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A B S T R A C T

A version of the one-particle exchange approximation has been suggested by Chew in which the Regge-trajectory nature of the exchanged particle is taken into account. The prescription is explained in detail, and applied to the $\eta\eta$, πN and $\pi\omega$ amplitudes. Comparison is made with earlier one-particle exchange calculations in which the amplitude, even though analytic in the direct channel angular momentum, comes from a force which represents the exchange of an elementary particle. We find the corrections to $\pi\pi$ and πN scattering to be insignificant on the scale of accuracy of such models; but for $\pi\omega$ scattering, the Regge-trajectory nature of the exchanged ρ meson may be sufficient to change the sign of the force in the 1^+ channel from a repulsion into an attraction, sufficient to create a resonance. Some kinematical properties of unequal mass scattering amplitudes are discussed, especially the poles of the partial waves at the origin.

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1. INTRODUCTION

Model dynamical calculations based on the analytic and unitary properties of the S matrix are most conveniently, and most often, done with the partial wave scattering amplitudes. One may write these as

$$A_\ell(s) = A_\ell^L(s) + A_\ell^R(s) = N_\ell(s) D_\ell^{-1}(s) \quad (1.1)$$

where A_ℓ^L and N_ℓ have only "left-hand" cuts. A_ℓ^L is the potential or force, and is in principle determined from the physical t and u channel scattering amplitudes. (Here s , t and u are the standard Mandelstam variables.) Analyticity in s and unitarity are not enough, however; it is well known that even given $A_\ell^L(s)$ or the discontinuities of $N_\ell(s)$, there are ambiguities in the solution related to CDD poles and subtraction constants. In addition, there are ambiguities in the definition of the force A_ℓ^L itself.

The hypothesis that the solutions are analytic in the complex angular momentum plane (maximal analyticity of the second degree) solves, in principle, both these problems: requiring that low ℓ partial waves are continuations from high ℓ eliminates subtraction and CDD pole ambiguities; requiring only moving (Regge) pole singularities in the crossed channel restricts the nature of the forces, as we shall discuss below.

A convenient and sometimes sensible rough approximation to the dynamics is to take A_ℓ^L to be just the contribution from one-particle (or resonance) exchange. Ambiguities in defining this potential $A_\ell^L(s)$ for low ℓ exist when the spin of the exchanged or external particles is greater than $1/2$.

Chew has shown how to choose the one-particle exchange force so as to approximate best ℓ analyticity in both the direct and exchange channels, and has discussed explicitly the ρ exchange contributions to the s wave $\pi\pi$ amplitude ¹⁾. His prescription, in brief, is to set up the ℓ analytic form of the N/D equations, with the force given by the exchange of a moving ρ trajectory ²⁾, and then to eliminate all but the one-particle exchange singularities by taking the limit $\epsilon \rightarrow 0$ of the slope ϵ of the exchanged trajectory. The result is that a term of the form $\delta_{\ell 0}$ is added to the continuation from high ℓ . We explain this more fully below. In the present work we extend the discussion to scattering of spinning particles with unequal masses, and as simple applications redo some old problems with Chew's idea.

We should like to test this prescription by applying it to those one-particle exchange models which can best be expected to give some kind of approximation to the gross features of A_{ℓ}^L at low energies. The two obvious criteria are the dominance of the exchanges considered because of their comparatively low mass and large coupling, and the absence of nearby communicating channels. Three of the "cleanest" problems in this sense are $\pi\pi$ scattering with ρ exchange, πN scattering with N and Δ exchange, and $\pi\omega$ scattering with ρ exchange in those states which do not communicate with $\pi\pi$.

We shall use elastic $\pi\pi$ scattering with ρ exchange to illustrate the approximation, and briefly compare the effect of the $\delta_{\ell 0}$ terms with the prediction of other methods. For πN scattering we find a $\delta_{\ell 0}$ type term which makes only a small repulsive correction to the Δ exchange contribution in the $P_{1/2}$ channel. The nucleon may still be obtained in this model as a πN bound state. For the $\pi\omega$ amplitude, however, the addition of the delta function term may change a repulsive force in the 1^+ state into an attractive one. Thus the possibility of a pseudovector $\pi\omega$ resonance, perhaps the B ³⁾, may be closely connected to the movement of the ρ pole in the ℓ plane.

In Section 2 we explain Chew's prescription for the $\pi\pi$ problem. In Section 3 we discuss the difficulties which appear in the extension to unequal mass, spinning external particles. We find, in particular, some ambiguities in the extension of Chew's approximation arising from the more complicated structure of the kinematic singularities. In Section 4 we present the results of our computations for πN and $\pi\omega$ scattering.

2. PION-PION SCATTERING

Let us begin by recalling the version of the N/D formalism for the $\pi\pi$ amplitude which guarantees ℓ analyticity of the solution whenever the force is ℓ analytic. With the normalization ⁴⁾

$$t_\ell(s) = \frac{v}{q} e^{i\delta(s)} \sin \delta(s) \quad (2.1)$$

one writes

$$\bar{t}_\ell(s) \equiv t_\ell(s)/q_s^{2\ell} = N_\ell(s)/D_\ell(s) \quad (2.2)$$

A form of the N/D equations, which allows continuation in ℓ , is

$$D_\ell(s) = 1 - \frac{1}{\pi} \int_R \frac{P_\ell(s') N_\ell(s') ds'}{s' - s} \quad (2.3a)$$

4.

$$N_l(s) = B_l^V(s) + \frac{1}{\pi} \int_R \frac{B_l^V(s') - B_l^V(s)}{s' - s} \rho_l(s') N_l(s') ds' \quad (2.3b)$$

where $\rho_l(s) = q_s^{2l+1}/W$ and the potential B_l^V may be written in the form of Wong ⁵⁾

$$B_l^V = \frac{1}{2q_s^{2l+2}} \int_0^\infty [Im Q_l(1+t/2q_s^2)] \tilde{A}(s,t) dt - \int_R \frac{disc(s')}{s' - s} ds' \quad (2.4)$$

+ same thing for the u channel,

where

$$\tilde{A}(s,t) = \frac{1}{\pi} \int_{t_0}^\infty \frac{D_t(t',s)}{t' - t} dt' \quad (2.5)$$

In Eqs. (2.3) we have written the integrals over an unspecified right-hand cut R , since we are not concerned here with complications arising from possible high energy divergences. Our remarks on the potential are applicable to any method which takes care of these divergences, for example, the strip approximation ²⁾, or even the crude cut-off model we use in Section 4 for numerical computations. In Eqs. (2.4) and (2.5), $D(t',s)$ is the absorptive part in the t channel at fixed s , and $disc(s')$ is the discontinuity of the first term across the cut R .

