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PRODUCTION OF LIGHT BOSONS BY CENTRAL COLLISIONS AT VERY HIGH ENERGIESG. Auberson ^{*)} and B. Escoubès

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ABSTRACT

The asymptotic behaviour (for $E \rightarrow \infty$) of the statistical model is investigated. It is shown that (if new particles with larger and larger masses do not appear) the masses of the particles may be neglected when $E \rightarrow \infty$. Then the phase-space integrals have a well-known analytical form and asymptotic expressions for sums over all channels - i.e., sums of the type $\sum_{n=2}^{\infty} \Omega^n \frac{1}{n!} \rho_n(E)$ - can be found. It is shown that large-angle elastic scattering will behave like $\frac{d\sigma}{d\Omega} \sim e^{-\alpha E}$ with $\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1$ according to what the detailed assumptions about the kind of particles and interactions are. Finally it is shown that the same results can be found from simple thermodynamical considerations.

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I. INTRODUCTION

In this article it is proposed to determine from the statistical model an asymptotic behaviour for nucleon-nucleon (or rather nucleon-antinucleon) scattering. The main results obtained are probably only realistic for energies higher than those produced by existing accelerators.

There are two characteristic facts about collisions at very high energies.

- A. The collisions become more and more peripheral as the energy increases. The ratio of the number of central collisions to the total number of inelastic collisions (i.e., central and peripheral) can be estimated at approximately ^{1),9)} :

$$\frac{\sigma_{\text{centr}}}{\sigma_{\text{inel}}} \approx \frac{4M^2}{E^2}$$

where

M = the mass of the incoming particles, assumed to be similar (nucleons)

E = total energy in the c.m. .

However, the statistical model correctly describes only the central collisions, because the interaction then lasts long enough (the impact parameter is small enough) for the various particles created within a certain volume of interaction Ω to form a statistical equilibrium where all states have the same a priori probability.

At very high energies the relative number of events likely to be described by this model therefore tends to zero as $1/E^2$.

- B. One can foresee (and our calculation will confirm it) that if the total energy is high enough, the mean energy per secondary particle is sufficient for the masses of these secondaries to be ignored. This approximation greatly simplifies the calculation of the asymptotic behaviour, which can be expressed then analytically.

Since, according to A., the statistical model applies asymptotically only to a minority of events, one might be led to doubt whether the model is of any interest in this case. Our study has been inspired by recent experimental results ²⁾ which show that in proton-proton collisions at high energy (primary momenta of 11 - 31 GeV/c) the differential elastic cross-section at large angles ($\sim 90^\circ$ in the c.m.), although very small, is clearly different from zero. Central collisions to which it must be possible to apply the statistical model correspond to this large angle scattering.

Last year, Fast, Hagedorn and Jones ^{3),4)} proposed a statistical theory of large angle scattering which agrees satisfactorily with recent experimental results ^{5),9)} if one takes account of the factor $4M^2/E^2$ mentioned above. The main point of this theory is that it introduces the branching ratio

$$g(E) = \frac{P_2(E)}{\sum_b P_b(E)}$$

between the probability $P_2(E)$ associated with the elastic channel ^{*)} and the sum of the probabilities $P_b(E)$ extended to all possible final channels. This ratio (on which attention will be concentrated in the following) is connected, in the case of proton-proton scattering, for instance, with the differential elastic cross-section at large angles, by the approximative formula ^{5),9)} :

$$\left(\frac{d\sigma_{el}}{d\Omega} \right)_{\text{large angles}} \simeq \frac{2M^2}{\pi E^2} g(E) \sigma_{inel}$$

^{*)} The arguments can be extended in an obvious way to give the large angle scattering into any two-body final channel related to the in-channel by strong interactions ⁹⁾.

A theoretical calculation of $g(E)$ according to Refs. ^{3),4)} for proton-proton scattering leads to the expression

$$g(E) = e^{-3.10(E-2)}$$

where $2 \leq E \leq 8$ (E expressed in units of nucleon mass M_p).

This exponential form, and also the constant 3.10, are the result of a large number of numerical calculations carried out at CERN in recent years. It remains to be seen whether such behaviour is the asymptotic limit of the statistical model (and can thus be extrapolated to higher energy values) or whether it applies specifically to the energy range under consideration. Our calculations show that the second possibility must be the true one, since the factor E of the exponent can be roughly replaced by $E^{\frac{1}{2}}$ at very high energies.

After summarizing in Section II and III the simplification hypotheses which seem legitimate in the asymptotic region, we shall derive the above result in Section IV and subject it to simple thermodynamical considerations in Section V.

II. HYPOTHESES AND DEFINITIONS

In our simplified model we assume that the production processes are all of the type

$$N + \bar{N} \rightarrow n \pi \quad (n = 2, 3, \dots)$$

where the n secondary particles ("pions") produced by central "nucleon-antinucleon" collision have the same mass m . We ignore the spin and the isospin, and also the production of $K\bar{K}$, $N\bar{N}$, etc., pairs.

Although the n bosons have the same mass, it is not obvious a priori that the best approximation consists in considering them indiscernible, at least as long as the energy E is not too high.

Indiscernibility applies, however, when E increases sufficiently for the average number of secondaries to be much larger than that of the different types of particles (baryons and mesons and their resonant states) among which these secondaries would be distributed in a more detailed model. We therefore envisage the two cases simultaneously. If

$$\rho_n(E, m) = \int \dots \int \prod_{i=1}^n d^3 p_i \delta\left(\sum_{i=1}^n \vec{p}_i\right) \delta\left[E - \sum_{i=1}^n (\vec{p}_i^2 + m^2)^{1/2}\right] \quad (\text{II.1})$$

the probability $P_n(E, m)$ of observing the creation of n particles of mass m sharing the available energy E (in the c.m.) is given, in the non-covariant statistical model, by :

$$P_n(E, m) = \begin{cases} \text{const} \left[\frac{\Omega}{(2\pi)^3} \right]^n \rho_n(E, m) & \text{if the particles are discernible} \\ \text{const} \frac{1}{n!} \left[\frac{\Omega}{(2\pi)^3} \right]^n \rho_n(E, m) & \text{if they are not.} \end{cases} \quad (\text{II.2})$$

The constant (of dimension E^4) which appears in these expressions is proportional to the unknown average squared matrix element. It depends, in general, on the spin and the isospin. In our simplified model it will therefore be an absolute constant (implying a possible dependence on energy) which will disappear in the ratio $g(E)$ and is not written.

The volume of interaction Ω will be :

* either a constant volume :

$$\Omega = \Omega_0 \equiv \frac{4\pi}{3 m_\pi^3} \quad (\hbar = c = 1) \quad (\text{II.3})$$

i.e., the volume of a sphere with radius equal to the pion Compton wave-length;

or a volume "globally" contracted in the direction of flight of the primary particles ⁶⁾ :

$$\Omega = \frac{\Omega_0}{\gamma} = \frac{2M}{E} \Omega_0 \quad (\text{II.4})$$

(the Lorentz factor γ relative to the movement of an incoming "nucleon" in the c.m. has the value $E/2M$).

