \frac{1}{2} I_{ff'} = \langle a_f^* a_{f'} \rangle \quad (2)$$

were combined into a supermatrix

$$\underline{G} = \begin{pmatrix} -G^* & F \\ -F^* & G \end{pmatrix} \quad (3)$$

and equations were found which determine $\underline{G}$ once the kinetic energy matrix

$$K_{ff'} = (f' | (p-A)^2 - \mu | f) \quad (4)$$

and the potential energy matrix $P_{f_1 f_2 f_3 f_4}$ are given. These equations are listed in the last paragraph of Section 4 of I and can be written in a slightly modified form as ⁹⁾:

$$\underline{G} = -\frac{1}{2} \tanh\left(\frac{1}{2}\beta \underline{E}\right) \quad (5)$$

$$\underline{E} = \begin{pmatrix} -E^* & D \\ -D^* & E \end{pmatrix} \quad (6)$$

$$E = K + Q \cdot (G + \frac{1}{2} I) \quad (7)$$

$$D = P \cdot F \quad (8)$$

$$Q_{f'f^2f^3f^4} = P_{f'f^3f^4f^2} - P_{f'f^3f^2f^4}. \quad (9)$$

The matrices $\underline{E}$ and $\underline{G}$ are Hermitian. In this form of the equations the main difficulty appears to be in the non-linear matrix relation (5). It is quite natural to replace (5) with the Cauchy integral

$$\underline{G} = \frac{1}{2\pi i} \oint \frac{f(\lambda)}{\lambda - \underline{E}} d\lambda \quad (10)$$

$$f(\lambda) = -\frac{1}{2} \tanh\left(\frac{1}{2}\beta \lambda\right) \quad (11)$$

where the contour of integration must encircle completely the (real) spectrum of the matrix $\underline{E}$.

It is possible, and it would be in fact more elegant and concise, to maintain the supermatrix notation throughout our calculation. We chose instead, for the sake of clarity, to work with the submatrices, such as F or G . It is easy to see that, in terms of submatrices, Eq. (10) becomes

$$F = \frac{1}{2\pi i} \oint f(\lambda) F_{\lambda} d\lambda \quad G = \frac{1}{2\pi i} \oint f(\lambda) G_{\lambda} d\lambda \quad (12)$$

$$F_{\lambda} = \frac{1}{\lambda + E^*} D G_{\lambda} \quad (13)$$

$$G_{\lambda} = \frac{1}{\lambda - E - D^* \frac{1}{\lambda + E^*} D} \quad (14)$$

In these equations the spin part of the matrices has been separated out. According to the conventions adopted in I, this separation gives rise to the minus sign appearing in front of the last term of the denominator of Eq. (14).

In conclusion, we shall take as our basic equations (7), (8), (12), (13), and (14), with the definitions (9) and (11). These basic equations are exact, within the limitations of the variational ansatz made in I. In their present form they allow one to establish very easily the connection with the theory of the thermal Green's functions. Because of the close connection with this theory, we shall refer to the functions F_{λ} and G_{λ} as propagators. The current density is expressed very simply in terms of G by the formula ¹⁰⁾

$$J(x) = \lim_{x' \rightarrow x} e \left[i(\nabla - \nabla') - 2A(x) \right] \langle \psi^*(x) \psi(x') \rangle \quad (15)$$

$$\langle \psi^*(x) \psi(x') \rangle = G(x, x') + \frac{1}{2} \delta(x - x'). \quad (16)$$

3. EXPANSION PROCEDURE

To derive the equations satisfied by a local superconductor, one must assume that the magnetic potential A , as well as the other quantities entering the problem, are slowly varying over distances of the order of the Pippard coherence length. We shall assume that A is slowly varying and small, or more precisely that one can set

$$A(R) = \sigma \tilde{A}(\sigma R) = \sigma \tilde{A}(\rho) \quad (17)$$

where $\tilde{A}(\rho)$ is a fixed function and σ is a small parameter, which will be our expansion parameter. This assumption has the remarkable property that, if one expands any gauge invariant quantity in powers of σ , the individual terms of the expansion will be separately gauge invariant. This is a consequence of the fact that a gauge transformation on A with a slowly varying gauge function

$$\chi(R) = \tilde{\chi}(\sigma R) = \tilde{\chi}(\rho) \quad (18)$$

changes A into

$$A \rightarrow A + \nabla_R \chi = \sigma \left(\tilde{A}(\rho) + \nabla_\rho \tilde{\chi} \right) \quad (19)$$

and preserves the character of A to be small and slowly varying. In other words, the parameter σ has a gauge invariant meaning.