What is the one ρ exchange contribution to B_l^V ? If one makes the approximation

$$D_t(s, t) = \frac{g^2}{2\pi} q_p^2 \delta(t - t_p) P_l(1 + s/2q_p^2) \quad (2.6)$$

where

$$\frac{g^2}{4\pi} = 3 m_p^2 \Gamma / 2 q_p^3$$

with Γ the full width at half maximum then the quantity $\tilde{A}(s, t)$ is given by

$$\tilde{A}(s, t) = \frac{g^2}{2\pi^2} q_p^2 \frac{P_l(1 + s/2q_p^2)}{t - t_p} \quad (2.7)$$

where

$$q_p^2 = \frac{m_p^2}{4} - m_\pi^2, \quad \text{and} \quad B_l^V(s) = B_l^F(s)$$

with

$$B_l^F(s) = \frac{g^2}{8\pi q_p^{2l+2}} (s + 2q_p^2) Q_l(1 + t_p/2q_p^2) \quad (2.8)$$

6.

B^σ is manifestly smooth in ℓ . It has been argued that $B_\ell^\sigma(s)$ is certainly correct for high ℓ and is therefore the potential one should use for all ℓ .

On the other hand, one might think of taking for the quantity $\tilde{A}(s,t)$ the amplitude for the one ρ exchange Feynman graph with the conserved current $\pi\pi\rho$ vertex $g[p^a + p^b]_\mu P^\mu$, where p^a and p^b are the momenta of the two pions and P^μ is the polarization vector of the ρ ⁶⁾. In that case,

$$\tilde{A}(s,t) = \frac{g^2}{4\pi} g_t^2 \frac{P_1(1 + s/2g_t^2)}{t_\rho - t} \quad (2.9)$$

and then from Eq. (2.4)

$$B_\ell^V = B_\ell^F \equiv B_\ell^\sigma - \frac{g^2}{8\pi} \delta_{\ell 0} \quad (2.10)$$

It appears at first glance that ℓ analyticity requires the potential B_ℓ^V to be analytic in ℓ at $\ell=0$ and hence that $B_\ell^V = B_\ell^\sigma$ is consistent with this principle while $B_\ell^V = B_\ell^F$ is not. Chew has discovered that the opposite result is obtained when the principle of ℓ analyticity is applied approximately in the crossed channel, as follows: D_t as given in Eq. (2.6) is the absorptive part for elementary ρ exchange in the sense that it represents the exchange of a stationary ℓ plane pole. But unitarity and ℓ analyticity imply that the pole must move in the ℓ plane, and for consistency one should consider one-trajectory exchange instead of one-particle exchange. Equation (2.6) is then replaced by⁷⁾

$$D_t(t,s) = \text{Im} \left\{ \frac{2\alpha(t)+1}{\sin \pi\alpha(t)} \beta(t) P_{\alpha(t)}(-1 - s/2 q_t^2) \right\} \quad (2.11)$$

As long as $\alpha(t)$ and $\beta(t)$ have no singularities other than the normal physical cut from t_0 to ∞ , one can obtain $\tilde{A}(s,t)$ simply by removing the symbol "Im" from Eq. (2.11). We find one-particle exchange as an approximation to Eq. (2.11) by looking at the limit in which α becomes $\alpha = 1 + \varepsilon(t-t_p)$, $\varepsilon \rightarrow 0$. Thus with

$$\beta(t) = (-q_t^2)^{\alpha(t)} \gamma(t) \quad (2.12)$$

an approximation to Eq. (2.11) is

$$\tilde{A}(s,t) = \frac{3\gamma(t)[q_t^2/\alpha'(t)]}{t_p - t} P_1(-1 - s/2 q_t^2) \quad (2.13)$$

If now $\gamma(t)$, which has only dynamic singularities, is approximated by a constant, the approximation to the Reggized B_{ℓ}^V , which we call B_{ℓ}^R , is the same as B_{ℓ}^F with $3\gamma(t_p)/\alpha'(t_p) = g^2/2\pi$. Thus, the flat trajectory limit of trajectory exchange gives a $\delta_{\ell 0}$ term. This Kronecker delta appears in B_{ℓ}^R and B_{ℓ}^F because in both the perturbation theory and the Regge theory the s dependence of $\tilde{A}(s,t)$ changes as t moves from the pole at t_p , whereas in the approach of Eq. (2.6) the s dependence is simply

$P_1(1+s/2q_s^2)$ for all t . [In contrast, for spin zero exchange the resulting B_l^R and B_l^F agree, since $P_0(z)$ is independent of z .] The identity between B_l^R and B_l^F is not necessarily maintained in the more complicated problems we discuss below.

The apparent violation of l analyticity in the direct channel is a result only of the approximation. Away from the flat trajectory limit the term in B_l^R is replaced by a function analytic in l and peaked at $l=0$ (with terms that look like, for example,

$$\frac{1}{q_s^2} J_l(\alpha'(t_p)/q_s^2) \quad).$$

Thus there is no contradiction between l analyticity in the s and t channels in the correct theory. Notice that we thus have another criterion for the usefulness of one-particle exchange models. They are valid only to the extent that $\alpha'(t)$ and $\beta'(t)$ are small.

