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ON A BREMSSTRAHLUNG THEORY FOR THE MULTIPLE PRODUCTION OF MESONSA. Białas ^{*)} and T. Ruijgrok ^{**)}

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A B S T R A C T

The bremsstrahlung theory of multiple meson production in very high energy nucleon-nucleon collisions is considered. We represent the motion of the nucleons by an orbit, chosen in such a way that the energy and momentum loss are in rough agreement with the observed values of the inelasticity K and the transverse momentum q . The field equation for the meson field is then solved in the neutral scalar meson approximation. Assuming that there is no other mechanism for the production of mesons it is then possible to calculate for the final state the average number of mesons for each momentum value. From this we computed the following quantities:

1. the average multiplicity;
2. the width of the diffraction peak for elastic scattering;
3. the distributions of longitudinal and transverse momentum;
4. the distributions of $\log \cotg \theta/2$ and of the azimuthal angle.

Where possible a comparison with the experimental results was made. Qualitatively the comparison is quite encouraging. Our main conclusion is that a more realistic version of the bremsstrahlung theory considered in this paper is likely to explain many features of multiple meson production at very high energies.

^{*)} On leave of absence from Jagellonian University, Cracow, Poland.

^{**)} Present address: Institute for Theoretical Physics, Maliesingel 23, Utrecht, Netherlands.

1. INTRODUCTION

A class of uncorrelated jet models was recently proposed by Van Hove ^{1),2)}. On this basis he could explain some essential features of high energy collisions of strongly interacting particles.

To describe nucleon-nucleon collisions at cosmic ray energies we will in the present paper consider in more detail a special model of this kind ³⁾. The basis of our method will be to use a "bremsstrahlung" approximation to calculate meson production in terms of the nucleon motion. We adopt for the final state vector of the mesons produced in the collision an ansatz of the following form

$$|\Psi\rangle = N^{1/2} \delta_4(\hat{P} - P) \exp \left[i \int d_4 k \, \rho(k) \delta(k^2 - m^2) \theta(k) \sqrt{2\omega} a^*(k) \right] |0\rangle \quad (1.1)$$

It is characteristic of uncorrelated meson production. $N^{1/2}$ is a normalization constant, $\hat{P}$ is the energy-momentum operator for the meson field and P is its actual value for given final energy-momentum of the nucleons ^{*)}. m is the meson mass, $a^*(k)$ the creation operator for meson of four-momentum $k = (\vec{k}, \omega)$. The function $\theta(k) = \theta(\omega)$ is equal to one for positive ω and zero otherwise. The state (1.1) is determined by the single meson wave function $\rho(k)$. For the sake of simplicity we treat all mesons as identical, neutral, spinless particles, and we shall take them to be scalar rather than pseudoscalar.

*) We use units such that $\hbar = 1$, $c = 1$ and the nucleon mass $M = 1$.

The form (1.1) comes about if we treat the mesons quantum mechanically and the nucleons classically ⁴⁾. The latter seems to be justified because of the high nucleon energy and of the fact that the energy and momentum loss of the nucleons is experimentally found to be small. Since we assume that the mesons are emitted independently [the exponential form of (1.1)] and since the multiplicity is rather large, it follows that in each emission the reaction on the nucleon must also be very small. As can be seen from (1.1) the final state of the mesons is completely characterized by the function $\rho(k)$. We have calculated this function using the theory of bremsstrahlung. Details will be discussed in the next section. Exactly the same method was used by Chyliński ⁵⁾. He also gives references to the previous literature on bremsstrahlung models.

Our model neglecting the reaction of the meson field on the nucleons, we cannot calculate the final state of the two nucleons. We therefore have the nucleons after the collision as free parameters. The pion-nucleon coupling constant we also consider as a parameter. Although its value is well known for interactions at low energies, there is no reason to believe that the same value can be valid for a bremsstrahlung model of the type here considered for cosmic ray energies.

For various values of these three parameters we calculated the high energy limit of the multiplicity and of several one-meson distributions which have been studied experimentally. By applying the idea of shadow scattering ²⁾ we also calculated the width of the diffraction peak for elastic proton-proton scattering. The multiplicity and the width of the diffraction peak approach constant values at high energy. The one-particle distributions also become constant or have a simple energy dependence.

All calculations are performed under the assumption that only symmetric jets are produced. This is because of two reasons. First of all the calculations become less complicated. Secondly there is very little experimental information about asymmetric jets at high energy.

In the next section we discuss the derivation of the $\mathcal{S}$ function and of some other quantities. Sections 3-8 are devoted to the discussion of the multiplicity (Section 3), elastic shadow scattering (Section 4), angular distribution (Section 5), distributions of longitudinal momentum (Section 6), of transverse momentum (Section 7) and of the azimuthal angle (Section 8). The results which follow from our model are compared with the existing experimental data. In Section 9 we give a survey of the results together with the conclusions. In an appendix a short account is given of a method for the calculation of phase space integrals, which is based on the work of Lurçat and Mazur⁶⁾.