In order to obtain a manifestly gauge invariant procedure, it is convenient to expand in powers of σ , rather than the propagators F_λ and G_λ , modified propagators with simpler transformation properties. In the co-ordinate representation, we define

the energy gap D will be taken as being of zero order in σ , but will be assumed to be slowly varying, or, more precisely, a function of ρ . As a consequence, the expansion in σ becomes simply a simultaneous expansion in A and in the gradient operator ∇_R . A considerable simplification in the calculations is obtained if, in the equation (7) for E , one drops the self-consistent term containing Q , so that

$$E = K = (p - A)^2 - \mu. \quad (30)$$

Furthermore, we shall neglect the dependence of $D(p, R)$ upon p . These two approximations, which are also made in all previous works, should not alter the basic properties of our equations. The second approximation corresponds to assuming that the interaction potential is effectively a constant in momentum space.

The transition to the gauge invariant propagators can be achieved, to a given order in σ , by going into the mixed representation and expanding (20) and (21) to that order. It is shown in Appendix A that, up to second order inclusive,

$$\bar{G}(p, R) = \left(1 + A(R) \cdot \nabla_p + \frac{1}{2} (A \cdot \nabla_p)^2 \right) G(p, R) \quad (31)$$

$$\bar{F}(p, R) = \left(1 + \frac{i}{4} \frac{\partial A_\lambda}{\partial R_\sigma} \frac{\partial}{\partial p_\lambda} \frac{\partial}{\partial p_\sigma} \right) F(p, R) \quad (32)$$

(sum over repeated indices).

Finally we observe that, in the mixed representation, the expression (15) for the current density becomes (after performing the sum over spin indices)

$$J(R) = \frac{4e}{V} \sum_p (p - A(R)) \left[G(p, R) + \frac{1}{2} \right] = \frac{4e}{V} \sum_p p \bar{G}(p, R). \quad (33)$$

This equality, which is exact, shows that an expansion for the gauge invariant propagator $\bar{G}$ will give immediately a gauge invariant expansion for the current density. Similarly, Eq. (8) can be transformed into an equation containing $\bar{F}$:

$$\bar{D}(p, R) = -\frac{1}{V} \sum_q P_{p-q} \bar{F}(q, R). \quad (34)$$

As already mentioned above, we shall neglect in (34) the momentum dependence of the interaction potential.

4. GENERALIZED GINZBURG-LANDAU EQUATIONS

The expansion of the gauge invariant propagators is performed in Appendix B. Using the results obtained there, one can derive immediately the generalized Ginzburg-Landau equations. It is clear, by symmetric integration, that only the first order term $\bar{G}_1$ contributes to the current density (33). Performing the λ integration and going over to a more conventional notation, we obtain

$$J(R) = \frac{e}{m} \left[-\frac{i}{2} (D^* \nabla_R D - D \nabla_R D^*) - 2eA |D|^2 \right] n h_1(|D|^2) \quad (35)$$

where

$$h_1(|D|^2) \equiv \int_0^\infty \frac{d}{d\varepsilon} \left(\frac{f(\varepsilon)}{\varepsilon} \right) \frac{d\omega}{\varepsilon} = \frac{1}{2|D|^2} \frac{\Lambda}{\Lambda_T} \quad (36)$$

and n is the number density. We have made the usual approximations in which slowly varying functions are replaced by their values on the Fermi surface. Equation (36) establishes the relation with the work of Bardeen, Cooper and Schrieffer¹²⁾, except that here the function h_1 will in

general vary with position. The above expression for the current density agrees exactly with the results of both Tewordt and Werthamer. As these authors have remarked, it is also very easy to see that the above expression goes over into Gor'kov's result if the temperature is very close to the critical temperature.