The δ_{l0} terms are proportional, like the rest of the potential, to the $\pi\pi\rho$ coupling constant g^2 . If the theory had an expansion in powers of g^2 , l analyticity would imply that each term must be analytic in l , and we would have a contradiction, since presumably the flat trajectory approximation would just identify the g^2 term in the expansion of the correct force. But the theory we are talking about, as well as the πN and $\pi\omega$ examples we discuss below are not expandable in the coupling constant; rather, g^2 is fixed by the condition that exchange of a Regge trajectory produce Regge trajectories in the direct channels which satisfy crossing symmetry, i.e., by the bootstrap condition. But now imagine applying Chew's prescription to vector-spinor scattering with spinor exchange. It follows from the work of Gell-Mann et al. ⁸⁾ that the amplitude given by field theory for this problem may be l analytic in both channels for any g^2 . This theory

already contains delta functions of order g^2 , in such a way as to be consistent with ℓ analyticity. The δ functions of the type we are discussing here must be included in these. It would be interesting to investigate the exact mechanism in detail.

We conclude this section with a brief discussion of this $\delta_{\ell 0}$ contribution in $\pi\pi$ scattering. Numerical calculations of the Chew-Jones strip approximation form of the potential for ρ trajectory exchange are being performed⁹⁾. Preliminary results show the order of magnitude of the smearing of the $\delta_{\ell 0}$ contribution when the exchanged ρ Regge trajectory is sufficiently non-flat so that $\alpha(\infty)$ is about 0.5; the $\delta_{\ell 0}$ is then replaced by a contribution, $d(\ell)$, appreciable in the range $\ell = -0.2$ to $\ell = +0.2$.

We may also see the effect of the contribution $d(\ell)$ to the $\pi\pi$ potential on the output trajectories by examining the N/D Eqs. (2.3). Since $d(\ell)$ is repulsive, the potential $B_{\ell}^{\sigma} + d(\ell)$ yields trajectories with larger α' in the neighbourhood of $\alpha = 0$ than the potential B_{ℓ}^{σ} . This is simply because, for fixed ℓ , the bound state energy is less for the more attractive B_{ℓ}^{σ} . To the extent that $d(\ell)$, like $\delta_{\ell 0}$, is constant in s , $\alpha(\infty)$ is the same for the two potentials. This follows from the fact that $\alpha(\infty)$ is that ℓ for which there is a solution of the homogeneous equation associated with the Fredholm integral equation (2.3a) for N ; the kernel $B_{\ell}^V(s') - B_{\ell}^V(s)$ is not affected by a constant, in s , contribution to B_{ℓ}^V .

3. KINEMATIC CONSIDERATIONS

To consider problems with spinning particles and unequal masses, we require a generalization of Eq. (2.12) of the form

$$\beta(s) = f(s) \gamma(s) \quad (3.1)$$

where $f(s)$ contains kinematic factors and $\gamma(s) = \text{const}$ is a reasonable approximation in the flat trajectory limit ($\alpha \rightarrow \ell$). In order to study the variation of the residue function $\beta(s)$ in this limit, we review here its analyticity and threshold properties, and the unitarity, analyticity, and threshold conditions for the ℓ 'th partial wave (which we wish to replace by just the pole contribution).

We begin by discussing the singularities to be expected in partial wave amplitude when the external particles have unequal masses. The trouble comes from the relation between the momentum and the energy squared variable:

$$q^2 = (s - s_t)(s - s_u)/(4s) \quad (3.2)$$

Here $s_t = (m_1 + m_2)^2$, $s_u = (m_1 - m_2)^2$, q^2 has a zero at $s = s_u$ and a pole at $s = 0$, both of which are absent in the equal mass case.

What poles may the partial waves have at $s = 0$? Consider first spinless (e.g. πK) elastic scattering. The invariant amplitude $A(s, t, u)$ must be regular at $s = 0$ for fixed t or u . The partial waves are defined as

$$\begin{aligned}
 t_\ell(s) &= \frac{\pi}{2} \int_{-1}^1 P_\ell(z) A(s,t) dz \\
 &= \frac{\pi}{2} \int_0^{\frac{-4q_s^2}{2q_s^2}} P_\ell(z_s(s,t)) A(s,t) dt
 \end{aligned} \tag{3.3}$$

where

$$z_s(s,t) = 1 + t/2q_s^2$$

As s increases to zero, $-4q_s^2$, the upper limit of integration in Eq. (3.3) tends to ∞ so that the behaviour of $t_\ell(s)$ at $s = 0$ is determined by the high t behaviour of the amplitude $A(s,t)$. The Froissart bound¹⁰⁾ for $A(\sim t \log^2 t)$ eliminates double, and higher order, poles at $s = 0$, but allows simple poles and branch points. As s decreases to zero the high u behaviour enters similarly with the same result.

In any case these singularities fall on the left-hand cut, which crosses $s = 0$ into the positive s axis. Therefore we may probably think of them as dynamical, and not have to put them in at the outset.

When the external particles have spin, there are more poles at $s = 0$ which are certainly "kinematical", i.e., not associated with the discontinuity of the invariant function. In the helicity representation, Hara¹¹⁾ has shown that if λ, μ are the initial and final differences of helicities, there is a pole in $A_{\lambda\mu}(s,t,u)$ at $s = 0$, for fixed t or u , of order $|\lambda - \mu|/2$. These certainly show up in the partial waves as well; we thus expect an $s^{-|\lambda - \mu|/2}$ in the partial wave helicity amplitudes.

The partial waves we consider here are not helicity amplitudes, but sums and differences of them of the form

$$t_{\lambda\mu}^{j\pm} = t_{\lambda\mu}^j \pm t_{-\lambda\mu}^j$$

which are the amplitudes between states of definite parity and magnitude of helicity. Both $t_{\lambda\mu}^{j+}$ and $t_{\lambda\mu}^{j-}$ will have poles of order $(|\lambda| + |\mu|)/2$. But there is a restriction on the residues: a certain linear combination of them must have a singularity of order $(|\lambda| - |\mu|)/2$. In the flat trajectory prescription for Eq. (3.3) we wish to construct a contribution from only one partial wave in the crossed channel, and set all the others equal to zero. This will be possible only if we allow no greater singularity at $s = 0$ than $(|\lambda| - |\mu|)/2$.