2. THE ONE-PARTICLE FUNCTION $\varphi(k)$

We will now derive the expression for the function $\varphi(k)$, which determines the final state (1.1) of the mesons. This function can be found by solving the equation for the meson field in which the nucleons are considered as classical particles. Because we neglect charges and assume the meson to be scalar, the equation for the meson field operator becomes

$$\begin{aligned} (\square - m^2)\varphi(x) &= -g \sum_{i=1}^2 \int_{-\infty}^{\infty} d\tau \delta_4[x^\mu - s_i^\mu(\tau)] = \\ &= -g \sum_{i=1}^2 \gamma_i^{-1} \delta_3(\vec{x} - \vec{s}_i(t)) \end{aligned} \quad (2.1)$$

For the source term in this equation we have taken the classical current of two particles describing orbits $\vec{s}_1(t)$ and $\vec{s}_2(t)$. We consider only symmetric jets. This means that in the c.m. system we have

$$\vec{s}_1(t) = -\vec{s}_2(t) \quad (2.2)$$

The factors γ_i^{-1} are given by

$$\gamma_i^{-1} = \sqrt{1 - |\dot{\vec{s}}_i(t)|^2} \quad (2.3)$$

The pion-nucleon coupling constant is denoted by g . Equation (2.2) can be solved exactly ^{4),5)}. From this solution the final state of the mesons can be calculated and it is found to be of the form (1.1) with a definite expression for $\varphi(k)$. This function is now completely determined by

$\vec{s}_1(t)$ and $\vec{s}_2(t)$ and we obtain the following

$$\rho(k) = \rho_1(k) + \rho_2(k) \quad (2.4)$$

with

$$\rho_{1,2}(k) = \frac{1}{2} ig \int_{-\infty}^{\infty} dt e^{i[\omega t - \vec{k} \cdot \vec{s}_{1,2}(t)]} \frac{d}{dt} \frac{|1 - |\vec{s}_{1,2}(t)|^2|}{\omega - \vec{k} \cdot \vec{s}_{1,2}(t)} \quad (2.5)$$

The same expression ^{*)} was found by Chyliński ⁵⁾. A formula for the average number of mesons with momenta in the volume element $d\vec{k}$ around $\vec{k}$, i.e., for $n(\vec{k})d\vec{k}$, can now easily be found. We obtain

$$n(\vec{k}) \equiv \langle \psi | a^*(\vec{k}) a(\vec{k}) | \psi \rangle = \frac{|\rho(k)|^2}{2\omega(k)} \frac{C(P-k)}{C(P)} \quad (2.6)$$

with

$$\omega(k) = \sqrt{|\vec{k}|^2 + m^2}$$

and

$$C(P) = \sum_{n=1}^{\infty} \frac{1}{n!} \Omega_n(P) \quad (2.7)$$

^{*)} The same form for $\rho(k)$ can also be found in the following way. Consider (2.1) as the equation for the classical field $\varphi_c(x)$. The function $\rho(k)$ is then completely determined by requiring that the expectation value of the meson field operator for the state (1.1), but without the δ_4 function, be equal to $\varphi_c(x)$.

6.

$P = (\vec{P}, E)$ is the energy-momentum loss of the two nucleons. $\Omega_n(P)$ is equal to the following phase space integral:

$$\Omega_n(P) = \int d^4k_1 \dots d^4k_n \prod_{i=1}^n |g(k_i)|^2 \delta(k_i^2 - m^2) \theta(k_i) \delta(k_1 + \dots + k_n - P) \quad (2.8)$$

From $n(\vec{k})$ one can calculate the multiplicity and all one-particle distribution functions of the mesons.

To calculate phase space integrals of the type (2.8) one should have an explicit expression for $g(k)$ in the region of $\vec{k}$ where $\omega(k) < E$. We assume that the collision takes place during a time interval t_0 around $t = 0$. Taking into account the Lorentz contraction of the nucleons one can expect that $t_0 \sim \frac{1}{E}$. Assuming that this is the case one can show that for $\omega(k) < E$

$$g_1(k) = \frac{i}{2} g e^{-i \vec{k} \cdot \vec{S}_1^{(0)}} \left[\frac{1}{k u_f} - \frac{1}{k u_i} \right] \quad (2.9)$$

u_i and u_f are the four-velocities for the initial and final state of the first nucleon. Equation (2.9) can also be written as

$$g_1(\vec{k}, \omega) = \frac{i}{2} g e^{-i \vec{k} \cdot \vec{S}_1^{(0)}} \left[\frac{v_i^2 - v_f^2}{2 \Omega \sqrt{1-v^2}} + \frac{\vec{k} (\vec{v}_f - \vec{v}_i) \sqrt{1-v^2}}{\Omega^2} \right] \quad (2.10)$$

$\vec{v}_i$ and $\vec{v}_f$ being the initial and final velocity of the first nucleon.
 Ω and $v = |\vec{v}|$ are defined by

$$\begin{aligned}\vec{v} &= \frac{1}{2} (\vec{v}_i + \vec{v}_f) \\ \Omega &= \omega - \vec{k} \cdot \vec{v}\end{aligned}\tag{2.11}$$

$\varrho_2(\vec{k}, \omega)$ is obtained from (2.10) by changing $\vec{k}$ into $-\vec{k}$ and $\vec{s}_1(0)$ into $-\vec{s}_1(0)$.

