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ON THE THEORY OF SCATTERING BY SINGULAR POTENTIALS *)

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ABSTRACT

Some characteristic features of the scattering by singular potentials are investigated both in the low and the high energy limit.

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I. INTRODUCTION

The problem of obtaining explicit expressions for the scattering amplitude from the Schrödinger equation with singular potentials has received recently some interest since the infinities appearing in the perturbation expansion have some analogy with those appearing in non-renormalizable field theories ¹⁾.

In particular, singular potentials have been used to test the soundness of the general peratization programme of Feinberg and Pais ²⁾.

A very interesting general method has been recently developed by Cornille and Predazzi ³⁾, which is based on the idea of extracting in an exact manner the most singular part of the wave function. One obtains the S matrix in the form of a convergent sequence of approximations.

In this paper we wish to discuss the singular potential problem from a different view-point. In particular, we are interested in finding expansions valid either at low or at high energies.

In this paper we shall limit our discussion to repulsive potentials of the form $-\lambda^2/r^n$ ^{*)}; in some cases we shall indicate how the results could be extended to more general cases.

In our case the Schrödinger equation is

$$r^2 \frac{d^2 \psi}{dr^2} + \left(K^2 r^2 - \lambda^2 / r^{n-2} \right) \psi = 0 \quad (1)$$

where, in order to avoid infra-red troubles, we must have $n > 1$. The parameter λ^2 has dimensions $(\text{length})^{n-2}$, so that clearly the case $n=2$, where the constant λ is dimensionless, plays a very special role : it divides [see Ref. ⁴⁾] the case of regular to singular potentials. It is now convenient to change the scale of r writing

*) For simplicity, we shall always consider only the $\ell = 0$ case.

2.

$$\mu = \kappa \mu_0$$

so that

$$x^2 \frac{d^2 \psi}{dx^2} + \left[(\kappa^2 \mu_0^2) x^2 - \frac{\lambda^2}{\mu_0^{n-2} x^{n-2}} \right] \psi = 0 \quad (2)$$

We choose now the free parameter μ_0 in order that the kinetic and the potential terms have the same coefficient

$$\frac{\lambda^2}{\mu_0^{n-2}} = \kappa^2 \mu_0^2 = f^2 \quad (3)$$

so that we finally get the canonical form

$$x^2 \frac{d^2 \psi}{dx^2} + f^2 \left[x^2 - \frac{1}{x^{n-2}} \right] \psi = 0 \quad (4)$$

$$\mu_0 = \left(\frac{\lambda}{\kappa} \right)^{2/n} \quad (5)$$

$$f = \lambda^{2/n} \kappa^{(1 - \frac{2}{n})} \quad (6)$$

Having written the Schrödinger equation in the form (4), we see that everything depends on the only dimensionless parameter f .

One sees again here the big difference between the singular and regular potentials. In the first case f increases with the momentum, in the latter it decreases. Therefore for the singular case in which we are interested, weak coupling means also low energy and strong coupling high energy.

Section III will be devoted to the first case, Section IV to the second one.

II. WEAK COUPLING LIMIT

As discussed in Section I, we are interested in the properties of the S wave scattering amplitude from a singular potential of the simple power form $V(r) = -\lambda^2/r^n$, $n > 2$. In this paper we shall consider only the very special case of the potential

$$V(r) = -\frac{\lambda^2}{r^4}$$

The reason for this choice is twofold : first the exact solution is already known and can be obtained in terms of Mathieu functions⁵⁾; we can therefore test our method of approximation by comparing our results with those obtained from the exact solutions.

Secondly, with this potential the Schrödinger equation possesses remarkable symmetry properties, as it will be explained in the following, which simplify very much the problem of obtaining the weak coupling limit for the S matrix.