The first order term $\bar{F}_1$ gives no contribution to the gap equation (34) since we have assumed that the interaction potential is momentum independent. Performing the λ integration and dropping terms which are odd in ω or can be written as derivatives with respect to ω , the gap equation can be transformed into the form

$$\begin{aligned} & \left[-\frac{6}{v_F^2} h_0 + h_1 (\nabla_R - 2ieA)^2 \right] D + \\ & + h_2 \left[D^* \left((\nabla_R - 2ieA) D \right)^2 + \frac{D}{3} \nabla_R^2 |D|^2 \right] + \\ & + h_3 \frac{D}{6} \left(\nabla_R |D|^2 \right)^2 = 0 \end{aligned} \quad (37)$$

where the functions appearing as coefficients are defined as

$$h_n = \frac{\partial}{\partial |D|^2} h_{n-1} \quad (38)$$

$$h_0(|D|^2) = 2 \int_0^\infty \left[\frac{f(\varepsilon)}{\varepsilon} - \frac{f(\varepsilon_0)}{\varepsilon_0} \right] d\omega \quad (39)$$

$$\varepsilon = \sqrt{\omega^2 + D^2} \quad \varepsilon_0 = \sqrt{\omega^2 + D_0^2}$$

and D_0 denotes the BCS gap at the given temperature. Except for slight differences in notation, Eq. (37) is identical to the equation obtained by

Werthamer. It differs from the equation obtained by Tewordt because he did not evaluate terms like those appearing in the second and third line of Eq. (37). At the same time Tewordt calculated correctly the total coefficient of those terms which are proportional to A^2 and attributed this coefficient completely to the $(\nabla_R - 2ieA)^2 D$ term. Therefore in his equation, in place of the function h_1 one finds the combination $h_1 + |D|^2 h_2$. Consequently, Tewordt's result concerning the magnetic field dependence of the gap at zero temperature is correct (see Section 5).

Near the critical temperature the terms in the second and third line become negligible and Tewordt's equation agrees with (37). At the same time Eq. (37) becomes identical to the equation derived by Gor'kov for this temperature range.

5. DEPENDENCE OF THE GAP UPON THE MAGNETIC POTENTIAL

Equation (37) can be used to study the dependence of the gap D upon the magnetic potential A , by expanding the solution D in powers of σ^2 . From (17) and (29) one sees that σ occurs in the equation only to the second power: therefore we shall use as expansion parameter $\sigma^2 \equiv \alpha$. However, it is clear that one should not take very seriously terms in the expansion of order higher than the second in σ , since in deriving Eq. (37) we have neglected terms of higher order. As a matter of fact, as long as one is interested only in the gap to order α , the function h_0 can be approximated by expanding it. Since, for $\alpha = 0$, Eq. (37) becomes

$$h_0(|D_0|^2) = 0, \quad (40)$$

and is solved by the BCS value of the gap D_0 , one can write approximately

$$\begin{aligned}
 h_0(|D|^2) &= h'_0(|D_0|^2) (|D|^2 - |D_0|^2) \\
 &= h_1(|D_0|^2) (|D|^2 - |D_0|^2). \quad (41)
 \end{aligned}$$

Incidentally, we observe that (41) has exactly the original Ginzburg-Landau form.

In order to expand Eq. (37), with or without the replacement (41), in a gauge independent way, it is convenient to express D in terms of its amplitude φ and its phase S

$$D = \varphi e^{iS} \quad (42)$$

and to write (37) as a set of two equations for φ and S . One then expands these two equations, setting

$$\varphi = \varphi_0 + \alpha \varphi_1 + \alpha^2 \varphi_2 + \dots \quad (43)$$

$$S = S_0 + \alpha S_1 + \alpha^2 S_2 + \dots \quad (44)$$

This calculation is performed in Appendix D. The result is that one can fix the gauge so that S_0 vanishes or D_0 is real (London gauge). For D_1 one finds the equation

$$\frac{3}{v_F^2} h_1 |D_1|^2 + 2e^2 A^2 (h_1 + |D_0|^2 h_2) = 0 \quad (45)$$

where the functions h_1 and h_2 are calculated at $|D_0|^2$. Now it is easy to see that, as the temperature approaches zero, the function $h_1 + |D|^2 h_2$ vanishes exponentially. Therefore, for $T=0$, our equation (37) gives no

dependence of the gap upon the magnetic potential. This agrees with the conclusions reached by Nambu and Tuan, and by Tewordt. To obtain a non-vanishing field dependence, one should improve on the equation (37) itself by going to higher orders in the σ expansion. This would introduce in the equation terms involving the gradient of the potential, or the magnetic field. Tewordt has calculated terms of this kind and found agreement with Nambu and Tuan's earlier calculation.