In this paper we will restrict ourselves to very high energies of the incident nucleons. Now, the function $|\varrho(k)|^2 = |\varrho_1 + \varrho_2|^2$, occurring in the phase space integrals, contains an interference term $2\text{Re } \varrho_1 \varrho_2$. This term will be neglected because it is of a higher order in $1/p$ than the terms $|\varrho_1|^2$ and $|\varrho_2|^2$ (p is the primary momentum of one of the nucleons, measured in nucleon masses). We get therefore

$$|\varrho(k)|^2 = |\varrho_1(\vec{k}, \omega)|^2 + |\varrho_1(-\vec{k}, \omega)|^2\tag{2.12}$$

After introducing the unit vector

$$\vec{e} = \frac{\vec{v}_f - \vec{v}_i}{|\vec{v}_f - \vec{v}_i|},\tag{2.13}$$

8.

$|g_1(\vec{k}, \omega)|^2$ takes the following form

$$|g_1(\vec{k}, \omega)|^2 = g^2 \left[\frac{F_1}{p^2 \Omega^2} + \frac{F_2 (\vec{k} \cdot \vec{e})}{p^3 \Omega^3} + \frac{F_3 (\vec{k} \cdot \vec{e})^2}{p^4 \Omega^4} \right] \quad (2.14)$$

with the following abbreviations

$$F_1 = \frac{(x^2 - 1)^2}{4[x^2 q^2 + 2(1 + x^2)]}$$

$$F_2 = \frac{1}{4} x q (x^2 - 1) \quad (2.15)$$

$$F_3 = \frac{1}{16} x^2 q^2 [x^2 q^2 + 2(1 + x^2)]$$

q is the transverse momentum (in nucleon masses) of one of the final nucleons.

x is related to the inelasticity, defined by

$$K = \frac{p - p_f}{p}$$

as follows

$$x = \frac{1}{1 - K}$$

p_f is the final momentum of one of the nucleons.

In the following sections the results of our calculations are presented. The method used for the determination of the phase space integrals is explained in the appendix.

3. MULTIPLICITY

From (2.6) one can calculate the average total multiplicity of mesons produced during the collision

$$N = \int n(\vec{k}) d^3\vec{k} = \frac{\sum_{n=1}^{\infty} n \frac{\Omega_n(E, \vec{P})}{n!}}{\sum_{n=1}^{\infty} \frac{\Omega_n(E, \vec{P})}{n!}} \quad (3.1)$$

The investigation of the high energy limit of this formula shows that the multiplicity tends to a constant value for the energy going to infinity. This result is known for scalar bremsstrahlung theories ^{5),7)}. We calculated $N_{\infty} = N(E = \infty)$ for different values of the inelasticity coefficient K , of the transverse momentum q of each nucleon after the collision and of the coupling constant $g^2/4\pi$. The calculations show that for q in the range

$$0 \leq q \leq 1.5 \quad (3.2)$$

the multiplicity is practically independent of q . The results for different values of K and $g^2/4\pi$ and with q in the range (3.2) are summarized in Table I.

TABLE I

$\frac{2}{g^2/4\pi} K$	0.3	0.4	0.5
15	7.8	12.0	17.6
45	10.5	16.5	24.4
75	12.1	19.2	28.5
150	14.8	23.8	
300	18		

MULTIPLICITY

For large multiplicities we derived the following asymptotic formula from (3.1)

$$N_{\infty} \simeq [4\pi\mu^2 F_1 g^2 K^2]^{1/3} \quad (3.3)$$

where F_1 is given by (2.15) and $\mu \simeq 6.7$ is the ratio of the nucleonic to the mesonic mass. The multiplicities in Table I can be reproduced with formula (3.3) with an error of at most 10 %.

We also calculated the distribution of the multiplicity. A typical curve, showing the probability $W(n)$ to find n mesons in the final state, is given in Figure 1. Without energy-momentum conservation the state (1.1) would give a Poisson distribution for $W(n)$. In Figure 1 it is seen that this conservation law narrows the distribution very much.

The experimental data ⁸⁾ concerning the multiplicity in high energy jets are, in our opinion, consistent with the results obtained here. The data favour slightly a slow increase of the multiplicity with energy, but a constant multiplicity cannot be excluded. The existence of correlation between the inelasticity coefficient and the multiplicity is also experimentally confirmed ⁹⁾. The very broad distribution of multiplicity in cosmic ray jets can therefore be reconciled with the result of our calculations by assuming a broad distribution in K .

On the basis of the present data ^{8),9)} one can expect that the multiplicity of charged particles lies between 14 and 20. Assuming that one third of the produced particles are neutrals we get 20-30 for N_{∞} . To some extent these limits fix the values of the parameters of our model. We will perform the further analysis only for those values of $g^2/4\pi$ and K for which N_{∞} , as given in Table I, is contained in the range $20 \leq N_{\infty} \leq 30$.