The more general problem of the potential $1/r^n$ is under investigation by Del Giudice and Galzenati⁶⁾. We shall therefore start from the equation

$$\left[\frac{d^2}{dx^2} + f^2 \left(1 - \frac{1}{x^4} \right) \right] \psi(x) = 0 \quad (7)$$

The asymptotic behaviours at the two boundaries are

$$x \rightarrow 0 \quad \psi \rightarrow x e^{\pm i f/x} \quad (8)$$

$$x \rightarrow \infty \quad \psi \rightarrow e^{\pm i f x} \quad (9)$$

Now the underlying idea of our procedure, which leads to a determination of the S matrix, is the following : we start from the origin and we construct through the Volterra method an iterative solution ψ_1 , with

the regular behaviour (2) (- sign). Analogously, starting from infinity we construct by iterations two solutions with the behaviours (3) or their combination, φ_2 , which involves the S matrix. Then we want to join φ_1 and φ_2 in an intermediate point r_1 asking the continuity of the function and of its first derivative, that is the vanishing of their Wronskian $W \{ \varphi_1, \varphi_2 \}_{r=r_1} = 0$. For instance, in the Jost function method this point is chosen to be the origin.

There are two peculiar difficulties inherent to this procedure. The first one is related to the choice of the point r_1 , which is in principle arbitrary, so that, although the final answer is independent of this parameter, an approximate result, obtained by a finite number of iterations, could sensibly depend on it.

Secondly, φ_1, φ_2 are both evaluated through a method of successive approximations and as a consequence it is difficult to be sure that these φ_1, φ_2 , when calculated at r_1 , are of the same order.

The problem we consider here presents particular properties of symmetry and exploiting them we can overcome in a very simple way the two difficulties discussed above. [However, in the more general case the fundamental idea will be the same that is the use of dimensionless variables and the natural choice of the point of connection.] We introduce then the new function

$$\psi(x) = x^{-\frac{1}{2}} \varphi(x) \quad (10)$$

which obeys the equation

$$\left\{ \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} - \frac{1}{4x^2} + \frac{f^2}{x^2} \left(x^2 - \frac{1}{x^2} \right) \right\} \psi(x) = 0 \quad (11)$$

We begin by observing that Eq. (11) is invariant with respect to the two transformations

$$\begin{aligned}
 \text{a)} \quad x &\rightarrow \pm \frac{i}{x} & f &\rightarrow f \\
 \text{b)} \quad x &\rightarrow \frac{1}{x} & f &\rightarrow \pm i f
 \end{aligned} \tag{12}$$

The first transformation maps, in the complex plane, the region at the interior of the circle $|x| = 1$ in its complementary and has the invariant points $x^2 = \pm i$, i.e., $x = \pm \sqrt{i}$, $\pm \sqrt{-i}$. The second one, which remains on the real axis, transforms the strip $0 \leq x \leq 1$ in the strip $1 < x < \infty$ and has the two symmetry points $x = \pm 1$; only $x = 1$ is in the physical region. To avoid the discussion in the complex plane where all the symmetry points are on the same footing (they are all complex), we prefer to use in the following the transformation (12b): this tells us that if $\Psi(x, f)$ is a solution of Eq. (11), $\Psi(1/x, \pm i f)$ is also a solution (they do not in general coincide).

In particular, if we consider the solution Ψ_0 regular at the origin, whose behaviour is

$$\Psi_0(x, f) \sim x^{1/2} e^{-f/x} \quad x \rightarrow 0 \tag{13}$$

it is transformed, with (12b), into the pair $\Psi_{\pm}$ of solutions whose asymptotic forms are

$$\Psi_{\pm}(x, f) = \Psi_0\left(\frac{1}{x}, \mp i f\right) \sim x^{-1/2} e^{\pm i f x} \quad x \rightarrow \infty \tag{14}$$

In other words, we are sure that if we consider in the region $x < 1$, the solution Ψ_0 with the behaviour (13), we can get through (12b) the solutions $\Psi_{\pm}$, defined for $x > 1$ and characterized by the boundary conditions (14).

Thus it will be sufficient to apply the standard Volterra method to construct for $x < 1$ the regular solution Ψ_0 . $\Psi_{\pm}$ are derived by symmetry. In this way we have two iterative expansions (for Ψ_0 and $\Psi_{\pm}$) in the same parameter "f" which guarantees that approximations of

the same order are done starting from the origin and infinity. Moreover, the choice of the point of connection is a quite natural one. We have to take $x = 1$, the invariant point of the transformation. To determine the S matrix we remember that the physical wave function at infinity is