We shall not consider here the case of a volume of interaction "individually" contracted in the direction of emission of each secondary particle ($\Omega = \frac{nm}{E} \Omega_0$), which is roughly equivalent to a constant volume Ω_0 combined with a covariant phase space ⁷⁾, and which has not been used in previous calculations.

Two indices i and k will be used to distinguish the four cases in which we are interested :

$$i = \begin{cases} 0 & \text{for discernible secondary particles} \\ 1 & \text{for indiscernible secondary particles} \end{cases}$$

$$k = \begin{cases} 0 & \text{for } \Omega = \Omega_0 \\ 1 & \text{for } \Omega = \Omega_1 \equiv \frac{2M}{E} \Omega_0 \end{cases}$$

Formulae (II.2) are therefore written :

$$P_n^{(ik)}(E, m) = \frac{1}{(n!)^i} \left[\frac{\Omega_k}{(2\pi)^3} \right]^n \rho_n(E, m) \quad (\text{II.5})$$

and the functions whose asymptotic behaviour is to be determined :

$$g^{(ik)}(E) = \frac{P_2^{(0,k)}(E, M)}{\sum_{n=2}^N P_n^{(ik)}(E, m)} \quad (\text{II.6})$$

the incoming particles being assumed discernible and $N = E/m$ limiting the number of secondaries.

III. APPROXIMATIONS

As the index n varies from 2 to N , we shall begin by dividing its range into three regions. Let

$$\eta_i = \varepsilon_i - m \quad (\varepsilon_i = (\vec{p}_i^2 + m^2)^{1/2})$$

be the kinetic energy of the i^{th} particle. The mean kinetic energy of a particle will be

$$\bar{\eta} = \frac{E - nm}{n}$$

1) The condition

$$\bar{\eta} \gg m$$

or more precisely

$$\bar{\eta} \geq \nu_1 m \quad (\nu_1 \gg 1)$$

which may also be written :

$$n \leq \lambda_1 N \quad \text{with} \quad \lambda_1 = \frac{1}{\nu_1 + 1} \ll 1$$

defines the ultra-relativistic region (UR). It will be assumed that the equipartition of the total kinetic energy $E - nm$ is sufficiently well achieved for the mass of all the particles to be neglected in that region, when the phase space integral $\rho_n(E, m)$ is being calculated. It then follows ⁸⁾ :

$$\rho_n(E, m) \longrightarrow \rho_n(E)_{UR} = \left(\frac{\pi}{2}\right)^{n-1} \frac{(4n-4)!(2n-1)}{(3n-4)![(2n-1)!]^2} E^{3n-4} \quad (\text{III.1})$$

2) The condition

$$\bar{\eta} \lesssim m$$

or

$$\bar{\eta} \leq \nu_2 m \quad (\nu_2 \sim 1)$$

which is written :

$$n \geq \lambda_2 N \quad \text{with} \quad \lambda_2 = \frac{1}{\nu_2 + 1} \lesssim 1$$

defines the non-relativistic region (NR) in which $\rho_n(E, m)$ takes the form ⁸⁾ :

$$\rho_n(E, m) \longrightarrow \rho_n(E, m)_{NR} = (2\pi)^{\frac{3n-3}{2}} \frac{m^{3n/2}}{(nm)^{3/2} \Gamma(\frac{3n-3}{2})} (E - nm)^{\frac{3n-5}{2}} \quad (\text{III.2})$$

3) Between these two regions ($\lambda_1 N < n < \lambda_2 N$) the calculation of $\rho_n(E, m)$ requires, for high values of n , numerical integration.

Let us note, however, that $\rho_2(E, M)$ can be calculated exactly :

$$\rho_2(E, M) = \frac{\pi}{2} E \sqrt{E^2 - 4M^2} \quad (\text{III.3})$$

Finally we have :

$$\sum_{n=2}^N P_n(E, m) = \sum_{h=2}^{\lambda_1 N} P_n(E)_{UR} + \Xi(E, m; \lambda_1, \lambda_2) + \sum_{n=\lambda_2 N}^N P_n(E, m)_{NR} \quad (\text{III.4})$$

$\Xi(E, m; \lambda_1, \lambda_2)$ denoting the unknown intermediary contribution.

Let us note further that the values of λ_1 and λ_2 depend only on the permitted relative error on $\bar{\eta}$ in each of the approximations. Since in fact the UR approximation consists in making the replacement $\xi = (p^2 + m^2)^{\frac{1}{2}} \simeq p + \frac{m^2}{2p} \rightarrow p$, one gets :

$$\left| \frac{\Delta \bar{\eta}}{\bar{\eta}} \right| \lesssim \frac{1}{2\nu_1^2}$$

and similarly in the NR approximation, where $\xi \simeq m + \frac{p^2}{2m} - \frac{p^4}{8m^3} \rightarrow m + \frac{p^2}{2m}$:

$$\left| \frac{\Delta \bar{\eta}}{\bar{\eta}} \right| \lesssim \nu_2$$

The following approximate formulae will moreover be used :

* the Stirling asymptotic formula limited to its first term :

$$\Gamma(n+1) \simeq \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \quad (n > 2) \quad (\text{III.5})$$

* a second asymptotic formula valid for $n \gg 1$:

$$(n-x)^{an+b} \simeq n^{an+b} e^{-ax} \quad (x, a, b = \text{const.}) \quad (\text{III.6})$$

* the asymptotic form (also limited to its first term) of the infinite sums of the type :

$$S_{\alpha, \beta}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^{\beta} \prod_{\nu=0}^{\alpha-1} (n+\nu)!} \quad (\alpha \text{ integer } \geq 1, \beta = \text{const.})$$

which is written [see Appendix A] :

$$\left\{ \begin{array}{l} S_{\alpha, \beta}(x) \simeq y_{\alpha, \beta}(x) \quad \text{for } x \gg 1 \\ y_{\alpha, \beta}(x) = \frac{1}{\sqrt{\alpha} (2\pi)^{\frac{\alpha-1}{2}}} x^{\frac{1-\alpha^2-2\beta}{2\alpha}} e^{\alpha x^{1/\alpha}} \end{array} \right. \quad (\text{III.7})$$

The approximations (III.5) and (III.6) are never very bad, even for relatively small values of n , and (III.7) is also a good approximation in the physically interesting region of the variable x [see in particular Fig. 1].

Finally we put $M_p = 1$. With this choice of units $m_{\pi} \simeq 1/7$, and E is expressed in terms of nucleon mass.

IV. CALCULATIONS AND RESULTS1) NR contribution

As an example, take the case where $\Omega = \Omega_0$ and the secondaries are considered as discernible ($i=k=0$). According to Eqs. (II.5) and (III.2) :

$$P_n^{(00)}(E, m)_{NR} = \left[\frac{\Omega_0}{(2\pi)^3} \right]^n \frac{2\pi m}{n^{3/2} \Gamma(\frac{3n-3}{2})} \left[2\pi m E \left(1 - \frac{n}{N} \right) \right]^{\frac{3n-5}{2}}$$

and an upper limit for that expression can be obtained by noting that $\frac{n}{N} \geq \lambda_2$:

$$P_n^{(00)}(E, m)_{NR} \leq \left[\frac{\Omega_0}{(2\pi)^3} \right]^n \frac{2\pi m}{n^{3/2} \Gamma(\frac{3n-3}{2})} \left[2\pi m E (1 - \lambda_2) \right]^{\frac{3n-5}{2}}$$

Using (II.3) and the approximations (III.5) and (III.6) one finds :

$$P_n^{(00)}_{NR} \lesssim \left(\frac{1}{6\pi^2 m_n^3} \right)^{5/3} \frac{2\pi m}{\left(\frac{2}{3} \right)^{3/2} p^{3/2} p!} \left[\left(\frac{1}{6\pi^2 m_n^3} \right)^{2/3} 2\pi m E (1 - \lambda_2) \right]^p$$

with $p = \frac{3n-5}{2}$.