4. ELASTIC SHADOW SCATTERING

Assuming that

- i) the only inelastic states which appear at high energy are those described by (1.1) and
- ii) that the elastic scattering amplitude is purely imaginary at high energy,

one can calculate the elastic scattering from the unitarity condition of the S matrix. As shown by Van Hove ²⁾ for a more general class of models, the resulting overlap function is exponential in t (the square of the momentum transfer) and the corresponding elastic differential cross-section takes the form

$$\frac{d\sigma_{el}}{dt} = \left(\frac{d\sigma_{el}}{dt} \right)_{t=0} e^{at+bt^2} \quad 0 < \frac{b}{a^2} \ll 1 \quad (4.1)$$

The value of the coefficient a depends on the details of the model. It is well known that for accelerator energies the formula (4.1) describes correctly the experimental data. For pp scattering we have $a \simeq 10(\text{GeV}/c)^2$. We calculate the coefficient a for the inelastic final state (1.1). The results are given in Table II.

TABLE II

$\frac{g^2}{4\pi} \backslash K$	0.3		0.4		0.5	
	$q = 0.5$	$q = 1$	$q = 0.5$	$q = 1$	$q = 0.5$	$q = 1$
15	8.8	10.5	10.2	14.2	11.3	18.2
45	7.5	9.5	8.5	12.4	9.3	15.6
75	6.8	8.8	7.7	11.4		14.0
150		7.8		9.8		
300		6.6				

WIDTH OF THE DIFFRACTION PEAK IN $(\text{GeV}/c)^2$

From this table it is seen that for all values of K and $\frac{g^2}{4\pi}$ the coefficient a is of the correct order of magnitude and does not change appreciably with K . Let us note, however, that in the calculation of a the presence of the nucleons in the final state is neglected¹⁰⁾. Since the inelasticity coefficient is rather small the overlap function for the nucleons may have a significant influence on the value of a . To estimate the contribution from the nucleons one should know their distribution after the collision. We are therefore not able to make a quantitative analysis of the problem in the frame of our model. One can only note that the resulting values of a will be larger than those given in Table II. A very rough estimate gives an increase of a by about $3(\text{GeV}/c)^2$ if the average transverse momentum of the final nucleons is $1(\text{GeV}/c)$. To get the correct experimental value of a the pion contribution to the overlap function in our model should be smaller than it actually is. We therefore expect the angular distribution of the pions to be too much peaked in the forward-backward direction. In the next sections this conclusion will be confirmed.

5. ANGULAR DISTRIBUTION

By integrating $n(\vec{k})$ of (2.5) over the length of the vector $\vec{k}$ and the azimuthal angle, one obtains the angular distribution of the pions

$$\begin{aligned} n(\theta) d\cos\theta &= d\cos\theta \int n(\vec{k}) k^2 dk d\varphi = \\ &= d\cos\theta \int n(\vec{k}') \delta(\cos\theta' - \cos\theta) d^3\vec{k}' \quad (5.1) \end{aligned}$$

To calculate $n(\theta)$ explicitly one should insert (2.6) into (5.1) and perform the corresponding integrations. We have done this, using the method described in the appendix, which is a slight generalization of the method proposed by Lurçat and Mazur⁶⁾. The resulting angular distributions are always highly peaked around the angles 0 and 180 degrees. It is therefore convenient to express them in the variable

$$\chi = \log_{10} \operatorname{ctg} \frac{\theta}{2} \quad (5.2)$$

which is also used in the experimental analysis. Let us write

$$n(\theta) d\cos\theta = M(\chi) dx \quad (5.3)$$

For an isotropic distribution $n(\theta) = \text{constant}$ one obtains

$$M(\chi) \sim \frac{1}{\cosh^2(\frac{\chi}{2.3})}$$

which is a bell-shaped distribution with a half-width $\sigma = 0.39$. In our model we calculated the function $M(x)$ for different values of the parameters K , $g^2/4\pi$ and q . The resulting distributions show, for high energy, a very pronounced two-maximum structure. A typical example of an angular distribution is shown in Figure 2. It corresponds to $K = 0.4$, $g^2/4\pi = 150$, $q = 1$ and a laboratory energy of the incident proton of 10^3 GeV. We find that the shape of each maximum does not depend on the energy. The distance between the maxima is

$$\bar{X} = x_0 + 2 \log_{10} P \quad (5.4)$$

where x_0 is energy independent and P is the c.m. system momentum of an incident nucleon, measured in nucleon masses.

The dependence of the angular distribution on the other parameters is very weak. To see this we calculated two quantities

- a) the dispersion σ_0 of one of the bumps

$$\sigma_0^2 = \frac{\int_0^\infty M(x) (x - x_m)^2 dx}{\int_0^\infty M(x) dx} \quad (5.5)$$

where x_m is the position of the maximum, and

- b) the average total dispersion σ of the jets

$$\sigma^2 = \frac{\int_{-\infty}^{\infty} M(x) x^2 dx}{\int_{-\infty}^{\infty} M(x) dx} \approx \left(\frac{\bar{X}}{2} \right)^2 + \sigma_0^2 \quad (5.6)$$

Our calculations show that

- i) both σ_0 and σ depend very little on the parameters of the model: the change in σ_0 and σ does not exceed 10 % when the parameters vary over the range considered in Section 3;
- ii) σ_0 is approximately equal to 0.38. This seems to be in agreement with the data from cosmic rays ¹¹⁾;
- iii) the approximate value of σ for an energy of 10^3 GeV is found to be $\sigma \simeq 1$. This result must be compared with cosmic ray data, which show ¹²⁾ σ to be of the order 0.7 in the energy region 10^3 - 10^5 GeV.