$$\psi(x, f) = A \{ \psi_-(x, f) - S(f) \psi_+(x, f) \} \quad (15)$$

and at the origin

$$\psi(x, f) = B \psi_0(x, f)$$

The continuity requirement at $x = 1$ amounts to

$$\left. \frac{\dot{\psi}_- - S \dot{\psi}_+}{\psi_- - S \psi_+} \right|_{x=1} = \left. \frac{\dot{\psi}_0}{\psi_0} \right|_{x=1} \quad (16)$$

and from that

$$\begin{aligned} S = \left. \frac{\dot{\psi}_- \psi_0 - \dot{\psi}_0 \psi_-}{\dot{\psi}_+ \psi_0 - \dot{\psi}_0 \psi_+} \right|_{x=1} &= \frac{\left\{ \left[\frac{d}{dx} \psi_0 \left(\frac{1}{x}, if \right) \right] \psi_0(x, f) - \left[\frac{d}{dx} \psi_0(x, f) \right] \psi_0 \left(\frac{1}{x}, if \right) \right\}}{\left\{ \left[\frac{d}{dx} \psi_0 \left(\frac{1}{x}, -if \right) \right] \psi_0(x, f) - \left[\frac{d}{dx} \psi_0(x, f) \right] \psi_0 \left(\frac{1}{x}, -if \right) \right\}} \Big|_{x=1} \\ &= \frac{\left\{ \left[\frac{d}{dy} \psi_0(y, if) \right] \psi_0(1, f) + \left[\frac{d}{dy} \psi_0(y, f) \right] \psi_0(1, if) \right\}}{\left\{ \left[\frac{d}{dy} \psi_0(y, -if) \right] \psi_0(1, f) + \left[\frac{d}{dy} \psi_0(y, f) \right] \psi_0(1, -if) \right\}} \Big|_{y=1} \end{aligned}$$

Finally we can write

$$S = \left. \frac{\left[\frac{d}{dx} \psi_0(x, if) \right] \psi_0(x, f)}{\left[\frac{d}{dx} \psi_0(x, -if) \right] \psi_0(x, f)} \right\} \Big|_{x=1} \quad (17)$$

We have now to write down the Volterra integral equation corresponding to Eq. (7), for $x = 1$. It is more convenient to do this introducing the new variable $g = 1/x$ and studying the function $\varphi(g) = g^{\frac{1}{2}} \psi(g)$. In other words, we consider the equation

$$\left[\frac{d^2}{dg^2} - \frac{f^2}{g^2} \left(g^2 - \frac{1}{g^2} \right) \right] \varphi(g) = 0 \quad (18)$$

We study it in the region $g > 1$ and we look for the regular solution with the behaviour

$$\varphi(g) \sim e^{-fg} \quad g \rightarrow \infty \quad (19)$$

The corresponding integral equation is

$$\varphi_0(g) = e^{-fg} + \frac{f}{2} \int_g^\infty \left[e^{f(g-s')} - e^{-f(g-s')} \right] \frac{ds'}{s'^4} \varphi_0(s') \quad (20)$$

and we want to derive a first few iterations. We have

$$\varphi_0^{(0)}(g) = e^{-fg}$$

Then in $\varphi_0^{(1)}$ we integrate repeatedly by part, adding each time a power of "f" in front, until we find a logarithmic term

$$\varphi_0^{(1)} = e^{-fg} \left\{ -\frac{f^2}{6g^2} + \frac{f^3}{3g} - \frac{2}{3} f^4 e^{2fg} \int_g^\infty e^{-2fs'} \frac{ds'}{s'} \right\}$$

If we put

$$E(2fg) = \int_{2fg}^\infty dy \frac{e^{-y}}{y}$$

using the well-known development of the exponential integral function, we get for small f

$$E(2fg) = -\gamma - \ln(2fg) + 2fg - f^2 g^2 + O(f^3) \quad (21)$$

(γ is the Euler-Mascheroni constant). Analogously the evaluation of the f^6 terms for the other iterations $\varphi_0^{(2)}$, $\varphi_0^{(3)}$ gives no troubles and finally the wave function is, to the f^6 order,

