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REMARKS ON AN ENLARGED POINCARÉ GROUP :
INHOMOGENEOUS $SL(6, \mathbb{C})$ GROUP

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INTRODUCTION

Many attempts ¹⁾ have been made in order to connect the Poincaré group $\mathcal{P}$ (inhomogeneous Lorentz group) with internal symmetries.

A solution for a much simpler problem ²⁾ has been suggested independently by several authors : the SU_3 group is embedded with the rotation group SU_2 in a simple Lie group, namely SU_6 . Relativistic generalizations of this (static) SU_6 group have been proposed ³⁾. In this paper we want to discuss some properties of an enlarged group $\mathcal{P}_E$, which contains both $\mathcal{P}$ and SU_6 as subgroups and has been suggested independently by many authors ⁴⁾ : the inhomogeneous $SL(6,C)$ group. At first sight the group $\mathcal{P}_E$, which is not semi-simple, does not seem very attractive : in fact it possesses 36 "translation operators" !

But it is important to remark that before deciding if a group is suitable for physics or not, one has first to set up a correspondence between the mathematical symbols and the physical observables. Therefore we begin in Section I by a mathematical description of the group with its commutation relations and representations. In Section II a natural interpretation of the group is given. It is seen that this interpretation possesses nice features; for example, many of the 36 possible momenta are automatically unphysical. Mass relations result without any breaking of the group.

- I. The enlarged Poincaré group $\mathcal{P}_E$ is a natural generalization of the Poincaré group $\mathcal{P}$. The group $\mathcal{P}$ (more exactly, its covering group $\overline{\mathcal{P}}$) is usually defined as the transformations (Λ, a) where Λ is a 2×2 complex unimodular matrix [the Λ matrices generate the group $SL(2,C)$] and a is a 2×2 Hermitian matrix. The transformations are written

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$$\begin{aligned}
 x &\rightarrow x' = \Lambda x \Lambda^+ + a \\
 \det \Lambda &= 1 \\
 x^+ &= x, \quad a^+ = a \\
 \det x' &= \det x
 \end{aligned}
 \tag{1}$$

The canonical commutation relations are given in Eqs. (2.1), (2.2), (2.3) below. A finite dimensional (non-unitary) representation is given in Table 1.

The group $\overline{\mathcal{P}}_E$ can also be defined by means of Eq. (1), but now x, Λ, a are 6×6 matrices. This group has 106 generators: 70 corresponding to the homogeneous transformations Λ and 36 translations.

A useful 12 dimensional representation is written in Table 2⁵⁾. Commutation relations of the group $\mathcal{P}_E$ exhibiting the Lorentz $\otimes$ SU_3 indices can be readily derived.

$$[M^{\mu\nu}, M^{\rho\lambda}] = i(g^{\nu\rho} M^{\mu\lambda} - g^{\nu\lambda} M^{\mu\rho} - g^{\mu\rho} M^{\nu\lambda} + g^{\mu\lambda} M^{\nu\rho}) \tag{2.1}$$

$$[M^{\mu\nu}, P^\rho] = i(g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu) \tag{2.2}$$

$$[P^\mu, P^\nu] = 0 \tag{2.3}$$

$$[P^\mu, P^{\nu(m)}] = 0 \tag{2.4}$$

$$[P^{\mu(m)}, P^{\nu(m)}] = 0 \tag{2.5}$$

$$[M^{(m)}, M^{(n)}] = 2i f^{mnp} M_{(p)} \quad (2.6)$$

$$[M^{\mu\nu}, M^{(m)}] = [M^{\mu\nu}, N^{(n)}] = [M^{(m)}, P^\mu] = 0 \quad (2.7)$$

$$[N^{(m)}, P^\mu] = 2i P^\mu M_{(m)} \quad (2.8)$$

$$[M^{\mu\nu}, M^{\rho\lambda(m)}] = i(g^{\nu\rho} M^{\mu\lambda(m)} - g^{\nu\lambda} M^{\mu\rho(m)} - g^{\mu\rho} M^{\nu\lambda(m)} + g^{\mu\lambda} M^{\nu\rho(m)}) \quad (2.9)$$