There seems to be no possibility of getting a smaller σ in the frame of our model, unless one assumes a much bigger inelasticity coefficient ($K \sim 1$) or very high transverse momentum of the nucleons after the collision ($q > 2$). Both possibilities seem rather unphysical to us.

It is interesting to note that we find $\sigma_0 \simeq 0.38$ and that this number is very insensitive to the values of our parameters. For a quite different value of $K (\sim 1)$ and for very small $g^2/4\pi$ Chyliński ⁵⁾ found the same σ_0 . Also in the model of Czyżewski and Krzywicki ³⁾ σ_0 is of the same order of magnitude.

The two-maximum structure of the angular distribution with σ_0 equal to 0.39 is predicted by the fireball or two-centre model ¹³⁾. In this model it is assumed that in the collision two regions in space are created where the meson field is highly excited. These regions, or fireballs, move apart with high velocities and almost in the directions determined by the in- and outgoing nucleons. The two-maximum structure with $\sigma_0 = 0.39$ one obtains

if the fireballs decay isotropically in their own rest system and if they are not too slow. At this point we want to stress the well-known fact that the existence of the two maxima is consistent only with a very strong peaking of the angular distribution. With

$$n(\theta) \sim \frac{1}{\sin^2 \theta}$$

one finds, e.g., $M(x)$ to be constant and for

$$n(\theta) \sim \frac{1}{|\sin \theta|}$$

one gets only one maximum, viz.

$$M(x) \sim \frac{1}{\cosh \frac{x}{2.3}}$$

with $\sigma = 0.70$. That our angular distribution turns out to have two maxima is nevertheless no surprise, since the Weizsäcker-Williams theory for bremsstrahlung can be regarded as a fireball theory (although the fireballs do not decay isotropically). We find, however, a width σ , which is somewhat too big as compared with the experimental value. Also the azimuthal angular distribution is not isotropic (Section 8).

6. LONGITUDINAL MOMENTUM DISTRIBUTION

With increasing energy of the incoming nucleons the average energy of the secondaries in our model also increases indefinitely. This is a consequence of the fact that the inelasticity coefficient is fixed and that the multiplicity tends to a constant value at high energy. In order to calculate the longitudinal momentum distribution in the limit of extremely high energy one should therefore introduce the variable

$$w = \frac{|k_{||}|}{P} \quad (6.1)$$

One can prove that in the high energy limit the distribution of the variable w does not depend on the primary energy P . We calculated this distribution for a few values of the parameters. Figure 3 shows a plot of the function $N(w)/w^2 \int N(w)dw$ is the average number of particles with longitudinal momentum between Pw and $P(w+dw)$. The parameters are the same as in Figs. 1 and 2. It is seen that the resulting curve (on a logarithmic scale) can be reasonably approximated by a straight line and therefore the distribution of w can be well approximated by

$$N(w)dw = w^2 e^{-bw} dw \quad (6.2)$$

The average value of w in the limit of high energy is given by a simple approximate formula

$$\bar{w} = \frac{2K}{N_{\infty}} \quad (6.3)$$

which has an obvious physical meaning. Table III gives the average longitudinal momentum of one pion in GeV/c for the laboratory energy 10^3 GeV.

TABLE III

$\begin{array}{c} K \\ g^2/4\pi \end{array}$	0.3	0.4	0.5
15	1.8	1.5	1.3
45	1.3	1.1	0.9
75	1.1	1.0	0.8
150	0.9	0.8	
300	0.8		

AVERAGE LONGITUDINAL MOMENTUM IN GeV/c FOR $E_{\text{lab}} = 10^3$ GeV

It is interesting to note that the high energy accelerator data ($E_{\text{lab}} \gtrsim 20$ GeV) show an exponential decrease of the energy spectrum of the secondaries ¹⁴⁾.

On the other hand, the distribution (6.2) is different from the one obtained in the I.C.E.F. collaboration ¹⁵⁾. Some data ¹⁶⁾ indicate that the average π meson energy is constant in a wide interval of primary energies. This can only be reconciled with (6.3) if we assume that $1/K$ and/or N_{∞} increase with energy.

7. TRANSVERSE MOMENTUM DISTRIBUTION

From (2.6) we get the transverse momentum distribution of a single pion by calculating

$$n(k_{\perp}) dk_{\perp} = k_{\perp} dk_{\perp} \int n(\vec{k}) d\varphi = dk_{\perp} \int n(\vec{k}) \delta(k'_{\perp} - k_{\perp}) d^3\vec{k}' \quad (7.1)$$

It can be shown that in the limit of very high energy the function $n(k_{\perp})$ ceases to depend on energy. A typical high energy transverse momentum distribution is shown in Fig. 4. The parameters K , q and $g^2/4\pi$ are the same as in Figs. 1-3.

The function $n(k_{\perp})$ is found to have a rather long tail. For large $k_{\perp}$ it behaves like $k_{\perp}^{-3}$. This behaviour extends only up to values $k_{\perp} \lesssim P$, so that

$$\langle k_{\perp}^2 \rangle \sim \log P \quad (7.2)$$

A separate calculation of the integral

$$\langle k_{\perp}^2 \rangle = \int_0^{\infty} n(k_{\perp}) k_{\perp}^2 dk_{\perp}$$

for large, but finite P , gave formula (7.2) as a result. We also calculated the average transverse momentum $\langle k_{\perp} \rangle$. In the range $0.3 \leq K \leq 0.5$, $0 \leq q \leq 1.5$ and $15 \leq g^2/4\pi \leq 300$ the average transverse momentum changes between 220 MeV/c and 240 MeV/c.