$$\begin{aligned} \varphi_0(s) = e^{-fs} \left\{ 1 - \frac{f^2}{6s^2} + \frac{f^3}{3s} + \frac{2}{3} f^4 [\gamma + \ln(2fs)] + \frac{4}{3} f^5 s [\gamma + \ln(2fs)] \right. \\ \left. - 1 \right] + \frac{4}{3} f^6 s^2 [\gamma + \ln(2fs) - \frac{3}{2}] + \frac{f^4}{120s^4} - \frac{f^5}{30s^3} - \\ \left. - \frac{f^6}{9s^2} \ln(2fs) - A f^6 \right\} \end{aligned} \quad (22)$$

where A is a real constant which is not important for the determination of the S matrix. Moreover, for convenience, we do not develop the e^{-fs} factor in front. We can then calculate φ_+ , φ_- from (22), go to ψ_0 , ψ_+ , ψ_- and writing the S matrix in the form

$$S = \frac{1 + i \operatorname{tg} \delta}{1 - i \operatorname{tg} \delta}$$

we get

$$\operatorname{tg} \delta = -f^2 + \frac{\pi}{3} f^4 + \left(\frac{8}{3} \gamma - \frac{28}{9} \right) f^6 + \frac{8}{3} f^6 \ln(2f) \quad (23)$$

Equation (23) is the small f expansion for the phase shift. We want to recall that, as discussed in the previous Section, this means either a weak coupling limit, or an effective range expansion. The correctness of this result can be checked using the calculations done starting from the exact solutions of the Schrödinger equation ⁷⁾.

There are some features we want to remark in these results. First of all, $\operatorname{tg} \delta$ is not analytic at $f=0$ where it presents a branch point of the logarithmic type. The origin of this singularity can be recognized as due both to the long range and to the singular behaviour at the origin

of the potential, i.e., it is both an infra-red and an ultra-violet divergence. In fact, if we remember that $f = \sqrt{k\lambda}$ the logarithmic singularity has the form $(\ln k + \ln \lambda)$. The first term is a display of the fact that in the complex k plane the cut due to the potential comes up to the origin. If we improved the behaviour at infinity (for instance, using a potential $e^{-\mu r}/r^4$) we should get a term $\sim \ln(\mu\lambda)$, without singularity as $k \rightarrow 0$. The $\ln \lambda$ part is a consequence of the highly singular form of the potential at the origin: regularizing it at small distances ($V \sim 1/(r+a)^4$) we could eliminate the $\lambda = 0$ singularity. On the other hand, aside this singularity, we have always a dependence on even powers of " f " that is on k , not on $\sqrt{k}$ which should introduce an extra branch point at $k=0$. We want to call the attention on the analogy that Eq. (23), owing to the presence of the $f^6 \ln f$ term, shows with the results of Lee and Yang⁸⁾ in their investigation of the non-renormalizable electrodynamics of vector mesons.

For what concerns the k dependence of the wave function φ_0 we know, on the basis of the Poincaré theorem, that it has to be an entire function of k . The introduction of f masks a bit this property [see, e.g., Eq. (23) where there is a dependence on $f, f^3 \dots$] but it can be easily verified that if we re-express everything through the old variable r with dimensions, the dependence on k is restored.

The simplicity of our analysis rests on the transformation properties of the Schrödinger equation with respect to $x \rightarrow 1/x$. However, these results maintain their validity also in the presence of another potential $V'(x)$ provided it verifies the relation $x^4 V'(x) = V'(1/x)$, i.e., $V'(x) = \alpha^2/x^2$. This shows that the effect of higher waves can be easily included. [It would be sufficient to add $-\ell(\ell+1)/x^2$ in the kernel of the Volterra equation (20).]

III. THE HIGH ENERGY LIMIT

Let us now discuss the high energy limit of potential scattering. We start, for sake of definiteness, from Eq. (24), valid for a restricted class of potentials (the discussion will then be extended to the general case) :

$$\left[\frac{d^2}{dx^2} + f^2 V(x) \right] \phi(x) = 0 \quad (24)$$

$$V(x) = 1 - \frac{1}{x^n} \quad n > 1 \quad (25)$$

$$f = \lambda^{\frac{2}{n}} K^{1 - \frac{2}{n}} \quad (26)$$

Equations (25) and (26) show clearly the fundamental difference between regular and singular potentials for what concerns the high energy limit.

In the first case ($1 - 2/n < 0$), large k means small f and so we understand why the Born approximation gives the right high energy limit.