These remarks hold for bosons. For fermions one must consider functions of W . The amplitude for negative real W represents the amplitude for positive W in a state of opposite parity. For fermions, the extra constraint on poles in linear combinations of partial wave amplitudes is a consequence of the generalized MacDowell reflection principle¹²⁾ applied at $W = 0$! Therefore, our partial wave amplitudes, which must satisfy the MacDowell type condition, may have kinematical poles in W of order $|\lambda| + |\mu|$ at $W = 0$.

In addition to the singularities at the origin, we want to factor out the threshold behaviour. This is not so unambiguous as in the equal mass case, because $q^2(s)$ vanishes in two places, s_t and s_u . The partial wave amplitude certainly behaves like $(s-s_t)^{2\ell}$ in the vicinity of the physical threshold. Some graphs, including one-particle exchange graphs, also have the $q^{2\ell}$ behaviour at $s = s_u$, but, as has been shown by Frye and Warnock, graphs which contain a u channel discontinuity extending to the u channel threshold will behave like $(s-s_u)^{1/2}$ at $s \approx s_u$ ¹³⁾.

This can be seen by taking the u channel contribution to the projection of the s channel partial wave from the fixed s dispersion relation:

$$\frac{1}{\pi} \int_{u_t}^{\infty} \frac{D_u(s, u')}{2q^2} Q_l \left(1 + \frac{s_t + s_u - s - u'}{2q^2} \right) du' \quad (3.4)$$

Near the u channel threshold u_t , D_u is proportional to $(u' - u_t)^{1/2}$, so there is a contribution

$$\frac{1}{\pi} \int_{u_t}^{u_t + \epsilon} \frac{(u' - u_t)^{1/2}}{2q^2} Q_l \left(1 + \frac{s_t + s_u - s - u'}{2q^2} \right) du' \quad (3.5)$$

where ϵ is small and fixed. The remaining contribution behaves like $(s - s_u)^{2l}$. The argument of the Q_l in (3.5) does not become infinite uniformly as $s \rightarrow s_u$ throughout the range of integration, since $u_t = s_t$. Indeed, letting $y = (u' - u_t)/2q^2$ we may transform (3.5) into

$$\frac{\sqrt{2}}{\pi} q \int_0^{-\epsilon/2q^2} y^{1/2} Q_l \left(1 + \frac{2s}{s_t - s} + y \right) dy \quad (3.6)$$

from which one sees directly that near $s = s_u$, $A_l \sim C_l q$ for $l \geq 1/2$ with

$$C_l = \frac{\sqrt{2}}{\pi} \int_0^{\infty} y^{1/2} Q_l \left(1 + \frac{2s_u}{s_t - s_u} + y \right) dy \quad (3.7)$$

We now turn to the behaviour of the residue function $\beta(s)$. Cheng and Sharp have shown that the functions $\alpha(s)$ and $\gamma(s) = \beta(s)/q^{2\alpha(s)}$ have only the right-hand cuts¹⁴⁾. Thus $\beta(s)$ has a branch point of the form $(s-s_u)^{\alpha(s)}$ at s_u but not the square root branch point which is found in the partial waves. Their argument is easily generalized to problems with spins. One cannot, however, show in this way that $\gamma(s)$ is analytic at $s=0$, and therefore poles at the origin seem to be possible. It is certainly analytic in any annulus about $s=0$, and so cannot have a branch point there.

Let us see how this works out for πK scattering with $1^-(K^*)$ exchange in the u channel. This is the unequal mass generalization of the example in the previous section. If $\alpha(u)$ and $\beta(u)$ are the parameters of the K^* trajectory then its contribution to the absorptive part in the u channel is

$$D_u(s, u') = \text{Im} \left\{ \frac{2\alpha(u') + 1}{\sin \pi \alpha(u')} \beta(u') P_{\alpha(u')} \left(-1 - \frac{s_2 + s_u - s - u'}{2 q_u^2} \right) \right\} \quad (3.8)$$

Now $\beta(u) = q^{2\alpha(u)} \gamma(u)$ where $\gamma(u)$ has no kinematical singularities at $u=0$. This time we cannot simply remove the "Im" to get $\tilde{A}(s, u)$, since $\beta(u)$ has singularities outside the region of u' integration; this is true even in the flat trajectory limit provided it is $\gamma(u)$ which we approximate by a constant, because of the pole at $u=0$ in q_u^2 .

It is simple to see what happens in this case, however. The quantity

$$\tilde{A}(s, u) = \frac{1}{\pi} \int_{u_0}^{\infty} \frac{D_u(s, u')}{u' - u} du' \quad (3.9)$$

is just equal to the expression in braces in Eq. (3.8) minus any singularities this has outside the limit of integration. Before this limit is taken, $\beta(u)$ presumably decreases fast enough to ensure convergence of (3.9), so the result for $\tilde{A}(s,u)$, in the flat limit, is

$$\tilde{A}(s,u) = \frac{g^2}{2\pi^2} g_u^2 P_1\left(-1 - \frac{s_t + s_u - s - u}{2g_u^2}\right) \frac{1}{m^{*2} - u} \quad (3.10)$$

minus the pole of this expression at $u = 0$ for fixed s . Here we have set

$$\alpha(u) \approx 1 + \varepsilon(u - m^{*2}),$$

$$3\gamma(u)/\varepsilon \approx g^2/2\pi^2$$

and then let $\varepsilon \rightarrow 0$ as before.