The independence of $n(k_{\perp})$ on energy is in excellent agreement with accelerator and cosmic ray data. Our value of $\langle k_{\perp} \rangle$, is however too small as compared with the experimental value ^{16),17)} $\langle k_{\perp} \rangle = 0.39$ GeV/c. This is, of course, another formulation of the fact that our distribution of pions is too strongly peaked in the forward and backward directions, as compared with the experimental data. Also the long tail of $n(k_{\perp})$ seems to be spurious ¹⁸⁾.

Finally, we investigated whether our model gives a correlation between the longitudinal and transverse momentum distributions. For this purpose we calculated the average transverse momentum for different (fixed) values of the longitudinal momentum. The result is shown in Fig. 5 for longitudinal momenta smaller than 2KP. The parameters are the same as in Figs. 1-4. It is seen that the more energetic pions have a slightly larger transverse momentum than the slower ones. The absence of a clear correlation agrees with experiment ¹⁹⁾.

8. THE DISTRIBUTION OF THE AZIMUTHAL ANGLE

Integration (2.6) over the length of the vector k and the angle θ gives the distribution of the azimuthal angle φ

$$n(\varphi) d\varphi = d\varphi \int n(\vec{k}) k^2 dk d\cos\theta \quad (8.1)$$

This distribution is not necessarily isotropic because the final state of the pions depends, in our model, on the final state of the nucleons. Only $q = 0$ would give an isotropic $n(\varphi)$ distribution. Our calculations show that there is a strong preference for meson emission in the scattering plane of the nucleons. This is to be expected in any bremsstrahlung theory. Two distributions (evaluated for $q = 0.5$ and $q = 1$) are shown in Fig. 6. Both are strongly peaked in the neighbourhood of $\varphi = 0$. (The symmetric peak around $\varphi = \pi$ has been omitted.) The peaking increases with q . To get an idea of the quantitative significance of this effect, we calculated the value of the angle φ_0 , defined by the equation

$$\int_0^{\varphi_0} n(\varphi) d\varphi = \frac{1}{2} \int_0^{\pi} n(\varphi) d\varphi \quad (8.2)$$

The calculations show that φ_0 is rather insensitive to the values of K and $g^2/4\pi$. It depends mainly on q . For $q = 0.5$ we find $\varphi_0 \sim 35^\circ - 45^\circ$ and for $q = 1$ we get $\varphi_0 \sim 25^\circ - 30^\circ$. The experimental evidence concerning the distribution $n(\varphi)$ is rather poor. No significant deviation from isotropy was found, however. It would therefore appear that our result is inconsistent with the present data.

9. CONCLUSIONS

In this paper we have tried to explain a number of properties of cosmic ray jets by a bremsstrahlung theory. By adopting such a theory many features were deliberately left out of the picture. Among them is the scattering of the nucleons. Instead of trying to calculate this scattering an orbit for the two nucleons was prescribed. For this reason the inelasticity and the transverse momentum of the final nucleons were to be considered as parameters, so that the distribution of the inelasticity coefficient and the angular distribution of the leading particles cannot be predicted in our model. Also the composition of the final particles ²⁰⁾ was left out of consideration, because we assumed that only neutral pions were produced. This was a result of the fact that a neutral scalar bremsstrahlung theory was adopted. The relative abundance of symmetric and asymmetric jets is also outside the scope of the present paper. In principle, we could have calculated the meson production in the case of a non-symmetric motion of the nucleons. Since, however, even the symmetric jets are not described in a completely satisfactory way, we did not consider this possibility. We summarize which quantities were calculated and how they compare with experiment.

- 1) The average multiplicity N was found to be of the correct order of magnitude for reasonable values of the inelasticity. Whereas experimentally a slow increase of N with energy is favoured, we found a finite limit for N with the energy approaching infinity. This is a well-known ⁷⁾ result for a scalar meson theory.
- 2) A shadow scattering calculation could give the shape of the diffraction peak for elastic nucleon-nucleon scattering. We only calculated the width of the peak and found it to be in agreement with experiment. It is constant with increasing energy. Because of the unknown contribution of the overlap function of the nucleons we have some doubt about the value of the width, however.

- 3) The angular distribution shows a clear two-maximum structure in a $\log \tan \theta$ plot. The width of each maximum is in agreement with a two-centre model. The total width, however, is too big. This will certainly be reduced in a theory where the multiplicity increases with energy.
- 4) We find that the longitudinal momentum distribution falls off exponentially with increasing longitudinal momentum, whereas experimentally it behaves more like a power law with negative exponent. The average pion longitudinal momentum increases linearly with the primary energy. This could again be improved if the multiplicity were growing with energy.
- 5) In agreement with the experimental results we find that the transverse momentum distribution is independent of energy. Our mesons have, however, an average transverse momentum, which is too small by a factor between 1.5 and 2. Also the tail of the distribution is too long. Here also an improvement may be obtained in a theory with an increasing multiplicity.
- 6) The calculation of the azimuthal angular distribution showed that the secondaries are concentrated too much in the plane in which the nucleons are scattered.