In the second case ($1 - 2/n > 0$), large k means large f and therefore we have to make use of an expansion in inverse powers of f .

Using the techniques developed in the preceding paper ⁹⁾, we obtain

$$\frac{1}{f} \left(\delta - \frac{\pi}{4} \right) = a_0 + \frac{1}{f^2} a_1 + \frac{1}{f^4} a_2 + \dots \quad (27)$$

where

$$a_0 = -\frac{1}{2} \int_1^{\infty} \frac{dx}{x^n} \frac{1}{(1 - \frac{1}{x^n})^{1/2}} = -\frac{1}{2n} B(1 - \frac{1}{n}, \frac{1}{2}) \quad (28)$$

$$a_1 = -\frac{n+1}{24} \int_1^{\infty} \frac{dx}{x^2} \frac{1}{(1 - \frac{1}{x^n})^{1/2}} = -\frac{n+1}{24n} B(\frac{1}{n}, \frac{1}{2}) \quad (29)$$

The preceding discussion has shown us that the behaviour of the phase shift at high energy is of the form

$$\delta \sim K^{1 - \frac{2}{n}} \quad (30)$$

This leads, for $n \rightarrow \infty$, to the repulsive core case in which the phase shift is proportional to momentum. This peculiar behaviour is completely different from that of the regular potentials, where the scattering amplitude goes nicely to zero for $k \rightarrow \infty$.

From this discussion and the one of the previous Section, we see that the high energy region is the place where the physical difference between regular and singular potentials shows up. Indeed, the effective range expression obtained in Section II does not differ so strongly from those of regular potentials.

This property is of course due to the fact that only at high energy the particle can penetrate so deep as to feel the effect of the singular part of the potential.

The high energy behaviour obtained in Eqs. (28) and (29) is actually a general property of singular potentials. Taking any angular momentum l , and for example

$$V(r) = -\lambda^2 \frac{e^{-\mu r}}{r^n} \quad (31)$$

we can write

$$\left\{ x^2 \frac{d^2}{dx^2} + f^2 \left(x^2 - \frac{\ell(\ell+1)}{x^2} - \frac{e^{-\gamma x}}{x^{n-2}} \right) \right\} \phi(x) = 0 \quad (32)$$

$$\gamma = \mu r_0 = \mu \left(\frac{\lambda}{\kappa} \right)^{2/n} \quad (33)$$

We can apply the W.K.B. expansion to Eq. (32), the leading term in f^2 being the first one

$$\delta = -f \int_{x_0}^{\infty} x dx \frac{d}{dx} \left[1 - \frac{e^{-\gamma x}}{x^n} \right]^{1/2} \quad (34)$$

since the centrifugal term can be treated in a perturbation manner.

In the limit $k \rightarrow \infty$ the effect of the exponential $e^{-\gamma x}$ tends to zero, so we get the identical result of Eq. (29). Note that an expansion in inverse powers of k cannot, in general, be done since the expansion of $e^{-\gamma x}$ in powers of γ causes the integral (34) to break down.

This high energy behaviour is very peculiar, since when the energy increases, the phase shift continues to yield resonances, which very likely have very little to do with what actually happens¹⁰⁾. This means that also if one succeeds in peratizing the non-renormalizable field theories, the results would be believable only for not too high energies.

It is probable that the theories of higher spin particles could actually generate "regular potentials" by means of self-damping mechanisms, as for instance that discussed by Delbourgo and Salam¹¹⁾ or a Regge pole mechanism.

Nevertheless, we think that the mathematical study of those singular models is still very interesting. It is in fact very likely that the self-damping mechanism (if it exists) would be very complicated, and that it would be very difficult to recognize all its details.

Therefore, when treating a field theory with the usual perturbative methods, one would arrive to results depending in an essential way on all the details of the self-damping mechanism.

The application of new mathematical procedures, as those developed in the last years, allows on the contrary to assert that in fact a great number of physical results do not depend at all on the detailed damping mechanism at small distances. The effect of the damping would be effective only at higher energies in such a way as to exclude the unphysical asymptotic behaviour of singular theories.

RIASSUNTO

Si studiano alcune proprietà caratteristiche della diffusione da potenziali singolari, nel limite sia di alte che di basse energie.

R E F E R E N C E S

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