$$[M^{\mu\nu(m)}, M^{\rho\lambda(n)}] = \frac{2}{3} i S^{mn} (g^{\nu\rho} M^{\mu\lambda} - g^{\nu\lambda} M^{\mu\rho} - g^{\mu\rho} M^{\nu\lambda} + g^{\mu\lambda} M^{\nu\rho}) \\ + \frac{i}{2} f^{mnp} (g^{\mu\rho} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\rho}) M_{(p)} \\ + i d^{mnp} (g^{\nu\rho} M^{\mu\lambda}_{(p)} - g^{\nu\lambda} M^{\mu\rho}_{(p)} - g^{\mu\rho} M^{\nu\lambda}_{(p)} + g^{\mu\lambda} M^{\nu\rho}_{(p)}) \quad (2.10)$$

$$[M^{\mu\nu(m)}, M^{(n)}] = 2i f^{mnp} M^{\mu\nu}_{(p)} \quad (2.11)$$

$$[M^{\mu\nu(m)}, N^{(n)}] = i \varepsilon^{\mu\nu\rho\lambda} f^{mnp} M_{\rho\lambda(p)} \quad (2.12)$$

$$[M^{(m)}, N^{(n)}] = 2i f^{mnp} N_{(p)} \quad (2.13)$$

$$[N^{(m)}, N^{(n)}] = -2i f^{mnp} M_{(p)} \quad (2.14)$$

$$[M^{\mu\nu}, P^\rho(m)] = i(g^{\nu\rho} P^\mu(m) - g^{\mu\rho} P^\nu(m)) \quad (2.15)$$

$$[M^{\mu\nu(n)}, P^\rho] = i(g^{\nu\rho} P^\mu(m) - g^{\mu\rho} P^\nu(m)) \quad (2.16)$$

$$[M^{\mu\nu(m)}, P^\rho(n)] = \frac{2}{3} i S^{mn} (g^{\nu\rho} P^\mu - g^{\mu\rho} P^\nu) + i \varepsilon^{\mu\nu\rho\lambda} f^{mnp} P_{\lambda(p)} \\ + i d^{mnp} (g^{\nu\rho} P^\mu_{(p)} - g^{\mu\rho} P^\nu_{(p)}) \quad (2.17)$$

$$[M^{(m)}, P^\rho(n)] = 2i f^{mnp} P^\rho_{(p)} \quad (2.18)$$

$$[N^{(m)}, P^\rho(n)] = -\frac{4i}{3} S^{mn} P^\rho - 2i d^{mnp} P^\rho_{(p)} \quad (2.19)$$

where $\varepsilon_{0123} = 1$, $g_{00} = -g_{ii} = 1$.

The coefficients f^{mnp} and d^{mnp} are those defined by Gell-Mann ⁶⁾. The 36 translation operators P_μ and $P_{\mu(m)}$ may be written in the matrix form

$$\hat{P} = \sum_{\mu=0}^3 \sum_{m=0}^8 \sigma^\mu \otimes \lambda^m P_{\mu(m)} \quad (3)$$

The determinant of this matrix is an invariant of the group and is one of the operators characterizing the irreducible representations of $\mathcal{P}_E$. A necessary condition for obtaining SU_6 as a little group is that $\det \hat{P} > 0$ ⁷⁾. In such a case the irreducible representations are characterized by $\det \hat{P}$ and the Casimir operators of SU_6 .

Let us turn now to the representations of the homogeneous part of $\mathcal{P}_E$, namely the $SL(6, \mathbb{C})$ group. They can be labelled by two numbers corresponding to dimensions of irreducible representations of SU_6 in the form $D(n, m)$, with the following defining transformation properties

$$\begin{aligned} D(6, 1): \quad \psi &\rightarrow \psi' = \Lambda \psi \\ D(1, 6): \quad \psi &\rightarrow \psi' = (\Lambda^\dagger)^{-1} \psi \\ D(\bar{6}, 1): \quad \psi &\rightarrow \psi' = (\Lambda^\dagger)^{-1} \psi \\ D(1, \bar{6}): \quad \psi &\rightarrow \psi' = \Lambda^* \psi \end{aligned} \quad (4)$$

The matrices M, N of Table 2 are given in the reducible representation $(6, 1) \oplus (1, 6)$. [Note that they are reducible with respect to $\mathcal{P}_E$.] The P_μ and $P_{\mu(m)}$ matrices provide a basis of the representation $(\bar{6}, 6)$.