On the other hand, if we had kept only the terms which appear in the first line of (37), which amounts to neglecting the rather unusual A^2 term in the second line, we would have to neglect the h_2 term in Eq. (45). This would give

$$|D_1|^2 = -\frac{2}{3} e^2 v_F^2 A^2 \quad (46)$$

a result in agreement with an older calculation of Gupta and Mathur. In this way we can understand to a certain degree the origin of the disagreement between these authors and Nambu and Tuan. It is interesting to observe that equation (46) does not contain any temperature dependent coefficients.

It is not necessary to emphasize that all results discussed above suffer from the basic limitation due to the expansion in A . Even if one is willing to accept this expansion, it is clear that local superconductivity cannot be valid at very low temperatures where the penetration depth is small and the assumption that the potential is slowly varying will fail. It appears to be rather difficult to give a reliable estimate of the range of validity of the local theory. Werthamer has attempted such an estimate and we refer to his paper ⁴⁾ for a discussion of this point.

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A P P E N D I X A

Expansion of Equations (20) and (21)

Consider first Eq. (20). The integral in the exponent is extended along a straight line and can be written as

$$I_1 = \int_{x'}^x A \cdot dx = \int_{-1}^1 A(R + \tau \frac{\xi}{2}) \cdot \frac{\xi}{2} d\tau. \quad (\text{A.1})$$

Making use of (17) this becomes

$$\int_{-\sigma}^{\sigma} \tilde{A}(\rho + \tau \frac{\xi}{2}) \cdot \frac{\xi}{2} d\tau \quad (\text{A.2})$$

and this expression can be easily expanded in powers of σ , by calculating its successive derivatives. Up to third order inclusive one obtains

$$I_1 = \xi \cdot \left\{ A(R) + \frac{1}{24} (\xi \cdot \nabla_R)^2 A(R) \right\}. \quad (\text{A.3})$$

The corresponding operator in the mixed representation is obtained by means of the substitution

$$\xi \rightarrow -i \nabla_p. \quad (\text{A.4})$$

With this substitution, the operator transforming a propagator like G into the corresponding gauge invariant propagator $\bar{G}$, is given by

$$\exp[i I_1] \quad (\text{A.5})$$

up to third order. The third order contribution is not needed in the main body of this paper: we use it in Appendix C where the propagator is needed up to third order inclusive. Notice that up to second order our result is

$$\exp[iI_1] = \exp(A \cdot \nabla_p) \quad . \quad (A.6)$$

This operator is equivalent to a substitution $p \rightarrow p+A$ and has a simple classical meaning: it can be interpreted as a change of independent variable from the canonical momentum p to the kinetic momentum $p-A$. In third order this simple classical transformation is no longer valid.

The expansion of (21) is performed in the same way. We have

$$\begin{aligned} I_2 &= \int_{x'}^R A \cdot dx + \int_x^R A \cdot dx = - \int_0^1 \left[A(R + \tau \frac{\xi}{2}) - A(R - \tau \frac{\xi}{2}) \right] \cdot \frac{\xi}{2} d\tau \\ &= - \int_0^\sigma \left[\tilde{A}(p + \tau \frac{\xi}{2}) - \tilde{A}(p - \tau \frac{\xi}{2}) \right] \cdot \frac{\xi}{2} d\tau \quad . \end{aligned} \quad (A.7)$$

Expanding in powers of σ one obtains, to second order,

$$\begin{aligned} I_2 &= - \frac{1}{4} A_{\lambda, \sigma} \xi_\lambda \xi_\sigma \\ A_{\lambda, \sigma} &\equiv \frac{\partial A_\lambda}{\partial R_\sigma} \end{aligned} \quad (A.8)$$

A P P E N D I X B

Expansion of the Propagators

In this Appendix we simplify the notation by dropping the subscript λ on propagators. In order to expand Eq. (14) we first observe that one has, exactly, in the mixed representation,

$$K(p, R) = (p - A(R))^2 - \mu \quad (\text{B.1})$$

$$K^*(p, R) = (p + A(R))^2 - \mu. \quad (\text{B.2})$$

We need first the expansion of the operator N defined by

$$NM = I \quad (\text{B.3})$$

$$M = \lambda + K^* \quad (\text{B.4})$$

or, in the mixed representation, according to (27),

$$(\theta_0 + \theta_1 + \theta_2 + \dots)(N_0 + N_1 + N_2 + \dots)(M_0 + M_1 + M_2) = I. \quad (\text{B.5})$$