Near $u = 0$ the pole in (3.10) is

$$-\frac{g^2}{2\pi^2} \frac{s_t s_u}{4m^{*2}} \frac{1}{u} \quad (3.11)$$

so that

$$\begin{aligned} \tilde{A} &= -\frac{g^2}{2\pi^2} \frac{1}{m^{*2} - u} \left\{ g_u^2 + \frac{s_t + s_u - s - u}{2} - \frac{s_t s_u}{4u} + \frac{s_t s_u}{4m^{*2}} \right\} \\ &= -\frac{g^2}{4\pi^2} \frac{1}{u - m^{*2}} \left[2g_{m^*}^2 + s + \frac{s_t s_u}{m^{*2}} \right] - \frac{g^2}{16\pi^2} \end{aligned} \quad (3.12)$$

and the potential in the s channel partial waves is

$$B_l^V(s) = \frac{g^2}{8\pi q^{2l+2}} \left(5 + 2q^{*2} - \frac{s_z s_u}{2m^{*2}} \right) Q_l \left(1 + \frac{s_z + s_u - s - m^{*2}}{2q^2} \right) - \frac{g^2}{8\pi} \delta_{l0} \quad (3.13)$$

Here $q^{*2} = (m^{*2} - s_t)(m^{*2} - s_u)/4m^{*2}$. As in the $\pi\pi$ problem exactly the same answer is obtained by projecting a Feynman graph with a vertex of the form $g \int \bar{p}_\pi + p_K \gamma K_\mu^*$; and again, the term describes approximately the s dependence of the force away from $u = m^{*2}$.

But surely there is something arbitrary about this procedure of subtracting out the pole at $u = 0$. In effect, we have put in a P wave in the u channel which violated the analyticity conditions by having a pole at $s = 0$, and then fixed it up by adding the proper amount of u channel s wave.

One alternative might be simply to multiply our original guess by u/m^{*2} :

$$\tilde{A}(s, u) = \frac{g^2}{4\pi} \frac{u q^2}{m^{*2}} P_1 \left(-1 - \frac{s_z + s_u - s - u}{2q_u^2} \right) \frac{1}{m^{*2} - u} \quad (3.14)$$

This form does not differ much from the first one near $u = m^{*2}$. Of course, it violates unitarity for high u worse than (3.12) does, but has only P wave in the u channel. The potential derived from (3.14) is different from that from (3.12) but since some chopping up is required in any case, it is not clear that it is necessarily worse. We may prefer (3.12) to (3.14) because

in the equal mass case (3.13) goes over into the potential of Section 2 while (3.14) does not. Nevertheless, the extent to which (3.12) and (3.14) lead to different answers for the complete amplitude is a measure of the ambiguity inherent in the flat trajectory approximation.

Finally, we might have tried to approximate $A(s,u)$ in the physical u region by a form without $(u-s_u)^{\ell}$ appearing explicitly, since we know the amplitude does not really vanish linearly at $u = s_u$. A pole at $u = s_u$ then has to be subtracted from A as was the pole at $u = 0$ in the first method we described. We do not discuss this method in the following examples; the result is always similar or identical to the analogue of Eq. (3.12). More importantly, the $(u-s_u)^{\ell}$ behaviour is probably a good approximation to the partial wave amplitude, since resonance and particle exchanges do not contribute to the $(s-s_u)^{1/2}$ term.

4. $\pi - \omega$ SCATTERING

We discuss first the 1^+ channels, in which the Kronecker delta terms may play a crucial role, and then compare the force in this channel with that in the 2^- and 2^+ states.

The partial wave amplitude for $\pi\omega$ scattering with parity $(-1)^j$ is a single function $t^{j-}(s)$. The parity $-(-1)^j$ amplitude is a 2×2 matrix t^{j+} whose elements may be chosen to be the parity-conserving helicity amplitudes

$$t_{11}^{j+}, \quad t_X^{j+} = t_{01}^{j+} = t_{10}^{j+}, \quad \text{and} \quad t_{00}^{j+}$$

The subscripts refer to the magnitude of the helicity of the ω . According to our remarks in Section 3, $t_{00}^{j+}(s)$ and $Wt_{01}^{j+}(s)$ are analytic in s with no (kinematical) pole at $s = 0$. $t_{11}^{j\pm}$ have single poles, but the residues are not independent. See the appendix for details.

If we just factor out $(q^2)^\alpha$ from exchange of a ρ trajectory in the u channel, then we get, in the flat trajectory limit, before fixing up the $u = 0$ behaviour

$$t_{11}^{1-}(u) = -\frac{h^2}{4\pi} \frac{m_\rho^2}{3m_\pi^2} \frac{q_u^2}{u - m_\rho^2} \quad (4.1)$$

and all other $t^{j\pm}(u)$ vanish. As in the πK problem, we may either subtract out the resulting pole in the invariant amplitudes or else multiply (4.1) by u/m_ρ^2 . Unlike the πK problem, however, we cannot say that one procedure is simpler than the other in the equal mass case, $m_\pi = m_\rho$; even then, formula (4.1) as it stands violates a condition on the $t^{j\pm}$ at $s = 0$ [Eq. (A.9)] and results in illegal singularities in the invariant amplitudes.

First let us simply multiply out the pole in (4.1) by taking

$$t_{11}^{1-}(u) = -\frac{h^2}{4\pi} \frac{u}{3m_\pi^2} \frac{q_u^2}{u - m_\rho^2} \quad (4.2)$$

The potential is then zero in all other u channel states. This is the same as the amplitude obtained from perturbation theory with an elementary $\pi\rho\omega$ vertex

$$\frac{h}{m_\pi} \omega^\lambda \rho^\sigma p_\rho^\mu p_\omega^\nu \epsilon_{\mu\nu\lambda\sigma}$$