The work of Chyliński ⁵⁾ has left the impression that a bremsstrahlung model for the multiple production of mesons in high energy nuclear collisions had to be stretched too much in order to obtain some agreement with the experimental data. We have shown that this is not the case. Our calculations demonstrate that with reasonable values of the parameters it is possible to reproduce the general behaviour of the quantities which are not outside the scope of a bremsstrahlung theory. The lack of detailed agreement with experiment we would like to blame mainly on our simple choice of theory. It is known ⁷⁾ that a pseudoscalar theory gives an increasing multiplicity with energy and this would improve many of our results. Another possibility is to

consider the bremsstrahlung of the vector mesons (e.g., ρ mesons). Because of higher mass of ρ meson it is very likely that one would obtain a larger transverse momentum of secondaries. If this expectation can be realized the problem of multiple production will have been, if not solved, at least reduced to simpler proportions.

A P P E N D I X

In this appendix we want to apply the method of Lurçat and Mazur⁶⁾ to the calculation of the phase space integral

$$\Omega'_n(E, \vec{P}) = \int d\vec{k}_1 \dots d\vec{k}_n \prod_{i=1}^n \frac{f[\vec{k}_i, \omega(k_i)]}{2 \omega(k_i)} \cdot \delta(\omega(k_1) + \dots + \omega(k_n) - E) \delta(\vec{k}_1 + \dots + \vec{k}_n - \vec{P}) \quad (\text{A.1})$$

for the special case that

1. $\vec{P} = 0$
2. $f(\vec{k}, \omega) = f(-\vec{k}, \omega)$
3. $f(\vec{k}, \omega) > 0$

Although we take all functions in the product the same, what follows can also be generalized for n different functions.

Firstly we write the phase space integral as a $4n$ dimensional integral

$$\Omega'_n(P) = \int dk_1 \dots dk_n \prod_{i=1}^n f(k_i) \theta(k_i) \delta(k_i^2 - m^2) \delta(k_1 + \dots + k_n - P) \quad (\text{A.2})$$

with

$$k = (\vec{k}, \omega) \quad k^2 = \omega^2 - |\vec{k}|^2 \quad m \text{ is the meson mass}$$

$$P = (\vec{P}, E) \quad \theta(k) = \begin{cases} 1 & \text{for } \omega > 0 \\ 0 & \text{for } \omega \leq 0 \end{cases}$$

Now introduce the following function of the four-vector P

$$U_n^{(\alpha)}(P) = \frac{1}{\phi_n(\alpha)} \Omega'_n(P) e^{-\alpha P} \quad (\text{A.3})$$

with

$$\phi_n(\alpha) = \int dk_1 \dots dk_n \prod_{i=1}^n f(k_i) \theta(k_i) \delta(k_i^2 - m^2) e^{-\alpha k_i} \quad (\text{A.4})$$

$\alpha = (\alpha_0, \vec{\alpha})$ is a four-vector, which will be specified later. The scalar product is defined by

$$\alpha P = \alpha_0 E - \vec{\alpha} \vec{P}$$

The function $U_n^{(\alpha)}(P)$ is positive and normalized so that

$$\int U_n^{(\alpha)}(P) d_4 P = 1$$

It can therefore be interpreted as a probability distribution.

The same is true for the function

$$u_{\alpha}(k) = \frac{1}{\varphi(\alpha)} f(k) \theta(k) \delta(k^2 - m^2) e^{-\alpha k} \quad (\text{A.5})$$

with

$$\varphi(\alpha) = \int d_4 k f(k) \theta(k) \delta(k^2 - m^2) e^{-\alpha k} \quad (\text{A.6})$$

Since

$$\phi_n(\alpha) = [\varphi(\alpha)]^n \quad (\text{A.7})$$

we can write $U_n^{(\alpha)}(P)$ as

$$U_n^{(\alpha)}(P) = \int d k_1 \dots d k_n \prod_{i=1}^n u_{\alpha}(k_i) \delta_4(k_1 + \dots + k_n - P) \quad (\text{A.8})$$

$U_n^{(\alpha)}(P) d_4 P$ is therefore the probability that the sum of n independent variables k_i , which have each the distribution $u_{\alpha}(k_i)$, has a value P in the element $d_4 P$. A very good approximate expression for this probability can be obtained by using the central limit theorem⁶⁾. This reads

$$U_n^{(\alpha)}(P) \simeq (2\pi)^{-2} \{\det B(n)\}^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(P - \bar{P}(\alpha)) B^{-1}(n) (P - \bar{P}(\alpha))\right] \quad (\text{A.9})$$

with the following abbreviations

$$\overline{P}(\alpha) = n \overline{k} = n \int d_4 k u_\alpha(k) k \quad (\text{A.10})$$

$$B_{\mu\nu}(n) = n \left[\overline{k_\mu k_\nu} - \overline{k_\mu} \overline{k_\nu} \right]$$

$$\overline{k_\mu k_\nu} = \int d_4 k u_\alpha(k) k_\mu k_\nu \quad (\text{A.11})$$