Therefore :

$$P_n^{(00)}_{NR} \lesssim \frac{3}{2} A_{00} \frac{1}{p^{3/2} p!} \left(\frac{E}{\epsilon_{00}} \right)^p$$

the quantities ϵ_{00} and A_{00} being explicitly written below. It then follows that :

$$\begin{aligned}
S^{(00)}(E, m)_{NR} &\equiv \sum_{n=\lambda_2 N}^N P_{n NR}^{(00)} \simeq \int_{\lambda_2 N}^N P_{n NR}^{(00)} dn = \\
&= \frac{2}{3} \int_{\lambda_2 P}^P P_{n NR}^{(00)} dp \simeq \frac{2}{3} \sum_{p=\lambda_2 P}^P P_{n NR}^{(00)}
\end{aligned}$$

where $P = \frac{3N-5}{2}$. Then :

$$\begin{aligned}
S^{(00)}_{NR} &\lesssim A_{00} \sum_{p=\lambda_2 P}^P \frac{1}{p^{3/2} p!} \left(\frac{E}{\varepsilon_{00}} \right)^p < \\
&< A_{00} \sum_{p=1}^{\infty} \frac{1}{p^{3/2} p!} \left(\frac{E}{\varepsilon_{00}} \right)^p \equiv A_{00} S_{1, 3/2} \left(\frac{E}{\varepsilon_{00}} \right)
\end{aligned}$$

or, with the help of (III.7) :

$$S^{(00)}_{NR} < A_{00} \left(\frac{E}{\varepsilon_{00}} \right)^{-3/2} e^{\frac{E}{\varepsilon_{00}}}$$

Analogous calculations carried out in the other cases lead finally to the following results :

$$\left\{ \begin{aligned}
S^{(00)}(E, m)_{NR} &< A_{00} (E/\varepsilon_{00})^{-3/2} e^{E/\varepsilon_{00}} \\
S^{(10)}(E, m)_{NR} &< A_{10} (E/\varepsilon_{10})^{-11/5} e^{(E/\varepsilon_{10})^{3/5}} \\
S^{(01)}(E, m)_{NR} &< A_{01} (E/\varepsilon_{01})^{-13/6} e^{(E/\varepsilon_{01})^{1/3}} \\
S^{(11)}(E, m)_{NR} &< A_{11} (E/\varepsilon_{11})^{-12/5} e^{(E/\varepsilon_{11})^{1/5}}
\end{aligned} \right. \quad (IV.1)$$

with

$$\left\{ \begin{aligned} \varepsilon_{00} &= \left(\frac{9\pi}{2}\right)^{1/3} \frac{m_\pi^2}{m} \frac{1}{1-\lambda_2} \\ \varepsilon_{10} &= \frac{3}{5} \left(\frac{18\pi}{25}\right)^{1/3} \frac{m_\pi^2}{m} \frac{1}{1-\lambda_2} \\ \varepsilon_{01} &= \left(\frac{3}{2}\right)^2 \frac{\pi}{2} \frac{m_\pi^6}{m^3 M^2} \frac{1}{(1-\lambda_2)^3} \\ \varepsilon_{11} &= \left(\frac{3}{5}\right)^5 \frac{\pi}{2} \frac{m_\pi^6}{m^3 M^2} \frac{1}{(1-\lambda_2)^3} \end{aligned} \right. \quad (\text{IV.1'})$$

$$\left\{ \begin{aligned} A_{00} &= \left(\frac{1}{6\pi^2}\right)^{7/6} \frac{m}{m_\pi^5} \\ A_{10} &= \left(\frac{10\pi}{3}\right)^{1/6} \frac{125}{144\pi^3} \frac{m}{m_\pi^5} \\ A_{01} &= \left(\frac{2}{3}\right)^{9/2} \frac{\sqrt{6}}{\pi^4} \frac{m^6 M^5}{m_\pi^{15}} (1-\lambda_2)^5 \\ A_{11} &= \frac{5}{\sqrt{2}} \left(\frac{5}{3}\right)^6 \left(\frac{5}{3\pi}\right)^{9/2} \frac{m^6 M^5}{m_\pi^{15}} (1-\lambda_2)^5 \end{aligned} \right. \quad (\text{IV.1''})$$

It is to be noted that the inequalities

$$s_{\text{NR}}^{(00)} > s_{\text{NR}}^{(10)} > s_{\text{NR}}^{(01)} > s_{\text{NR}}^{(11)}$$

resulting directly from the definitions $[\Omega_0 > \Omega_1, P_n^{(0k)}(E, m) > P_n^{(1k)}(E, m)]$ are again fulfilled asymptotically by the limits (IV.1).

2) UR contribution

Assuming here that $\Omega = \Omega_0$ and that the secondaries are indiscernible, then according to (II.5) and (III.1) :

$$P_n^{(10)}(E)_{\text{UR}} = \frac{1}{n!} \left[\frac{\Omega_0}{(2\pi)^3} \right]^n \left(\frac{\pi}{2}\right)^{n-1} \frac{(4n-4)!(2n-1)}{(3n-4)! [(2n-1)!]^2} E^{3n-4}$$

which can be written, taking (II.3), (III.5) and (III.6) into consideration, as follows :

$$P_n^{(10)}(E)_{UR} \approx \left(\frac{2}{3\pi}\right)^{1/2} \frac{E^2}{81\pi^2 m_\pi^6} \frac{1}{p^{-3/2} p! (p+1)! (p+2)! (p+3)!} \left[\left(\frac{E}{\xi}\right)^3\right]^p \quad (\text{IV.2})$$

where $p = n-2$, $\xi = 3(3\pi/4)^{1/3} m_\pi$.