Experimentally h is determined from the ω decay width to be ¹⁵⁾

$$\frac{h^2}{4\pi} \approx 0.35$$

The projection of this amplitude into s channel partial waves, which we will take for our potential, is ¹⁶⁾

$$\begin{aligned} B_{00}^{j+A} &= \frac{h^2 m_\omega^2}{16\pi m_\pi^2} \left\{ (1-x^2) Q_j(x) + x \delta_{j0} + \frac{1}{3} \delta_{j1} \right\} \\ B_x^{j+A} &= \frac{h^2 m_\omega}{16\pi m_\pi^2} \left\{ -(E_\pi + x E_\omega) \frac{\sqrt{j(j+1)}}{2j+1} [Q_{j-1}(x) - Q_{j+1}(x)] \right. \\ &\quad \left. + \frac{\sqrt{2}}{3} E_\omega \delta_{j1} \right\} \quad (4.3) \\ B_{11}^{j+A} &= -\frac{h^2}{16\pi m_\pi^2} \left\{ (s - m_\pi^2 - m_\omega^2 - q^2 + E_\omega^2 x) [(j+1) Q_{j-1}(x) \right. \\ &\quad \left. + j Q_{j+1}(x)] / (2j+1) + [q^2(1-x) + m_\pi^2] Q_j(x) - \frac{2}{3} E_\omega^2 \delta_{j1} \right\} \end{aligned}$$

with

$$x = 1 + (s_t + s_u - m_\rho^2 - s) / 2q^2$$

There are terms smooth in j which are attractive in the $0^-, 2^-, \dots$ etc., states and repulsive in the $1^+, 3^+, \dots$ etc., states. In addition, there are δ function terms for $j = 0$ and $j = 1$. The δ functions for $j = 1^+$ exactly cancel the smooth-in- j terms at threshold, leaving an attractive potential with P wave type threshold behaviour.

We have estimated the effect of the potential in the 1^+ state by using the elementary method $t^{1+} = B^{1+}/D$ where

$$D = 1 - \frac{s - s_0}{\pi} \int_{s_t}^{s_1} \frac{q'}{W'} \frac{B^{1+}(s')}{(s' - s)(s' - s_0)} ds' \quad (4.4)$$

and choosing s at the left end of the short cut. The cut-off s_1 was varied from about $(2 \text{ GeV})^2$ to $(4 \text{ GeV})^2$, the position of the strip width in the strip approximation, and $h^2/4\pi$ from about 0.2 to 0.6. This is just the same calculation as in Reference ¹⁶⁾; we have provided some justification for it by considering Regge trajectory exchange. There is indeed a zero of $\det D$, hence a resonance, within the stated range of $h^2/4\pi$. There seems to be a numerical error in the values given in Reference ¹⁶⁾; for given $h^2/4\pi$ and s_1 , our present computations give a resonance position rather higher than those quoted there. For example, for $h^2/4\pi = 0.6$ and $s_1 = (3 \text{ GeV})^2$ we find a zero of $\det D$ at 1.6 GeV. We may expect the position of the zero of $\text{Re } \det D$ to be lower in the solution of the full N/D equation than is given by our approximation $N = B^V$, since $B_{\ell=0}^V = 0$ in Eq. (2.3b), being an increasing function, implies a positive contribution from the neglected integral term.

Even though this method agrees with the Feynman diagram method, with the simplest coupling, we do not have any reason to think it superior to the analogue of the first method we used for scattering in Section 3. Thus, we

may factor out only q_u^2 , i.e., take (4.1) for the force, and then subtract out the resulting poles in the invariant functions. Because of the conditions mentioned in the appendix which are not fulfilled by (4.1) such subtractions will have to be made even in the equal mass case. In this method, the 1^+ potentials in the s channel are slightly different from (4.3). They are $B^{1+} = B^{1+A} + \Delta B^{1+}$ where

$$\Delta B_{\infty}^{1+} = \frac{h^2}{32\pi m_{\pi}^2} \frac{1}{3} \left\{ \frac{q^2}{m_{\omega}^2} (s - 2m_{\omega}^2) + s \right\}$$

$$\Delta B_x^{1+} = \frac{h^2}{32\pi m_{\pi}^2} \frac{\sqrt{2}}{3} \frac{E_{\omega} s}{m_{\omega}} \quad (4.5)$$

$$\Delta B_{II}^{1+} = \frac{h^2}{32\pi m_{\pi}^2} \frac{2}{3} (s + 2g^2)$$

All of the expressions (4.5) are negative near threshold, so the potential is repulsive, although not as much so as the smooth-in- j term alone. In any case, we do not expect a potential which is not very small near threshold to be able to produce an s wave resonance.

Thus it appears that we cannot reach a firm prediction for the sign of the ρ trajectory-exchange force in 1^+ state without knowing more details about the behaviour of $\beta(u)$, even though we can argue that one-particle exchange and elastic unitarity are reasonable approximations. It is clear, however, that replacing elementary ρ exchange by a Regge trajectory makes the 1^+ force less repulsive. Even though we cannot predict that the B meson should be 1^+ , if it is, it can only be because the ρ lies on a moving trajectory.

We have also looked at the 2^- potential, which contains no delta function, and is therefore just the Regge term. The force is indeed attractive, as Peierls has shown in the static limit ¹⁷⁾, but a relativistic calculation by the same B/D method with the same range of s_1 and $h^2/4\pi$ as above shows the force to be too weak to produce a resonance, due to the relatively (compared with πN) small distance above threshold over which the static limit form is correct; indeed, the P wave (2^-) potential eventually becomes repulsive. For completeness, we note that the ρ exchange force is repulsive as well in the 2^+ state for which $\pi\omega$ is the lowest $J^P_{I^G} = 2^+_{1^+}$ strong threshold.

5. THE πN AMPLITUDE

The force due to Δ exchange in πN scattering will have delta functions of the type we have been discussing in the $j = \frac{1}{2}$ amplitude; we must ask whether putting the Δ on a Regge trajectory approximately in this way affects the conclusion that Δ exchange can provide the force to make a nucleon of $\pi + N$.

