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A TWO PARTICLE MODEL FOR HIGH-ENERGY COLLISIONS

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A B S T R A C T

This paper studies a very simplified model of high-energy collisions of strongly interacting particles. It introduces two particles A and B, where A can be regarded as stable and B as an unstable resonance of A. It assumes that all high-energy collisions can be described as taking place between the three channels A+A, A+B and B+B, so that the S matrix is 3×3 . Taking further A and B to be self-conjugate, it enables one to express all consequences of crossing symmetry in terms of the nine S matrix elements. From these crossing relations and from unitarity it is shown that for each orbital momentum l the S matrix depends on a single real parameter a_l which has a simple physical significance in terms of the K matrix. Constant high energy cross-sections are obtained when a_l is a function $a(\varphi)$ of the impact parameter. When $a(\varphi)$ is a smoothly decreasing function of φ approaching zero for $\varphi \rightarrow \infty$ more rapidly than any power of φ^{-1} , and has moderately large values for small φ , one finds that the elastic differential cross-sections derived from the model agree with the experimental shape of the diffraction peak and are insensitive to the detailed form of $a(\varphi)$. If $a(\varphi)$ is very large for all φ for which it does not vanish the elastic angular distributions become of the black sphere type. This insensitivity is due to the strong damping imposed by unitarity of the S matrix whenever the K matrix elements are appreciable.

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1. INTRODUCTION

High-energy collisions of hadrons (strongly interacting particles) have a number of properties which are qualitatively very simple and seem to be universally valid. The best known ones are

- i) the near constancy of total cross-section σ_T and small angle differential elastic cross-section $d\sigma_{el}/dt$ for increasing energy $\sqrt{t}$ denotes as usual $-(c.m. \text{ momentum transfer})^2$,
- ii) the fact that the real part of the scattering amplitude becomes small compared to its imaginary part ¹⁾,
- iii) the very simple t dependence of $d\sigma_{el}/dt$ on t , given by ²⁾

$$\frac{d\sigma_{el}}{dt} = \left(\frac{d\sigma_{el}}{dt}\right)_{t=0} e^{at+bt^2} \quad \text{with} \quad 0 < \frac{b}{a^2} \ll 1 \quad (1.1)$$

(this holds for an interval of t values of order $0 < -at \leq 10$ and the order of magnitude of a is $10(\text{GeV}/c)^{-2}$ for all known cases),

- iv) the fact that the transverse momentum distribution of secondaries in inelastic collisions is energy independent and limited to small transverse momentum values.

In the absence of a dynamical theory of strong interactions, it is natural to study the theoretical significance of these properties along phenomenological lines, by introducing tentative assumptions and models and discussing their implications on the basis of the few general principles we have good reasons to believe in. Thus, as one of us showed in a previous paper ³⁾, properties i) and ii) are closely related to each other if $C = 1$ exchange dominates in elastic scattering, and using ii) one is able to explain iii) under the assumption that the most common inelastic collisions give a large number of secondaries satisfying iv) and having no other correlations with each other than the ones imposed by momentum and energy conservation.

While these conditions may well be satisfied to a sufficient degree at very high energies, experiment suggests that the angular distribution iii) already prevails at energies where the multiplicity and momentum distribution of secondaries in inelastic collisions are such that the asymptotic situation described in Ref. ³⁾ cannot be fully realized. As shown by another of the present authors ⁴⁾, this is mainly due to the fact that most inelastic collisions contain a few (usually two) secondaries carrying away a large fraction of the incident energy. Furthermore, it may well be that even at high energy an appreciable fraction of the inelastic collisions is of two particle type, the particles being usually unstable resonances or isobars.

This led us to consider whether properties i) to iv) could hold in models of high-energy collisions which would be complementary to the high-multiplicity models discussed in Ref. ³⁾ and would put the main emphasis on two-particle inelastic collisions. The present paper discusses a rather extreme example of such a model, characterized by the fact that it possesses only two inelastic channels which are both of the two-particle type. In order to allow full exploitation not only of unitarity but also of crossing symmetry, our model introduces two spinless hadrons A, B, identical to their antiparticles,

$$A = \bar{A} \quad B = \bar{B}. \quad (1.2)$$

It considers the 9 possible reactions between the 3 two-particle channels

$$\text{channel 1} = A + A, \quad \text{channel 2} = B + B, \quad \text{channel 3} = A + B.$$

For each initial channel i , collisions lead to final channels i (elastic) or $j \neq i$ (inelastic), where of course $i, j = 1, 2, 3$. To give to this model some resemblance to reality, one might imagine A to be a stable particle and B to be an isobar of A, or a fireball of some sort, so that $A + A \rightarrow A + A$ would look like elastic scattering while $A + A \rightarrow A + B$ and $A + A \rightarrow B + B$

with subsequent decay of B would imitate what we usually call inelastic collisions. Clearly, however, the model is unrealistic in many respects, and it has been studied only because its simplicity allows a detailed theoretical analysis to be carried out ⁵⁾.