Let us consider :

$$\begin{cases} S^{(ik)}(E, \lambda_1)_{UR} = \sum_{n=2}^{\lambda_1 N} P_n^{(ik)}(E)_{UR} \\ S^{(ik)}(E)_{UR} = S^{(ik)}(E, \infty)_{UR} \quad (\text{convergent series}) \end{cases} \quad (\text{IV.3})$$

When $E \rightarrow \infty$, then also $\lambda_1 N \rightarrow \infty$ and :

$$S^{(ik)}(E, \lambda_1)_{UR} \longrightarrow S^{(ik)}(E)_{UR} \quad (\text{IV.4})$$

It is, however, important to note that this limiting relation is only valid if the label $\hat{n}$ of the term of $\sum_{n=2}^{\infty} P_n^{(ik)}(E)_{UR}$ supplying the maximum contribution increases less quickly, as a function of E , than the upper limit of summation $\lambda_1 N = \lambda_1 \frac{E}{m}$. Now, for $S^{(10)}(E)_{UR}$, it is easy to see - from (IV.2) - that the maximum term is the one with

$$n = \hat{n} \approx \left(\frac{E}{\xi}\right)^{3/4}$$

It therefore follows that for a sufficiently large E :

$$\lambda_1 N \gg \hat{n}$$

and the required condition is well satisfied. Asymptotically one can write :

$$S^{(10)}(E, \lambda_1)_{UR} \simeq S^{(10)}(E)_{UR} = \sum_{n=2}^{\infty} P_n^{(10)}(E)_{UR}$$

Then, inserting (IV.2) :

$$S^{(10)}(E)_{UR} = \left(\frac{2}{3\pi}\right)^{1/2} \frac{E^2}{81 \pi^2 m_\pi^6} S_{4,-3/2} \left[\left(\frac{E}{E}\right)^3\right]$$

Analogous expressions are obtained in the other cases ^{*)}, and the following results can be obtained by applying (III.7) :

$$\left\{ \begin{array}{l} S^{(00)}(E)_{UR} \simeq B_{00} (E/E_{00})^{-3/2} e^{E/E_{00}} \\ S^{(10)}(E)_{UR} \simeq B_{10} (E/E_{10})^{-5/2} e^{(E/E_{10})^{3/4}} \\ S^{(01)}(E)_{UR} \simeq B_{01} (E/E_{01})^{-7/3} e^{(E/E_{01})^{2/3}} \\ S^{(11)}(E)_{UR} \simeq B_{11} (E/E_{11})^{-3} e^{(E/E_{11})^{1/2}} \end{array} \right. \quad (IV.5)$$

with :

$$\left\{ \begin{array}{l} E_{00} = \left(\frac{3\pi}{4}\right)^{1/3} m_\pi = \left(\frac{\pi^2}{\Omega_0}\right)^{1/3} = 0.190 \\ E_{10} = \frac{3}{8} \left(\frac{3\pi}{2}\right)^{1/3} m_\pi = \frac{3}{4^{4/3}} \left(\frac{\pi^2}{\Omega_0}\right)^{1/3} = 0.059 \\ E_{01} = \frac{1}{2} \left(\frac{3\pi m_\pi^3}{2M}\right)^{1/2} = \left(\frac{\pi^2}{2M\Omega_0}\right)^{1/2} = 0.090 \\ E_{11} = \frac{9}{32} \left(\frac{\pi m_\pi^3}{2M}\right) = \frac{3^{3/2}}{16} \left(\frac{\pi^2}{2M\Omega_0}\right)^{1/2} = 0.019 \end{array} \right. \quad (IV.5')$$

^{*)} It must be mentioned that the justification of "passing to the limit" $S^{(ik)}(E, \lambda_1)_{UR} \rightarrow S^{(ik)}(E)_{UR}$ requires a much finer analysis when $i=k=0$.

$$\left\{ \begin{array}{l} B_{00} = \frac{(6\pi)^{1/6}}{12\pi^3} \frac{1}{m_\pi^4} \\ B_{10} = \frac{1}{3\sqrt{3}} \left(\frac{2}{\pi}\right)^3 \left(\frac{2}{3\pi}\right)^{1/3} \frac{1}{m_\pi^4} \\ B_{01} = \frac{2}{3\pi^3} \left(\frac{2}{3\pi}\right)^{1/2} \frac{M^2}{m_\pi^5} \\ B_{11} = \frac{1}{\sqrt{3}} \left(\frac{8}{3\pi}\right)^4 \frac{M^2}{m_\pi^5} \end{array} \right. \quad (\text{IV.5''})$$

Here, again, the inequalities

$$S_{UR}^{(00)} > S_{UR}^{(10)} > S_{UR}^{(01)} > S_{UR}^{(11)}$$

hold asymptotically.

3) Evaluation of the branching ratio $g(E)$

It will first be established that the UR contribution is the only one which exists when $E \rightarrow \infty$.

Let us write the inequalities (IV.1) (omitting the indices i, k in order to make the notation easier) :

$$S(E, m, \lambda_2)_{NR} < C(E, m, \lambda_2)$$

since the right-hand side of (IV.1) depends implicitly on λ_2 via A_{ik} and $\mathcal{E}_{ik}$.

Further, let us put

$$S(E, m) = \sum_{n=2}^N P_n(E, m)$$

in such a way that (III.4) can be written :

$$S(E, m) = S(E, \lambda_1)_{UR} + \Xi(E, m, \lambda_1, \lambda_2) + S(E, m, \lambda_2)_{NR} \quad (\text{IV.6})$$

From (IV.4) it results that for a sufficiently large E :

$$\frac{S(E, m, \lambda_2)_{NR}}{S(E, \lambda_1)_{UR}} \simeq \frac{S(E, m, \lambda_2)_{NR}}{S(E)_{UR}} < \frac{C(E, m, \lambda_2)}{S(E)_{UR}}$$

Now, and this is the important point in our evaluation, a comparison of formulae (IV.1) and (IV.5) shows that for $E \rightarrow \infty$ (whatever i, k may be) :

$$\frac{C^{(ik)}(E, m, \lambda_2)}{S^{(ik)}(E)_{UR}} \longrightarrow 0 \quad (\text{IV.7})$$

Therefore :

$$\frac{S(E, m, \lambda_2)_{NR}}{S(E, \lambda_1)_{UR}} \longrightarrow 0 \quad (\text{IV.8})$$

when $E \rightarrow \infty$. This result is to be compared with the inequality (although it cannot be derived from it since it is stronger)

$$P_n(E, m)_{NR} < P_n(E)_{UR}$$

which simply means that the accessible phase space is more extensive when the mass is zero.

According to (IV.8) there certainly exists a $\lambda = \lambda(E)$
 $[\lambda_1 < \lambda < \lambda_2 \text{ and } \lim_{E \rightarrow \infty} \lambda(E) = \lambda_1]$ so that (IV.6) can just as
 well be written :

$$S(E, m) = S(E, \lambda)_{UR} + S(E, m, \lambda)_{NR}$$

the intermediary contribution Ξ having been replaced by an adequate
 "continuation" of S_{UR} and S_{NR} .

Taking (IV.4) into account, it then follows that :

$$S(E, m) \approx S(E)_{UR} \left[1 + \frac{S(E, m, \lambda)_{NR}}{S(E)_{UR}} \right]$$

As, further

$$S(E, m, \lambda)_{NR} < C(E, m, \lambda) < C(E, m, \lambda_2)$$

Eq. (IV.7) leads to

$$\frac{S(E, m, \lambda)_{NR}}{S(E)_{UR}} \longrightarrow 0$$

when $E \rightarrow \infty$, and finally, for a sufficiently large E :

$$S(E, m) \approx S(E)_{UR} \quad (IV.9)$$

It therefore follows that the sum $\sum_{n=2}^N P_n(E, m)$ no longer depends,
asymptotically, on the mass m of the secondaries. Furthermore,
 numerical evaluations show that this m dependence soon disappears when
 the energy increases. In fact, the ratio (IV.7) quickly tends towards
 zero. For example, for $E = 5M_p$ (with $m = m_\pi$, $\lambda = \frac{1}{2}$), one finds :

$$\frac{C^{(11)}}{S^{(11)}_{UR}} \simeq 2.5 \cdot 10^{-2}$$

Generally speaking, one can say that the error introduced by neglecting the non-relativistic and intermediary contributions very rapidly becomes small.