Multiplication rules are

$$D(n,m) \otimes D(n',m') = D(n \otimes n', m \otimes m') \quad (5)$$

where the reduction $n \otimes n'$ is that of SU_6 .

According to Eq. (4) the conjugate of $D(n,m)$ is

$$D^*(n,m) = D(\bar{m}, \bar{n}) \quad (6)$$

With respect to the maximal subgroup SU_6 of $SL(6,C)$, $D(n,m)$ reduces according to

$$D(n,m) \xrightarrow{SU_6} n \otimes m \quad (7)$$

It is important for the physical interpretation of the group to stress the two following points :

- a. it is well known [Eqs. (4) and (6)] that bilinear Hermitian invariants cannot be built using the representation $D(n,m)$ alone unless $m=n$. In other cases we must use the reducible representation ⁸⁾ $D(n,m) \oplus D(m,n)$:
- b. many inequivalent parity operations Π (such that $\Pi^2 = 1$) may be defined for the representations of $SL(6,C)$. Each of them has to transform $D(m,n)$ into one of the eight following representations

$D(n,m)$	$D(m,n)$	(8)
$D(\bar{n}, \bar{m})$	$D(\bar{m}, \bar{n})$	
$D(\bar{n}, m)$	$D(m, \bar{n})$	
$D(n, \bar{m})$	$D(\bar{m}, n)$	

In order to identify the usual space parity, one makes the following assumptions.

- i) Π induces on the subgroup $SL(2, \mathbb{C})$ the usual space parity : this requirement implies that the order of labels m and n is changed under Π since for the Lorentz group Π turns $D(j, j')$ into $D(j', j)$. This discards the four cases in the second column of (8).
- ii) Π has to induce an automorphism of SU_6 : we are left with the following two solutions

$$D(m, n) \xrightarrow{\Pi} D(n, m) \quad (9)$$

$$D(m, n) \xrightarrow{\Pi'} D(\bar{n}, \bar{m}) \quad (10)$$

Remembering that P_μ and $P_{\mu(m)}$ belong to $(\bar{6}, 6)$ one has

$$D(\bar{6}, 6) \xrightarrow{\Pi} D(6, \bar{6}) \quad (11)$$

$$D(\bar{6}, 6) \xrightarrow{\Pi'} D(\bar{6}, 6) \quad (12)$$

This shows that if one chooses the parity Π , the P 's do not have a well-defined parity and one has to double the number of P 's following

$$D(\bar{6}, 6) \oplus D(6, \bar{6}) \quad (13)$$

This is not the case for Π' where there exist two non-equivalent representations of the parity extended group

$$D(\bar{6}, 6)^+ \quad \text{and} \quad D(\bar{6}, 6)^- \quad (14)$$

II. One has now to give a physical interpretation of the group $\mathcal{P}_E$. By interpretation we mean a correspondence between the elements of the Lie algebra or of its enveloping algebra, and physical observables.

The existence of a 36 dimensional Abelian invariant subgroup generated by the P_μ 's and the $P_{\mu(m)}$'s appears at first sight as a difficulty but the following remarks will lead to a reasonable and attractive interpretation.

- a. Suppose first that the physical four-momentum operators are represented by the four generators P_μ . Following Dirac's ⁹⁾ postulates of quantum mechanics, one has to complete the set of commuting observables. Among these observables we choose of course the usual operators like spin, charge, isospin. It may happen that after such a choice some $P_{\mu(m)}$'s cannot be added to this set of observables. We can no longer speak of a 36 dimensional momentum. Such an idea is not new and is already needed by the SU_6 theory ¹⁰⁾.
- b. The physical energy momentum operators Q_μ need not be generators of the group $\mathcal{P}_E$. In other words, the Q_μ 's can be chosen in the enveloping algebra instead of the Lie algebra. They must commute and behave as Lorentz vectors.

We turn now to a tentative interpretation of $\mathcal{P}_E$.