Lurçat and Mazur prove that for the cases which they treat $\det B(n)$ is always positive. We cannot do this, but it is sufficient to know that this necessary condition is indeed fulfilled in all cases where we apply it. We must now choose α . It is shown in Ref. ⁶⁾ that the central limit theorem gives the best approximation to the exact phase space integral if, for each P , the α is so chosen that

$$\overline{P}(\alpha) = P \quad (\text{A.12})$$

These are four equations for $\alpha = (\alpha_0, \vec{\alpha})$. With the α so determined the phase space integral $\Omega'_n(P)$ becomes, using (A.3)

$$\Omega'_n(P) \simeq [\varphi(\alpha)]^n e^{\alpha P} (2\pi)^{-2} [\det B(n)]^{-\frac{1}{2}} \quad (\text{A.13})$$

The problem is therefore reduced to the solution of (A.12) and the calculation of a few integrals of the types (A.6), (A.10) and (A.11). For $P = 0$ and $f(\vec{k}, \omega) = f(-\vec{k}, \omega)$ Eqs. (A.12) are simplified considerably. We write them out in detail.

$$\int d\omega d\vec{k} \vec{k} f(\vec{k}, \omega) \theta(\omega) \delta(\omega^2 - \omega^2(k)) e^{-\alpha_0 \omega + \vec{\alpha} \cdot \vec{k}} = 0 \quad (\text{A.14})$$

Because $f(\vec{k}, \omega)$ is invariant for $\vec{k} \rightarrow \vec{k}' = -\vec{k}$ the solution of this equation for $\vec{\alpha}$ is

$$\vec{\alpha} = 0$$

There remains therefore only one equation for α_0 , which is

$$\frac{1}{\varphi(\alpha_0)} \int d\omega d\vec{k} \omega f(\vec{k}, \omega) \theta(\omega) \delta(\omega^2 - \omega^2(k)) e^{-\alpha_0 \omega} = \frac{E}{n} \quad (\text{A.15})$$

This equation can also be written as

$$-\frac{d \ln \varphi(\alpha_0)}{d \alpha_0} = \frac{E}{N} \quad (\text{A.16})$$

where

$$\varphi(\alpha_0) = \int d\omega d\vec{k} f(\vec{k}, \omega) \theta(\omega) \delta(\omega^2 - \omega^2(k)) e^{-\alpha_0 \omega} \quad (\text{A.17})$$

In the paper of Jurek and Mazur all elements of the matrix $B(n)$ could also be expressed as derivatives of the function $\varphi(\alpha_0)$. In our case this is only true for the element $B_{00}(n)$. All others have to be calculated as separate integrals, some of which are identically zero, however. The integrals which we encounter are all of the form (A.17). The integration over the angles of $\vec{k}$ is always elementary, over the length of k is trivial, due to the δ function and the only more complicated integration is over ω . We found, however, that this could always be reduced to integrals of the type

$$I_{nm}(\alpha) = \int_0^{\infty} \frac{x^n e^{-\alpha x}}{(1+x^2)^m} dx \quad (\text{A.18})$$

which can all be written in terms of

$$I_{01}(\alpha) = P(\alpha) = \int_{-\infty}^{\infty} \frac{\sin(x-\alpha)}{x} dx$$

and

$$I_{11}(\alpha) = Q(\alpha) = \int_{-\infty}^{\infty} \frac{\cos(x-\alpha)}{x} dx$$

These integrals are related to the integral - sine and - cosine functions in the following way

$$\begin{aligned} P(\alpha) &= \left\{ \frac{\pi}{2} - \text{Si}(\alpha) \right\} \cos \alpha + \sin \alpha \text{Ci}(\alpha) \\ Q(\alpha) &= \left\{ \frac{\pi}{2} - \text{Si}(\alpha) \right\} \sin \alpha - \cos \alpha \text{Ci}(\alpha) \end{aligned} \quad (\text{A.19})$$

In the numerical calculations we used the series expansion for these functions for $\alpha < 1$ and an approximate rational function ²¹⁾ for $\alpha \geq 1$.

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FIGURE CAPTIONS

- Figure 1 Distribution of the number of mesons for $K = 0.4$;
 $(g^2/4\pi) = 150$ and $q = 1$ (full curve) and Poisson distribution
 with the same average $N_{\infty} = 23.8$ (dashed curve).
- Figure 2 Angular distribution $M(x)$ for $K = 0.4$, $(g^2/4\pi) = 150$ and
 $q = 1$. The zero point of the x scale is arbitrarily put in
 the middle. The distance between the two maxima corresponds to
 a laboratory energy of 10^5 GeV.
- Figure 3 Distribution of the longitudinal momentum.
- Figure 4 Transverse momentum distribution for $K = 0.4$, $(g^2/4\pi) = 150$,
 $q = 1$ and for infinite energy of the incident nucleons.
- Figure 5 Average transverse momentum $\langle k_{\perp} \rangle$ for given longitudinal
 momentum $k_{\parallel} = wP$.
- Figure 6 Distribution of the azimuthal angle.

FIG.1

$$K=0.4 \quad \frac{q^2}{4\pi}=150 \quad q=1$$

FIG. 2

FIG. 3

FIG. 4

FIG. 5

FIG. 6