The branching ratio can thus be written :

$$g^{(ik)}(E) \simeq \frac{P_2^{(ok)}(E, M)}{S^{(ik)}(E)_{UR}}$$

with

$$P_2^{(ok)}(E, M) = \left[\frac{\Omega_k}{(2\pi)^3} \right]^2 \rho_2(E, M)$$

$$\rho_2(E, M) \simeq \frac{\pi}{2} E^2$$

[Eq. (III.3) with $E \gg 2M$]. One obtains

$$\left\{ \begin{array}{l} g^{(00)}(E) = G^0 (E/E_{00})^{7/2} e^{-E/E_{00}} \\ g^{(10)}(E) = G^1 (E/E_{10})^{9/2} e^{-(E/E_{10})^{3/4}} \\ g^{(01)}(E) = G^0 (E/E_{01})^{7/3} e^{-(E/E_{01})^{2/3}} \\ g^{(11)}(E) = G^1 (E/E_{11})^3 e^{-(E/E_{11})^{1/2}} \end{array} \right. \quad (\text{IV.10})$$

with

$$\left\{ \begin{array}{l} G^0 = \frac{1}{4} \sqrt{\frac{\pi}{6}} = 0.181 \\ G^1 = 9\pi\sqrt{3} \left(\frac{1}{2}\right)^{13} = 0.006 \end{array} \right.$$

Of these four results the last two, in particular, will be retained. They correspond in effect to a contracted interaction volume, and that hypothesis was already the basis of previous detailed calculations ^{3),4)}, which gave satisfactory results up to an energy of $E=8$ GeV. The two curves, $g^{(01)}(E)$ and $g^{(11)}(E)$, are shown in Fig. 2.

It must be pointed out that the position of these curves depends rather strongly on the choice of Ω_0 (which occurs in the expressions for E_{01} and E_{11}). We have chosen $\Omega_0 = 4\pi/3m_\pi^3$ but it is probable that at very high energies the mass of the π must be replaced, in this formula, by a mass somewhere between that of the π and that of the K. For the energy range under consideration ($E < 30$ GeV), it would appear reasonable to keep m_π .

It appears that the ratio $g(E)$ takes very different values according to whether the secondaries are assumed to be discernible or indiscernible. Since these are extreme assumptions, the two corresponding functions form in fact upper and lower bounds, respectively, for the true function $g(E)$. Extrapolation of the "exact" curve (straight in the logarithmic scale) is therefore certainly incorrect for energies $E \gtrsim 8$ GeV and soon gives values several orders of magnitude too small.

With our extremely simplified model it seems difficult to fix the function $g(E)$ more precisely between these two bounds. If it is probable that the discernibility hypothesis is the most realistic at low energies, one cannot very well locate the energy at which this hypothesis must be abandoned (if at all), and even worse that at which indiscernibility gives a satisfactory approximation. An important remark may, however, be made in this connection: the preceding calculations are not essentially modified if one uses an hypothesis of "partial" discernibility, in which n particles produced are distributed among r equally populated groups, the n/r particles belonging to each group being indiscernible. If r (which is in fact a difficult function to evaluate from n and E) is taken as constant, one finds that the form of the exponential factor is not perturbed, the general expression being of the type:

$$g^{(rk)}(E) = G^r (E/E_{rk})^{a_{rk}} e^{-(E/E_{rk})^{b_k}} \quad (\text{IV.11})$$

where

$$\left\{ \begin{array}{l} a_{rk} = \frac{r+11}{8} (3-k) \\ b_k = \frac{3-k}{4} \quad \text{for all } r \geq 1 \\ E_{rk} = E_{1k} r^{-\frac{1}{3-k}} \\ G^r = G^1 \left(\frac{\pi}{2}\right)^{\frac{r-1}{2}} \frac{1}{r^{r/2+2}} \end{array} \right.$$

If, for example, there are only three groups and if $\Omega = \Omega_1$ ($r=3, k=1$) :

$$g^{(31)}(E) = \frac{\pi^2}{3 \cdot 2^{14}} (E/E_{31})^{7/2} e^{-(E/E_{31})^{1/2}} \quad (\text{IV.12})$$

with

$$E_{31} = \frac{E_{11}}{\sqrt{3}} = 0.011$$

This function is also illustrated in Fig. 2. Clearly r could be larger than 3, to take into account the resonances at high energies. [If, however, in reality the strongly interacting particles should have an infinity of excited states, then r could possibly increase linearly with E , in that case we fall back essentially on discernible particles.]

In any case these considerations lead to the conjecture that the branching ratio will have the general form :

$$\left\{ \begin{array}{l} g^{(ro)}(E) = G_{ro}(E) e^{-(E/E_{ro})^{3/4}} \quad \text{for } \Omega = \Omega_0 \\ g^{(r1)}(E) = G_{r1}(E) e^{-(E/E_{r1})^{1/2}} \quad \text{for } \Omega = \frac{2M}{E} \Omega_0 \end{array} \right. \quad (\text{IV.13})$$

where $G_{rk}(E)$ are functions which increase very slowly as compared to the decreasing exponential, as will be confirmed by a purely thermodynamical evaluation.

4) Multiplicities

For purposes of comparison, the average number of secondaries produced as a function of E will again be determined. It is defined by :

$$\bar{n}(E) = \frac{\sum_{n=2}^N n P_n(E, m)}{\sum_{n=2}^N P_n(E, m)} \quad (\text{IV.14})$$

or, in our approximation :

$$\bar{n}^{(ik)}(E) \simeq \frac{\sum_{n=2}^{\infty} n P_n^{(ik)}(E)_{UR}}{\sum_{n=2}^{\infty} P_n^{(ik)}(E)_{UR}}$$

From what has been seen, this expression takes asymptotically the form :

$$\bar{n}^{(ik)} = \frac{S_{\alpha, \beta-1} [(E/E^{(ik)})^{\delta}]}{S_{\alpha, \beta} [(E/E^{(ik)})^{\delta}]} \simeq \frac{y_{\alpha, \beta-1} [(E/E^{(ik)})^{\delta}]}{y_{\alpha, \beta} [(E/E^{(ik)})^{\delta}]}$$

where α , β and γ depend on the choice of the hypothesis (ik).
According to (III.7) :

$$\bar{n}^{(ik)} = (E/E'_{ik})^{\gamma/\alpha} \quad (\text{IV.15})$$

where

$$E'_{ik} = \alpha^{\alpha/\gamma} E_{ik} = \varepsilon_k$$

independent of i. One thus finds :

$$\left\{ \begin{array}{l} \varepsilon_0 = 3 E_{00} = 4^{4/3} E_{10} = 0.570 \\ \varepsilon_1 = 3^{3/2} E_{01} = 4^2 E_{11} = 0.304 \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \bar{n}^{(00)} = E/\varepsilon_0 \\ \bar{n}^{(10)} = (E/\varepsilon_0)^{3/4} \\ \bar{n}^{(01)} = (E/\varepsilon_1)^{2/3} \\ \bar{n}^{(11)} = (E/\varepsilon_1)^{1/2} \end{array} \right. \quad (\text{IV.16})$$