- The $M_{\mu\nu}$, (2), are assumed to be the generators of the homogeneous Lorentz group.
- The $M_{(m)}$ are assumed to be the generators of the usual SU_3 group $[M_{(1)} \ M_{(2)} \ M_{(3)}]$ are the generators of isospin and $M_{(8)}$ the hypercharge.]
- We require that the physical energy momenta Q_μ obey the commutation rules (2.2) and (2.3) and commute with $M_{(3)}$ and $M_{(8)}$ (if we neglect only weak interactions).
- As an example, let us assume that the Q_μ belong to a representation $(\bar{6}, 6)$ and, to be definite, are the generators P_μ , $P_{\mu(m)}$.

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These four assumptions lead to a very important and restrictive result : only 12 among the 36 P_μ 's may be included in the set of observables. The most general form of the Q_μ is then

$$Q_\mu = \alpha P_\mu + \beta P_{\mu(8)} + \gamma P_{\mu(3)} \quad (15)$$

Moreover, it is important to note that once a vector Q_μ is given (i.e., coefficients α, β, γ are chosen) no other momentum can be added to the set of commuting observables if this set already includes the polarization operator :

$$W^2 = W_\mu W^\mu \quad (16)$$

where

$$W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\lambda} M^{\rho\lambda} Q^\nu \quad (17)$$

Let us define the following gradation :

$$1) \quad \underline{\beta = \gamma = 0} \quad Q_\mu = \alpha P_\mu$$

Then C_2 and C_3 (the Casimir operators of SU_3) and $\vec{I}^2$ belong to the set of commuting observables. This describes the very strong interactions. Masses are constants within a given SU_3 multiplet.

$$2) \quad \underline{\gamma = 0} \quad Q_\mu = \alpha P_\mu + \beta P_{\mu(8)}$$

C_2 and C_3 are no longer sharp quantum numbers, $\vec{I}^2$ stays sharp. For the limit of β smaller than α , the mass operator squared

$$Q_\mu Q^\mu = \alpha^2 P_\mu P^\mu + 2\alpha\beta P_\mu P^\mu_{(8)} + \beta^2 P_{\mu(8)} P^\mu_{(8)} \quad (18)$$

apart from a term proportional to β^2 which may be neglected has the right transformation properties to give mass formulae "à la Gell-Mann Okubo" ¹¹⁾. This approximation includes the medium-strong interactions.

$$3) \quad Q_\mu = \alpha P_\mu + \beta P_\mu(8) + \gamma P_\mu(3)$$

$\bar{I}^2$ is no longer a good quantum number and electromagnetic mass relations will result in an analogous fashion.

The unitary symmetry appears thus as a good approximation insofar as β and γ are small compared to α and can be neglected.

Finally we want to comment on the fourth hypothesis. We have chosen the Q_μ 's as belonging to an irreducible tensor P all components of which are commuting. This is by no means implied by a physical argument, since only the $4Q_\mu$ must commute. Moreover, the tensor P has been taken as a $(\bar{6}, 6)$ tensor ¹²⁾. Again this is not necessary. Considerations based on the generalized Klein-Gordon equation

$$(\det \hat{P} - m^6) \psi = 0 \quad (19)$$

(which is of sixth order in the P 's) suggest that one defines the physical Q_μ as trilinear functions of the P 's ¹³⁾.

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T A B L E 14-dimensional representation of the Poincaré group

rotations

$$M_{ij} = \frac{1}{2} \varepsilon_{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

pure Lorentz transformations

$$M^{0k} = \frac{1}{2} \begin{pmatrix} i\sigma^k & 0 \\ 0 & -i\sigma^k \end{pmatrix}$$

translations

$$P^\mu = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ \sigma^\mu & 0 \end{pmatrix}$$

where $\sigma^\mu = (1, \vec{\sigma})$ are the 2×2 unit matrix and the usual Pauli matrices.