Identifying terms of equal order, we obtain

$$\theta_0 N_0 M_0 = I \quad (\text{B.6})$$

$$\theta_0 N_0 M_1 + \theta_0 N_1 M_0 + \theta_1 N_0 M_0 = 0 \quad (\text{B.7})$$

$$\theta_0 N_0 M_2 + \theta_0 N_1 M_1 + \theta_0 N_2 M_0 + \theta_1 N_0 M_1 + \theta_1 N_1 M_0 + \theta_2 N_0 M_0 = 0 \quad (\text{B.8})$$

etc., where

$$M_0 = \lambda + p^2 - \mu = \lambda + \omega \quad (\text{B.9})$$

$$M_1 = 2 p \cdot A(R) \quad (\text{B.10})$$

$$M_2 = A^2(R) \quad , \quad (\text{B.11})$$

and there are no higher order terms. Equation (B.6) gives immediately

$$N_0 = \frac{1}{M_0} \quad . \quad (\text{B.12})$$

To proceed further, it is useful to observe the following general properties of the operator

$$\theta_1 = -\frac{i}{2} \left(\nabla_R \cdot \nabla_{p'} - \nabla_p \cdot \nabla_{R'} \right) . \quad (\text{B.13})$$

The first property is : if two functions are related by (B.12), then

$$\theta_1 N_0 M_0 = 0 \quad . \quad (\text{B.14})$$

Consequently, (B.7) gives

$$N_1 = -\frac{M_1}{M_0^2} \quad . \quad (\text{B.15})$$

Now, the second property of the operator θ_1 states that if (B.12) and (B.15) apply, then

$$\theta_1 N_0 M_1 + \theta_1 M_1 M_0 = 0 \quad . \quad (\text{B.16})$$

In our case we also have

$$\theta_2 N_0 M_0 = 0 \quad (\text{B.17})$$

since neither M_0 nor N_0 depend upon R . Consequently

$$N_2 = - \frac{N_0 M_2 + N_1 M_1}{M_0} = - \frac{M_2}{M_0^2} + \frac{M_1^2}{M_0^3} . \quad (\text{B.18})$$

The above-mentioned properties of the operator Θ_1 always come into play when one wants to solve equations like (B.3).

Having obtained N to second order, one can calculate the denominator of the right-hand side of (14) to the same order and then, following the pattern above, its reciprocal. Finally, using Eq. (31), one obtains the gauge invariant propagator $\bar{G}$. The calculation is straightforward, although a little tedious. We give therefore only the result

$$\bar{G}_0 = \frac{\lambda + \omega}{\lambda^2 - \varepsilon^2} \quad (\text{B.19})$$

$$\bar{G}_1 = \frac{-p \cdot [4A|D|^2 + iD^* \nabla_R D - iD \nabla_R D^*]}{(\lambda^2 - \varepsilon^2)^2} \quad (\text{B.20})$$

$$\begin{aligned} \bar{G}_2 = & -\frac{1}{4} \frac{\nabla_R^2 |D|^2}{(\lambda^2 - \varepsilon^2)^2} \left(1 + \frac{4\omega(\lambda + \omega)}{\lambda^2 - \varepsilon^2}\right) - \frac{1}{2} \frac{(\nabla_R |D|^2)^2}{(\lambda^2 - \varepsilon^2)^3} \left(1 + \frac{2\omega(\lambda + \omega)}{\lambda^2 - \varepsilon^2}\right) \\ & - \frac{(\nabla_R + 2iA)D^* \cdot (\nabla_R - 2iA)D}{(\lambda^2 - \varepsilon^2)^2} + \frac{[p \cdot (4A|D|^2 + iD^* \nabla_R D - iD \nabla_R D^*)]^2}{(\lambda + \omega)(\lambda^2 - \varepsilon^2)^3} \\ & - 2(p \cdot \nabla_R)^2 |D|^2 \frac{\lambda + \omega}{(\lambda^2 - \varepsilon^2)^3} \left(1 + \frac{2\omega^2}{\lambda^2 - \varepsilon^2}\right) \\ & + \frac{(p \cdot \nabla_R |D|^2)^2}{(\lambda + \omega)(\lambda^2 - \varepsilon^2)^3} \left(1 - \frac{2(\lambda + \omega)^2}{\lambda^2 - \varepsilon^2}\right) \\ & + \frac{4p \cdot (\nabla_R + 2iA)D^* \cdot p \cdot (\nabla_R - 2iA)D}{(\lambda + \omega)(\lambda^2 - \varepsilon^2)^2} . \end{aligned} \quad (\text{B.21})$$