Numerically one has, for example :

$$\bar{n}^{(11)} = 1.81 E^{1/2}$$

and, since the mean energy of a secondary is $\bar{\varepsilon} = E/\bar{n}$:

$$\bar{\varepsilon}^{(11)} = 0.55 E^{1/2}$$

For $E = 10M = 10$:

$$\bar{n}^{(11)} = 5.72$$

$$\bar{\xi}^{(11)} = 1.74$$

This last value shows that our results are consistent with the fact of having kept only the UR contribution in the calculation of $S(E, m)$, for which

$$\bar{\xi} \gg m \approx \frac{1}{7}$$

It should be noted that the above multiplicity is due to central collisions only, the contribution of which goes down as $4M^2/E^2$. Hence at high energies the contribution of the $\bar{n}$ calculated from the statistical model becomes negligible, as compared to the sum of all non-centrally produced particles.

V. THERMODYNAMICS

Consider an assembly in statistical equilibrium ("gas") consisting of massless particles without internal variables. If the particles interact for a sufficiently long time and if the available energy E is sufficiently high, the secondaries produced in the central collisions NN can be assimilated to such a gas, momentarily contained in a volume V of about the same size as the interaction volume. To this system, two kinds of statistics can be applied (in which the number of particles is not conserved) according to whether they are indiscernible or not.

1) Indiscernible particles

If the particles are considered as indiscernible, the statistics will be of the well-known Bose-Einstein type, with the following results :

* energy distribution function (Planck's law)

$$\frac{dn}{d\varepsilon} = \frac{V}{2\pi^2} \frac{\varepsilon^2}{e^{\beta\varepsilon} - 1}, \quad \beta = 1/KT \quad (V.1)$$

(K being the Boltzmann constant);

* total energy and energy dependence on the temperature

$$\left\{ \begin{array}{l} E = \frac{3\xi(4)V}{\pi^2\beta^4} \\ KT = \left[\frac{\pi^2 E}{3\xi(4)V} \right]^{1/4} \end{array} \right. \quad (V.2)$$

$\xi(s)$ being the Riemann zeta function:

* average energy

$$\bar{\varepsilon} = \frac{E}{\langle N \rangle} = \frac{3\xi(4)}{\xi(3)} \frac{1}{\beta} = 2.88 KT \quad (V.3)$$

At this point one can calculate two quantities which have to be compared to the statistical results, the entropy and the average multiplicities.

a) The thermodynamical entropy is defined as :

$$\sum_{th} = \int \frac{dE}{T(E)}$$

and can be calculated immediately from (V.2) in both cases of interest : constant volume $V = V_0$ and contracted volume $V = (2M/E)V_0$. $\sum$ has also its statistical definition :

$$\sum_{st} = K \text{Log } \Delta \Gamma$$

$\Delta\Gamma$ represents in fact the accessible phase space volume of the system, and it has been seen that :

$$\Delta\Gamma \simeq S(E)_{UR}$$

* In the constant volume case we have :

$$\sum_{th}^{(10)} = K \int \frac{dE}{\left(\frac{\pi^2}{3\xi(4)} \frac{1}{V_0}\right)^{1/4} E^{1/4}} = K \frac{E^{3/4}}{\left[\left(\frac{3}{4}\right)^{4/3} \left(\frac{\pi^2}{3\xi(4)V_0}\right)^{1/3}\right]^{3/4}} \equiv K (E/e_{10})^{3/4}$$

This rather sophisticated form has to be compared to the statistical result, given by Eqs. (IV.5), by forgetting the slowly varying logarithmic function :

$$\sum_{st}^{(10)} \simeq K (E/E_{10})^{3/4} = K \frac{E^{3/4}}{\left[\left(\frac{3}{4}\right)^{4/3} \left(\frac{\pi^2}{3\Omega_0}\right)^{1/3}\right]^{3/4}}$$

The agreement is obtained for :

$$V_0 = \frac{\Omega_0}{\xi(4)} = 0.92 \Omega_0 \quad (V.4)$$

* In the contracted volume case :

$$\left\{ \begin{array}{l} \sum_{th}^{(11)} = K \frac{E^{1/2}}{\left[\frac{1}{4\sqrt{3}\xi(4)} \left(\frac{\pi^2}{2MV_0}\right)^{1/2}\right]^{1/2}} \equiv K (E/e_{11})^{1/2} \\ \sum_{st}^{(11)} = K (E/E_{11})^{1/2} = K \frac{E^{1/2}}{\left[\frac{9}{16\sqrt{3}} \left(\frac{\pi^2}{2M\Omega_0}\right)^{1/2}\right]^{1/2}} \end{array} \right.$$

the agreement is obtained for :

$$V_0 = \left[\frac{4}{9\sqrt{\xi(4)}}\right]^2 \Omega_0 = 0.18 \Omega_0 \quad (V.5)$$

b) The average multiplicity is defined as :

$$\bar{n} \equiv \langle N \rangle = \frac{E}{\bar{\epsilon}} = \frac{E}{2.88 \text{ KT}}$$

and when one has :

$$* V = V_0$$

$$\left\{ \begin{array}{l} \bar{n}_{th}^{(10)} = \frac{E^{3/4}}{\left[\left(\frac{4\xi(4)}{\xi(3)} \right)^{4/3} e_{10} \right]^{3/4}} \\ \bar{n}_{st}^{(10)} = (E/\epsilon_0)^{3/4} \end{array} \right.$$

the agreement is obtained for :

$$V_0 = \frac{[\xi(4)]^3}{[\xi(3)]^4} = 0.79 \Omega_0 \quad (\text{V.6})$$

$$* V = \frac{2M}{E} V_0$$

$$\left\{ \begin{array}{l} \bar{n}_{th}^{(11)} = \frac{E^{1/2}}{\left[\left(\frac{6\xi(4)}{\xi(3)} \right)^2 e_{11} \right]^{1/2}} \\ \bar{n}_{st}^{(11)} = (E/\epsilon_1)^{1/2} \end{array} \right.$$

the agreement is obtained for the same value :

$$V_0 = 0.79 \Omega_0 \quad (\text{V.7})$$

This is encouraging. One can speak of quantitatively satisfactory agreement ($V_0 \simeq \Omega_0$), taking into account the roughness of some approximations made in the statistical method when E is not too big and the fact that in thermodynamics momentum conservation is neglected.