Consider the usual amplitudes $f_1^I(W)$ and $f_2^I(W)$ for πN scattering. The Regge trajectories must be constructed in such a way as to preserve the reflection rule $f_1(W) = -f_2(-W)$. The contribution of a given trajectory to f_1 and f_2 has been worked out by Singh, Gribov and Dombey ¹⁸⁾. In addition to a factor $(q^2)^{\alpha(W) \pm \frac{1}{2}}$ in the relation between $\beta(W)$ and an analytic function $\gamma(W)$, factors $(E \pm M)/W$ appear explicitly. In the flat trajectory limit, we obtain for the f_i in the u channel, for exchange of the Δ trajectory

$$f_1^{3/2}(W_u) = \frac{M^* \Gamma q_u^2}{2 q_u^3 W_u} \frac{1}{E^* + M} \left\{ \frac{E_u - M}{W_u + M^*} - 3 z_u \frac{E_u + M}{W_u - M^*} \right\}$$

$$f_2^{3/2}(W_u) = \frac{M^* \Gamma q_u^2}{2 q_u^3 W_u} \frac{1}{E^* + M} \left\{ \frac{E_u + M}{W_u - M^*} - \frac{E_u - M}{W_u + M^*} 3 z_u \right\} \quad (5.1)$$

The f 's are normalized to $e^{i\sigma} \sin \sigma / q$. According to our discussion in Section 3 the f_1 may have the double pole from $(E_u + M)/W_u$ at $W_u = 0$ but the extra $1/u$ from q_u^2 violates the Mandelstam representation.

To calculate with this force, we have proceeded as in our first πK example, by constructing the invariant amplitudes from f_1^I and subtracting out the illegal pole at $u = 0$. Projecting the partial waves we obtain in this way into the s channel, we find

$$\begin{aligned} f_{l\pm}^I(W) &= \frac{C^I \Gamma M^*}{12 q^* q^2} \left\{ \left[\frac{2M - M^* - W}{E^* - M} + \frac{3 z_u(s)(W - 2M - M^*)}{E^* + M} \right] \right. \\ &\quad \times (E + M) Q_l(x) + \left[\frac{M^* - 2M - W}{E^* - M} + \frac{3 z_u(s)(2M + W + M^*)}{E^* + M} \right] \\ &\quad \times (E - M) Q_{l\pm 1}(x) \\ &\quad \left. + \frac{q^2}{q^{*2}} \frac{1}{W} \frac{1}{E^* + M} \left[(E + M)(M^* - 2W + 2M) \delta_{l0} \right. \right. \\ &\quad \left. \left. - (E - M)(2W + 2M + M^*) \delta_{l\pm 1,0} \right] \right\} \quad (5.2) \end{aligned}$$

Here $c^{1/2} = 4$, $c^{3/2} = 1$, Γ = width of Δ , $Z_u = 1 + \frac{S_t + S_u - S - M^{*2}}{2g^{*2}}$,

$$X = 1 + \frac{S_t + S_u - S - M^{*2}}{2g^2}$$

and the rest of the notation is standard.

In the nucleon channel $f_{1-}^{1/2}$, the smooth in b term is attractive. Again, the delta function is of opposite sign, and lessens the attraction, but only by about 15% near threshold and less above. We conclude that the calculation of the N as a πN bound state will not be qualitatively affected by the inclusion of the Kronecker delta terms which result from the approximate "Reggeization" of the exchanged Δ .

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APPENDIX

We recall here a convenient formalism for $\pi\omega$ amplitudes.

Let the initial and final pion four-momenta be p^μ and p'^μ , and the ω four-momenta and polarization vectors be q^μ , e^μ , q'^μ , e'^μ . The states may be defined by the pion and omega momenta, and the omega helicity $\mu = 0, \pm 1$. We normalize the states to delta functions in three-momentum and Kronecker deltas in helicity (or spin direction). The S matrix $S(s, t, u)$ is a 3×3 matrix, and we write

$$S = \delta_3(p - p') \delta_3(q - q') + \frac{2i \delta_4(p + q - p' - q')}{4 \sqrt{p^0 q^0 p'^0 q'^0}} T \quad (A.1)$$

T is Lorentz invariant and can therefore be written ¹⁹⁾

$$\begin{aligned} T = & T_1 (e'^* \cdot e) + T_2 (e \cdot q') (e'^* \cdot q) \\ & + \frac{T_3}{2} [(e \cdot p) (e'^* \cdot q) + (e \cdot q') (e'^* \cdot p)] \\ & + T_4 (e'^* \cdot p') (e \cdot p) \end{aligned} \quad (A.2)$$

The T_i are the scalar invariant functions. They satisfy the crossing relations

$$T_1(s, t, u) = T_1(u, t, s)$$

$$T_2(s, t, u) = T_2(u, t, s) + T_3(u, t, s) + T_4(u, t, s)$$

$$T_3(s, t, u) = -T_3(u, t, s) - 2T_4(u, t, s) \quad (\text{A.3})$$

$$T_4(s, t, u) = T_4(u, t, s)$$

Let us call the matrix elements of T between given helicity states $T_{\mu'\mu}$. In the centre of mass, the corresponding cross-section is

$$\frac{d\sigma_{\mu'\mu}}{d\Omega} = \frac{\pi^2}{s} |T_{\mu'\mu}|^2 \quad (\text{A.4})$$

Introduce, with $z = \cos \theta$

$$T_{01}' = T_{01} / \sin \theta \quad (\text{A.5})$$

$$T_{11}^{\pm} = \frac{T_{11}}{1+z} \pm \frac{T_{-1,1}}{1-z}$$

These combinations of centre-of-mass frame helicity amplitudes are related to the invariant T_i by

$$T_{00} = \frac{T_1}{m_\omega^2} [q^2 - E_\omega^2 z] + \frac{T_2}{m_\omega^2} E_\omega^2 q^2 (1-z^2) + \frac{T_3}{m_\omega^2} E_\omega W q^2 (1-z) + \frac{T_4}{m_\omega^2} q^2 s \quad (A.6)$$

$$T_{01}' = \frac{T_1}{\sqrt{2} m_\omega} E_\omega + \frac{T_2}{\sqrt{2} m_\omega} E_\omega q^2 (1-z) + \frac{T_3}{\sqrt{2} m_\omega} \frac{q^2 W}{2}$$

$$T_{11}^+ = T_2 q^2 z - T_1 \quad T_{11}^- = -T_2 q^2$$

We need define them, as here, only for zero azimuthal angle.