TABLE 2

12-dimensional representation of $\mathcal{P}_E$

$$\begin{aligned}
 & \left\{ \begin{aligned} M^{ij} &= \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k \otimes \mathbb{1} & 0 \\ 0 & \sigma^k \otimes \mathbb{1} \end{pmatrix} \\ M^{(m)} &= \begin{pmatrix} \mathbb{1} \otimes \lambda^m & 0 \\ 0 & \mathbb{1} \otimes \lambda^m \end{pmatrix} \\ M^{ij(m)} &= \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k \otimes \lambda^m & 0 \\ 0 & \sigma^k \otimes \lambda^m \end{pmatrix} \end{aligned} \right. \\
 & \left\{ \begin{aligned} M^{\mu k} &= \frac{1}{2} \begin{pmatrix} i \sigma^k \otimes \mathbb{1} & 0 \\ 0 & -i \sigma^k \otimes \mathbb{1} \end{pmatrix} \\ N^{(m)} &= \begin{pmatrix} i \mathbb{1} \otimes \lambda^m & 0 \\ 0 & -i \mathbb{1} \otimes \lambda^m \end{pmatrix} \\ M^{\mu k(m)} &= \frac{1}{2} \begin{pmatrix} i \sigma^k \otimes \lambda^m & 0 \\ 0 & -i \sigma^k \otimes \lambda^m \end{pmatrix} \end{aligned} \right. \\
 & \left\{ \begin{aligned} P^\mu &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ \sigma^\mu \otimes \mathbb{1} & 0 \end{pmatrix} \\ P^{\mu(m)} &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ \sigma^\mu \otimes \lambda^m & 0 \end{pmatrix} \end{aligned} \right.
 \end{aligned}$$

35 generators of SU_6

35 generators of pure "Lorentz transformations"

36 "translations"

where $a \otimes b$ means the direct product of a 2×2 matrix a by a 3×3 matrix b . The indices ijk run from 1 to 3, μ from 0 to 3 and (m) from 1 to 8.

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- 4) B. Sakita, see Ref. 2);
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- 5) It is easily seen from Table 1 that the group $\overline{\mathcal{P}}$ is a subgroup of U(2,2), the group of 2x2 complex matrices leaving invariant the following quadratic form :

$$\sum_{i=1}^2 |x_i|^2 - \sum_{i=3}^4 |x_i|^2$$

SU(2,2) is the covering group of the conformal group, usually defined as the group of all space-time transformations preserving the Maxwell equations.

In complete analogy with $\overline{\mathcal{P}}$, the group $\overline{\mathcal{P}}_E$ is a subgroup of $U(6,6)$, the group of 6×6 complex matrices leaving invariant the quadratic form :

$$\sum_{i=1}^6 |x_i|^2 - \sum_{i=7}^{12} |x_i|^2$$

The generators of $U(6,6)$ can be obtained as direct products $\Gamma \times \lambda$ where Γ is one of the sixteen Dirac matrices and λ one of the nine 3×3 U_3 - matrices.

- 6) M. Gell-Mann, Phys.Rev. 125, 1067 (1962).
- 7) All possible little groups of $\overline{\mathcal{P}}_E$ can be shown to be isomorphic to the subgroups of $SL(6, \mathbb{C})$ generated by matrices Λ [Eq.(1)] which preserve a diagonal 6×6 matrix with elements +1, -1, or 0. The SU_6 group is obtained if this matrix is taken to be the unit matrix.
- 8) See, e.g., M. Hamermesh, Group Theory (Addison Wesley, 1962), p. 136.
- 9) P.A.M. Dirac, The Principles of Quantum Mechanics (Oxford, 1958).
- 10) SU_6 is a compact Lie group of rank 5; therefore there exist 5 additive quantum numbers, but only three of them - namely J_z , I_3 and Y commute with the operators $C_2, C_3, \vec{J}^2, \vec{I}^2$ where C_2 and C_3 are the Casimir operators of SU_3 and $\vec{J}^2$ and $\vec{I}^2$ the Casimir operators of spin and isospin groups.
- 11) M. Gell-Mann, see Ref. 6);
S. Okubo, Prog.Theor.Phys. 27, 349 (1962).
- 12) Note that the tensor $(\overline{6}, 6)$ has a definite parity only for the operation Π .
- 13) The Q_μ 's can then, for example, belong to the representation $(20, 20)$. Note that the representation $(20, 20)$ has a definite parity as well under operation Π as under operation Π' .