2) Discernible particles

The Maxwell-Boltzmann distribution supposes essentially that the number of particles is conserved. If one does not want this, it is necessary to construct a new statistics for distinguishable particles. We did that and found the following results [see Appendix B] :

* energy distribution function :

$$\frac{dn}{d\varepsilon} = \frac{V}{2\pi^2} \frac{\varepsilon^2 e^{-\beta\varepsilon}}{1 - \frac{V}{\pi^2\beta^3}} \quad (\text{V.1'})$$

* total energy and energy dependence on the temperature :

$$\left\{ \begin{array}{l} E = \frac{3V}{\pi^2\beta^4} \frac{1}{1 - \frac{V}{\pi^2\beta^3}} \\ KT = \left(\frac{\pi^2}{V}\right)^{1/3} \left[1 - \left(\frac{\pi^2}{V}\right)^{1/3} \frac{1}{E} + 3 \left(\frac{\pi^2}{V}\right)^{2/3} \frac{1}{E^2} + \dots \right] \end{array} \right. \quad (\text{V.2'})$$

This series converges fast if :

$$E \gg \left(\frac{\pi^2}{V}\right)^{1/3} \simeq \left(\frac{\pi^2}{\Omega_0}\right)^{1/3} = E_{00} = 0.19 ,$$

a condition which is evidently fulfilled. Thus :

$$KT \simeq \left(\frac{\pi^2}{V}\right)^{1/3}$$

* average energy :

$$\bar{\varepsilon} = \frac{E}{\langle N \rangle} = 3KT \quad (\text{V.3'})$$

It is possible to calculate the entropy and the average number of particles :

a) Entropy :

$$* V = V_0$$

$$\left\{ \begin{array}{l} \sum_{th}^{(00)} = K \int \frac{dE}{\left(\frac{\pi^2}{V_0}\right)^{1/3}} = K \frac{E}{\left(\frac{\pi^2}{V_0}\right)^{1/3}} \equiv K E/e_{00} \\ \sum_{st}^{(00)} = K E/E_{00} = K \frac{E}{\left(\frac{\pi^2}{\Omega_0}\right)^{1/3}} \end{array} \right.$$

The agreement is obtained for :

$$V_0 = \Omega_0 \quad (V.4')$$

$$* V = \frac{2M}{E} V_0$$

$$\left\{ \begin{array}{l} \sum_{th}^{(01)} = K \frac{E^{2/3}}{\left[\left(\frac{2}{3}\right)^{3/2} \left(\frac{\pi^2}{2M V_0}\right)^{1/2}\right]^{2/3}} \equiv K (E/e_{01})^{2/3} \\ \sum_{st}^{(01)} = K (E/E_{01})^{2/3} = K \frac{E^{2/3}}{\left[\left(\frac{\pi^2}{2M \Omega_0}\right)^{1/2}\right]^{2/3}} \end{array} \right.$$

The agreement is obtained for :

$$V_0 = 0.30 \Omega_0 \quad (V.5')$$

b) Average number of particles :

$$* V = V_0$$

$$\left\{ \begin{array}{l} \bar{n}_{th}^{(00)} = \frac{E}{3e_{00}} \\ \bar{n}_{st}^{(00)} = E/\varepsilon_0 \end{array} \right.$$

$$* V = \frac{2M}{E} V_0$$

$$\left\{ \begin{array}{l} \bar{n}_{th}^{(01)} = \frac{E^{2/3}}{\left[3^{3/2} \left(\frac{\pi^2}{2MV_0} \right)^{1/2} \right]^{2/3}} \\ \bar{n}_{st}^{(01)} = (E/\varepsilon_1)^{2/3} \end{array} \right.$$

In both cases the agreement is obtained for :

$$V_0 = \Omega_0$$

(V.6'-7')

The thermodynamical, as well as the statistical, calculations are approximate [first term of the development of $KT(E)$]. The exact agreement of the calculations in three cases over four seems due to a coincidence and could be an indication that the "same" approximations have been made in both cases.

VI. CONCLUSIONS

This statistical approach of very high energy collisions is compatible with a thermodynamical description of the phenomena. The comparison between the two models depends essentially on the relative magnitude of the two "interaction volumes", which are the free parameters of both theories. The agreement has been found to be good, and volumes of about the same size lead roughly to the same results in the models.

Are the secondary particles produced by central collisions at very high energies discernible or not ? It seems more realistic to choose an intermediate description where the particles are indiscernible within r groups containing the same number of particles. This assumption leads to Eq. (IV.11), which does not differ appreciably from those given by the hypothesis of total indiscernibility [second and last Eq. (IV.10)].

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A P P E N D I X A

The asymptotic form of the infinite sums of the type

$$S_{\alpha, \beta}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^{\beta} \prod_{v=0}^{\alpha-1} (n+v)!} \equiv \sum_{n=1}^{\infty} \sigma_n \quad (\text{A.1})$$

will be determined. One can easily verify that the terms σ_n , which provide the maximum contribution, are those for which $n \simeq x^{1/\alpha}$. It is therefore legitimate to use the approximations (III.5) and (III.6) for all the σ_n , since the region in which these approximations are the least valid (i.e., when $n \ll x^{1/\alpha}$) is of very little importance.

Thus one finds :

$$\sigma_n \simeq \frac{x^n}{n^{\frac{2\beta+\alpha^2-1}{2}} (\sqrt{2\pi})^{\alpha-1} \sqrt{2\pi} n^{\alpha n + 1/2} e^{-\alpha n}}$$

By writing

$$n^{\alpha n + 1/2} = \left(\frac{1}{\alpha}\right)^{\alpha n + 1/2} (\alpha n)^{\alpha n + 1/2}$$

$$x^n = (x^{1/\alpha})^{\alpha n}$$

one obtains, with the help of (III.5) :

$$\sigma_n \simeq \frac{\sqrt{\alpha}}{(2\pi)^{\frac{\alpha-1}{2}}} n^{-\gamma} \frac{(\alpha x^{1/\alpha})^{\alpha n}}{(\alpha n)!}$$

where

$$\gamma = \frac{1}{2} (\alpha^2 + 2\beta - 1)$$

To calculate the sum, one will write :

$$\sum_{n=1}^{\infty} G_n \approx \int_0^{\infty} G_n dn = \frac{1}{\alpha} \int_0^{\infty} G_{\frac{m}{\alpha}} dm \approx \frac{1}{\alpha} \sum_{m=1}^{\infty} G_{\frac{m}{\alpha}}$$

therefore :

$$\sum_{n=1}^{\infty} G_n \approx \frac{\alpha^{\frac{\gamma-1}{2}}}{(2\pi)^{\frac{\alpha-1}{2}}} \sum_{m=1}^{\infty} \frac{m^{-\gamma} (\alpha x^{1/\alpha})^m}{m!}$$

To evaluate

$$\sum_{m=1}^{\infty} \frac{m^{-\gamma} z^m}{m!} \quad (z = \alpha x^{1/\alpha})$$

it is sufficient to note that (supposing here that the γ is a positive integer, which is a non-essential condition) :

$$z^{\gamma} \sum_{m=1}^{\infty} \frac{z^m}{(m+\gamma)!} = e^z - \sum_{m=0}^{\gamma} \frac{z^m}{m!}$$

and that, according to (III.5) :

$$(m+\gamma)! \approx m^{\gamma} m!$$

Whence :

$$\sum_{m=1}^{\infty} \frac{m^{-\gamma} z^m}{m!} \approx z^{-\gamma} \left(e^z - \sum_{m=0}^{\gamma} \frac{z^m}{m!} \right)$$

Finally :

$$S_{\alpha,\beta}(x) \simeq \frac{1}{\sqrt{\alpha} (2\pi)^{\frac{\alpha-1}{2}}} x^{\frac{1-\alpha^2-2\beta}{2\alpha}} \left[e^{\alpha x^{1/\alpha}} - \sum_{m=0}^{\frac{1}{2}(\alpha^2+2\beta-1)} \frac{(\alpha x^{1/\alpha})^m}{m!} \right] \quad (\text{A.2})$$

The last sum is negligible as compared to $e^{\alpha x^{1/\alpha}}$ if x is large, or more exactly if the maximum value of m is still very small compared to $\alpha x^{1/\alpha}$, that is when

$$x \gg \left(\frac{\alpha^2 + 2\beta - 1}{2\alpha} \right)^\alpha$$

Note that for $\alpha = 1$, $\beta = 0$, one obtains the trivial case of the exponential e^x , and for $\alpha = 2$, $\beta = 0$, the asymptotic behaviour of $I_1(2\sqrt{x})/\sqrt{x}$ where $I_1(x)$ is the modified Bessel function of order 1.