The partial wave amplitudes we used are normalized to $\text{Im } t = \frac{q}{W} t t^+$. t^{j-} is the amplitude for states of parity $(-1)^j$, and t^{j+} is a 2×2 matrix amplitude for parity $-(-1)^j$. They are related to the helicity amplitudes by ²⁰⁾

$$T_{00} = \frac{1}{\pi} \sum_{j=0}^{\infty} (2j+1) P_j(z) t_{00}^{j+}(s)$$

$$T_{01}' = \frac{1}{\pi \sqrt{2}} \sum_{j=1}^{\infty} \frac{(2j+1)}{\sqrt{j(j+1)}} P_j'(z) t_{01}^{j+}(s) \quad (A.7)$$

$$T_{11}^{\pm} = \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{(2j+1)}{j(j+1)} \left\{ [P_j'(z) + z P_j''(z)] t_{11}^{j\pm}(s) - P_j''(z) t_{11}^{j\mp}(s) \right\}$$

or

$$t_{00}^{j+} = \frac{\pi}{2} \int_{-1}^1 P_j(z) T_{00} dz$$

$$t_{01}^{j+} = -\frac{\pi}{2} \frac{\sqrt{j(j+1)}}{2j+1} \int_{-1}^1 [P_{j-1}(z) - P_{j+1}(z)] T_{01}' dz \quad (A.8)$$

$$t_{11}^{j\pm} = \frac{\pi}{2} \int_{-1}^1 \left[\frac{(j+1) P_{j-1}(z) + j P_{j+1}(z)}{2j+1} T_{11}^{\pm} dz + P_j(z) T_{11}^{\mp} \right]$$

By inverting (A.6) one obtains the threshold behaviour of the $t_{ij}^{j\pm}$, as well as the fact that t_{00}^{j+} , Wt_{01}^{j+} , and $st_{11}^{j\pm}$ are analytic functions of s with only dynamical singularities, provided the T_i have only dynamical singularities (e.g., satisfy the Mandelstam representation). The condition that $-T_1 = T_{11}^+ + zT_{11}^-$ be regular at $s = 0$ for fixed t or u puts one condition among the residues of the poles of all the $t_{11}^{j\pm}$. All these conditions are examples of the kinematic conditions on the partial waves discussed in Section 3 for general spin.

In the equal mass case, $m_\pi = m_\omega$, $t_{11}^{j\pm}$ can have no poles at $s = 0$, while $t_{01}^{j+}(s)$ must vanish like W . The requirement that $T_4(s, t, u)$ be regular at $s = 0$, which follows from the absence of singularities in the physical region in the t and u channels, corresponds to a condition on the partial waves for each j :

$$t_{00}^{0+} - t_{00}^{2+} = \frac{3}{2} t_{11}^{1+} - \frac{3}{2} t_{11}^{2-}$$

$$t_{00}^{(j-1)+} - t_{00}^{(j+1)+} = \frac{j-1}{j} t_{11}^{(j-1)-} - \frac{j+2}{j+1} t_{11}^{(j+1)-} + \frac{2j+1}{j(j+1)} t_{11}^{j+} \quad (A.9)$$

$(j \geq 2)$

REFERENCES

- 1) G.F. Chew, private communication.
- 2) G.F. Chew, Phys. Rev. 129, 2363 (1963);
G.F. Chew and C.E. Jones, Phys. Rev. 135, B208 (1964).
- 3) M. Abolins, R.L. Lander, W.A.W. Mehlhop, N. Xuong and P.M. Yager,
Phys. Rev. Letters 11, 381 (1963).
- 4) Here W and q are the centre-of-mass frame total energy and momentum
of each particle, $s = W^2$. We ignore I spin and omit signature
factors in the formulae for Regge trajectory contributions.
- 5) D.Y. Wong, private communication to G.F. Chew; see Ref. 2).
- 6) This was the method of, for example, F. Zachariasen and C. Zemach,
Phys. Rev. 128, 849 (1962).
- 7) See, for example, G.F. Chew, S.C. Frautschi and S. Mandelstam,
Phys. Rev. 126, 1202 (1962).
- 8) M. Gell-Mann, M.L. Goldberger, F.E. Low, E. Marx and F. Zachariasen,
Phys. Rev. 133, B145 (1964).
- 9) P.D.B. Collins, private communication.
- 10) M. Froissart, Phys. Rev. 123, 1053 (1961).
- 11) Y. Hara, Phys. Rev. 136, B507 (1964).
- 12) S.W. MacDowell, Phys. Rev. 116, 774 (1959). For general spins see,
for instance, Y. Hara, Ref. 11).
- 13) G. Frye and R.L. Warnock, Phys. Rev. 130, 478 (1963), (particularly
Appendix E).

- 14) Hung Cheng and D. Sharp, *Ann. Phys. (New York)* 22, 481 (1963).
- 15) M. Gell-Mann, D. Sharp and W.G. Wagner, *Phys. Rev. Letters* 8, 261 (1962);
N. Gelfand, D. Miller, M. Nussbaum, J. Ratau, J. Schultz, J. Steinberger,
T.-H. Tan, L. Kirsch and R. Plano, *Phys. Rev. Letters* 11, 436 (1963).
- 16) E. Abers, *Phys. Rev. Letters* 12, 55 (1964).
- 17) R.F. Peierls, *Phys. Rev. Letters* 12, 50, 119(e), (1964);
See also T.K. Kuo, *Phys. Rev. Letters* 12, 465 (1964) and
J. Franklin, *Phys. Rev.* 137, B994 (1965).
- 18) N. Dombey, unpublished;
V.N. Gribov, 1962 International Conference on High Energy Physics
at CERN, p. 547;
V. Singh, *Phys. Rev.* 129, 1889 (1963).
- 19) A.C. Hearn, *Nuovo Cimento* 21, 233 (1961).
- 20) Except for numerical factors these are just particular cases of the
formulae derived in Ref. ⁸⁾, Section II and Appendix A.