In a similar way one can expand Eq. (13) and, using Eq. (32), go over to $\bar{F}$. The result is

$$\bar{F}_0 = \frac{D}{\lambda^2 - \varepsilon^2} \quad (\text{B.22})$$

$$\bar{F}_1 = \frac{-2i\lambda p \cdot (\nabla_R - 2iA) D}{(\lambda^2 - \varepsilon^2)^2} \quad (\text{B.23})$$

$$\begin{aligned} \bar{F}_2 = & -\frac{1}{2} \frac{\omega (\nabla_R - 2iA)^2 D}{(\lambda^2 - \varepsilon^2)^2} - \frac{\omega D \nabla_R^2 |D|^2}{(\lambda^2 - \varepsilon^2)^3} - \frac{\omega D (\nabla_R |D|^2)^2}{(\lambda^2 - \varepsilon^2)^4} \\ & - [p \cdot (\nabla_R - 2iA)]^2 D \left[\frac{2}{(\lambda^2 - \varepsilon^2)^2} + \frac{1}{2} \frac{\partial^2}{\partial \omega^2} \frac{1}{\lambda^2 - \varepsilon^2} \right] \\ & - \frac{4 D^* [p \cdot (\nabla_R - 2iA) D]^2}{(\lambda^2 - \varepsilon^2)^3} - \frac{2}{3} D (p \cdot \nabla_R)^2 |D|^2 \left[\frac{2}{(\lambda^2 - \varepsilon^2)^3} + \frac{\partial}{\partial \omega} \frac{\omega}{(\lambda^2 - \varepsilon^2)^3} \right] \\ & - \frac{2 D (p \cdot \nabla_R |D|^2)^2}{(\lambda^2 - \varepsilon^2)^4} \quad (\text{B.24}) \end{aligned}$$

A P P E N D I X C

Landau Diamagnetism

The techniques developed in this paper provide a very simple derivation of the Landau diamagnetism of free electrons; this derivation avoids the complications involved in the study of the energy spectrum in presence of a magnetic field.

For free electrons, we can take $D=0$. Equation (14) then becomes exactly

$$G_{\lambda} = \frac{1}{\lambda - K} \quad (C.1)$$

where K is given by (30). To obtain a non-trivial result, one must expand (C.1) up to third order. Making use of the results of Appendix A, one can then calculate the function $\bar{G}_{\lambda}$ up to third order. Since the procedure is completely straightforward, using the method of Appendix B, we give only the result

$$\bar{G}_{\lambda} = \frac{1}{\lambda - \omega} - \frac{2}{3} \frac{1}{(\lambda - \omega)^3} \beta_{\alpha} \bar{F}_{\alpha\beta,\beta} \quad (C.2)$$

where

$$\bar{F}_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta} \quad (C.3)$$

is the magnetic field. This result can be used to calculate the current density, according to Eqs. (12) and (33). By symmetric integration, the zero order term does not contribute and we obtain

$$J_{\alpha}(R) = -\frac{4e}{9V} \sum_p p^2 f''(\omega) F_{\alpha\beta,\beta} \quad (\text{C.4})$$

This current density corresponds to a susceptibility

$$\begin{aligned} \chi &= -\frac{e^2}{9\pi^2} \int_{-\mu}^{\infty} (\omega + \mu)^{3/2} f''(\omega) d\omega \\ &= \frac{e^2}{3\pi^2} \int_0^{\infty} p^2 f'(\omega) dp \end{aligned} \quad (\text{C.5})$$

which, considering our choice of units, is in agreement with previous calculations¹²⁾.