A P P E N D I X B ^{*)}

Consider an assembly of N particles contained in a box of volume V . Let $\epsilon_1, \epsilon_2, \dots$ be the energy levels of one particle. For a given configuration with n_1 particles on the level ϵ_1 , n_2 on $\epsilon_2, \dots$, the energy of the system will be :

$$E_a = \sum_{\alpha} n_{\alpha} \epsilon_{\alpha}$$

"a" labelling the configuration with the occupation numbers $\{n_1, n_2, \dots\}$.

If $T = 1/\beta K$ is the temperature of the box and if the thermal equilibrium is reached, the probability for an energy E_a of the system is proportional to

$$e^{-\beta E_a}$$

according to the "canonical ensemble" description. The partition function will be the sum over all states of the system, weighting each state by its probability $e^{-\beta E_a}$:

$$Z_{ind} = \sum_a e^{-\beta E_a} \quad (B.1)$$

This expression for Z applies when, to each $a \equiv \{n_1, n_2, \dots\}$, there corresponds one and only one state of the whole system, i.e., when the particles are indiscernible.

If the particles are discernible, to each "a" correspond

$$\frac{N!}{n_1! n_2! \dots} \quad \left(N = \sum_{\alpha} n_{\alpha} \right)$$

^{*)} The considerations of this section are the result of several discussions of A. Bialas, B. Escoubès and R. Hagedorn.

different states (of the same energy E_a), obtained from the $N!$ permutations of the N particles by taking into account that permutations, within a same energy level, do not lead to a new state. Therefore :

$$Z_{dis}(N) = \sum_a \frac{N!}{n_1! n_2! \dots} e^{-\beta \sum_{\alpha} n_{\alpha} \epsilon_{\alpha}} \quad (\text{B.2})$$

where the sum is extended to all terms for which $\sum_{\alpha} n_{\alpha} = N$. As we do not conserve the number of particles :

$$Z_{dis} = \sum_N Z_{dis}(N) \quad (\text{B.3})$$

Now putting

$$z_{\alpha} = e^{-\beta \epsilon_{\alpha}}$$

we have formally :

$$\left\{ \begin{array}{l} Z_{ind} = \sum_a z_1^{n_1} z_2^{n_2} \dots = \prod_{\alpha} \left(\sum_{n_{\alpha}} z_{\alpha}^{n_{\alpha}} \right) = \prod_{\alpha} \frac{1}{1 - z_{\alpha}} \\ Z_{dis} = \sum_N \left(\sum_a \frac{N!}{n_1! n_2! \dots} z_1^{n_1} z_2^{n_2} \dots \right) = \sum_N \left(\sum_{\alpha} z_{\alpha} \right)^N = \frac{1}{1 - \sum_{\alpha} z_{\alpha}} \end{array} \right. \quad (\text{B.4})$$

The total energy is given by :

$$E = - \frac{\partial}{\partial \beta} \text{Log } Z$$

and (B.4) involves :

$$\left\{ \begin{array}{l} E_{ind} = \frac{\partial}{\partial \beta} \sum_{\alpha} \text{Log}(1 - z_{\alpha}) = \sum_{\alpha} \frac{\epsilon_{\alpha}}{e^{\beta \epsilon_{\alpha}} - 1} \\ E_{dis} = \frac{\partial}{\partial \beta} \text{Log}(1 - \sum_{\alpha} z_{\alpha}) = \sum_{\alpha} \frac{\epsilon_{\alpha} e^{-\beta \epsilon_{\alpha}}}{1 - \sum_{\alpha} e^{-\beta \epsilon_{\alpha}}} \end{array} \right.$$

Next, we have to substitute to the sums the integrals over all states, via the density of the states for massless particles :

$$g(\varepsilon) d\varepsilon = \frac{V}{2\pi^2} \varepsilon^2 d\varepsilon \quad (k=c=1)$$

Then

$$\left\{ \begin{array}{l} E_{ind} = \int_0^\infty \frac{V}{2\pi^2} \frac{\varepsilon^3}{e^{\beta\varepsilon} - 1} d\varepsilon \\ E_{dis} = \int_0^\infty \frac{V}{2\pi^2} \frac{\varepsilon^3 e^{-\beta\varepsilon}}{1 - \frac{V}{2\pi^2} \int_0^\infty \varepsilon^2 e^{-\beta\varepsilon} d\varepsilon} d\varepsilon = \int_0^\infty \frac{V}{2\pi^2} \frac{\varepsilon^3 e^{-\beta\varepsilon}}{1 - \frac{V}{\pi^2 \beta^3}} d\varepsilon \end{array} \right. \quad (B.5)$$

Since we have by definition :

$$E = \int_0^\infty \varepsilon \frac{dn}{d\varepsilon} d\varepsilon$$

the energy distribution functions will be :

$$\left(\frac{dn}{d\varepsilon} \right)_{ind} = \frac{V}{2\pi^2} \frac{\varepsilon^2}{e^{\beta\varepsilon} - 1} \quad (B.6)$$

$$\left(\frac{dn}{d\varepsilon} \right)_{dis} = \frac{V}{2\pi^2} \frac{\varepsilon^2 e^{-\beta\varepsilon}}{1 - \frac{V}{\pi^2 \beta^3}} \quad (B.7)$$

Whereas Eq. (B.6) is the well-known energy distribution for Bose-Einstein statistics, the distribution (B.7) characterizes a new one, of the Maxwell-Boltzmann "type", but for which the special condition

$$\frac{V}{\pi^2 \beta^3} < 1 \quad (B.8)$$

must hold [Eq. (B.8) ensures also the convergence of the series

$$\sum_N \left(\sum_{\alpha} z_{\alpha} \right)^N$$

which appears in the second equation (B.4)]. For constant V , this rather surprising condition limits the temperature to values :

$$KT < \left(\frac{\pi^2}{V} \right)^{1/3}$$

the limit $(\pi^2/V)^{1/3}$ being approached asymptotically when $E \rightarrow \infty$ [see Eq. (V.2')].

R E F E R E N C E S

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FIGURE CAPTIONS

Figure 1 Example of the accuracy of the asymptotic expression
 $y_{\alpha,\beta}(x)$ for the sum $S_{\alpha,\beta}(x)$.

Figure 2 The ratio $g(E)$ under various assumptions.

FIG.1

FIG.2