A P P E N D I X D

Expansion of Equation (37)

If one substitutes Eq. (42) into Eq. (37) and separates the real and imaginary parts, one obtains the two equations

$$h_1 \left[\nabla^2 \varphi - \varphi (\nabla S - 2A)^2 \right] + h_2 \left[\varphi (\nabla \varphi)^2 - \varphi^3 (\nabla S - 2A)^2 + \frac{1}{3} \varphi \nabla^2 \varphi^2 \right] + h_3 \frac{\varphi}{6} (\nabla \varphi^2)^2 - \frac{6}{v_F^2} \varphi h_0 = 0 \quad (\text{D.1})$$

and

$$h_1 \left[\varphi \nabla \cdot (\nabla S - 2A) + 2(\nabla S - 2A) \cdot \nabla \varphi \right] + 2h_2 \varphi^2 \nabla \varphi \cdot (\nabla S - 2A) = 0 \quad (\text{D.2})$$

where the functions h_n depend upon φ^2 . As expected, these equations are invariant under the gauge transformation

$$A \rightarrow A + \nabla \chi \quad (\text{D.3})$$

$$S \rightarrow S + 2\chi \quad (\text{D.4})$$

$$\varphi \rightarrow \varphi \quad (\text{D.5})$$

Now, making use of Eqs. (17), (29), (43), and (44), we can expand in powers of α . To order zero in α , we obtain from (D.1)

$$h_0(\varphi_0^2) = 0 \quad (\text{D.6})$$

after which, since φ_0 is a non-vanishing constant, (D.2) gives

$$\nabla(\nabla S_0 - 2A) = 0. \quad (\text{D.7})$$

To first order in α , (D.1) gives

$$\left[h_1(\varphi_0^2) + \varphi_0^2 h_2(\varphi_0^2) \right] (\nabla S_0 - 2A)^2 + \frac{6}{v_F^2} h'_0(\varphi_0^2) \varphi_1^2 = 0 \quad (\text{D.8})$$

while (D.2) gives

$$h_1[\varphi_0 \nabla^2 S_1 + 2(\nabla S_0 - 2A) \cdot \nabla \varphi_1] + 2h_2 \varphi_0^2 \nabla \varphi_1 \cdot (\nabla S_0 - 2A) = 0 \quad (\text{D.9})$$

and so on.

Clearly the equations generated by this expansion are invariant under the transformation (D.3) and

$$S_0 \rightarrow S_0 + 2\chi \quad (\text{D.10})$$

while the functions S_n ($n \geq 1$) and all the functions φ_n are invariant. If one chooses the London gauge

$$\nabla A = 0, \quad (\text{D.11})$$

it follows that S_0 vanishes. Equation (D.8) then becomes identical to Eq. (45).

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- 6) See, e.g., J.B. Keller and B. Zumino, Phys.Rev. Letters 7, 164 (1961).
- 7) K.K. Gupta and V.S. Mathur, Phys.Rev. 121, 107 (1961).
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- 9) The label f stands for any label characterizing the one-particle state (for instance, momentum plus spin or co-ordinate plus spin) and $I_{ff'}$ denotes the unit matrix. We have used here the notation, introduced in I,

$$A_{f1f2f3f4}B_{f4f3} = (A \cdot B)_{f1f2}$$

$$B_{f1f2}A_{f2f1f3f4} = (B \cdot A)_{f3f4} .$$

The above equations follow immediately from those given in I if one eliminates the unitary matrix C. The reciprocal temperature is denoted by β . [There is a misprint in I in the equation defining G , the correct definition is Eq. (2) above.]7

- 10) We have used the same conventions as in I. In particular $2m\hbar=c=1$ and A denotes the conventional vector potential multiplied by the charge $e = -|e|$. In Eq. (15) one must understand a summation over the spin indices, which are not explicitly indicated.
- 11) W.R. Theis, Z. Physik 142, 503 (1955). See also, G.A. Baraff and S. Borowitz, Phys.Rev. 121, 1704 (1961). Observe, however, that the expansion of the operator Θ defined in (28) is interpreted here as an expansion in the slow variation parameter, rather than an expansion in Planck's constant.
- 12) See, J. Bardeen, L.N. Cooper and J.R. Schrieffer, Phys.Rev. 108, 1175 (1957), formula (C19); our notation differs slightly from that of BCS. Thus $\omega = \frac{p^2}{2m} - \mu$ while ϵ is the quasiparticle energy and the function f is defined in Eq. (11) above.
- 13) See, e.g., E.C. Stoner, Proc. Royal Soc. A152, 672 (1935).