The conditions of unitarity, crossing symmetry and constancy of cross-sections put severe limitations on the nine S matrix elements. For each value of the angular momentum l , they depend on one parameter a_l only which can be simply interpreted in terms of the corresponding K matrix. The only non-vanishing elements of the latter are found to belong to the transitions

$$A + A \longleftrightarrow B + B, \quad A + B \longleftrightarrow B + A \quad (1.3)$$

and have for even l real values a_l and $-a_l$, respectively. The l dependence of a_l cannot be obtained from general arguments and the procedure we follow is to adopt

$$a_l = D a_l^{(0)} \quad (1.4)$$

where $a_l^{(0)}$ is some smoothly decreasing function of l normalized to

$$\lim_{k \rightarrow \infty} \frac{1}{(kr)^2} \sum_{l \text{ even}} (2l+1) a_l^{(0)} = 1 \quad (1.5)$$

where the sum runs over even l . The quantity $\bar{r}^2$ is the logarithmic derivative of the K matrix elements with respect to t for $t = 0$. D is a factor with the significance of coupling strength. We first adopt for $a_l^{(0)}$ the general shape given by a single particle exchange mechanism for (1.3), but our main point is that for rather strong D , the angular distributions deduced from the S matrix become very insensitive to the details of $a_l^{(0)}$ and are in good agreement with (1.1). This is shown by studying a few numerical examples. We also remark that

the detailed shape of $a_1^{(0)}$ becomes completely immaterial for $D \rightarrow \infty$, the resulting diffraction peak having the shape which corresponds to shadow scattering by a black sphere.

In the next Section we give the partial wave decomposition of the S matrix and discuss the restrictions due to unitarity, crossing symmetry and analyticity. The general form of the S and K matrices satisfying these restrictions is given in Section 3. Section 4 deals with the one particle exchange model mentioned above. In Section 5 we consider two other models and we show that the results are essentially unchanged for sufficiently strong D . Our conclusions are listed in the last Section.

2. UNITARITY AND CROSSING SYMMETRY

In terms of the S matrix element for the process $i \rightarrow j$

$$S_{ij} = I_{ij} + iT_{ij} \quad (2.1)$$

the relativistic scattering amplitude $\mathcal{T}_{ij}(s,t)$ is defined by

$$T_{ij} = \frac{\tau_{ij}}{\mathcal{E}} \delta(\vec{P}_i - \vec{P}_j) \delta(E_i - E_j)$$

$(\vec{P}_i, E_i)$ and $(\vec{P}_j, E_j)$ are the total energy momentum four vectors of the states i and j . $\mathcal{E}^2$ is the product of the energies of the four particles involved in the reaction $i \rightarrow j$ (we take $\mathcal{E} > 0$). As usual s is the square of the c.m. energy of the reaction and t is defined as

$$t = -2k^2(1 - \cos\theta)$$

where k is the length of the momentum of each incident particle in the c.m. system. In the case of reaction $3 \rightarrow 3$ (i.e., $A + B \rightarrow A + B$) we define θ as the scattering angle of A . For all other processes $i \rightarrow j$, either i , or j , or both contain two identical particles, so that the definition of θ becomes ambiguous. We then take for θ that scattering angle which satisfies $0 \leq \theta \leq \frac{\pi}{2}$. The corresponding amplitude $\mathcal{T}_{ij}(s,t)$, however, is defined over the whole range of t , $-4k^2 \leq t \leq 0$, by requiring forward-backward symmetry. This means that $\mathcal{T}_{ij}(s,t)$ is invariant under the transformation

$$\cos\theta \rightarrow -\cos\theta$$

or

$$t \rightarrow u \equiv 4M^2 - s - t.$$

Here $4M^2$ denotes the sum of the masses squared of the four particles involved.

The partial wave amplitudes η_1^{ij} are defined by

$$\tau_{ij}(s, t) = \frac{i\sqrt{s}}{4\pi k} \sum_{l=0}^{\infty} (2l+1) \eta_l^{ij} P_l(\cos\theta). \quad (2.2)$$

They are functions of s .

We define a complete and orthogonal set of states $|\vec{k}_1 \vec{k}_2\rangle$ for each channel i . Writing $a^*(\vec{k})$, $b^*(\vec{k})$ for the creation operators of A and B particles of momentum $\vec{k}$ (A and B are of course bosons) we take

$$\begin{aligned} |1\vec{k}_1 \vec{k}_2\rangle &= |1\vec{k}_2 \vec{k}_1\rangle = a^*(\vec{k}_1) a^*(\vec{k}_2) |0\rangle && \text{for all pairs } \vec{k}_1 \vec{k}_2 \\ |2\vec{k}_1 \vec{k}_2\rangle &= |2\vec{k}_2 \vec{k}_1\rangle = b^*(\vec{k}_1) b^*(\vec{k}_2) |0\rangle && \text{for all pairs } \vec{k}_1 \vec{k}_2 \\ |3\vec{k}_1 \vec{k}_2\rangle &= a^*(\vec{k}_1) b^*(\vec{k}_2) |0\rangle && \text{for all } \vec{k}_1 \text{ and all } \vec{k}_2 \end{aligned}$$

$|0\rangle$ is the vacuum state. In terms of these states the identity operator can be written as

$$I = \sum_{m=1}^3 c_m \int d\vec{k}_1 d\vec{k}_2 |\vec{m}\vec{k}_1 \vec{k}_2\rangle \langle \vec{m}\vec{k}_1 \vec{k}_2|$$

with $c_1 = c_2 = \frac{1}{2}$ and $c_3 = 1$.

Unitarity of $S = I + iT$ implies

$$i(T^\dagger - T) = T^\dagger T.$$

Written in full this becomes

$$\begin{aligned} & \frac{1}{\epsilon} \delta(\vec{P}_i - \vec{P}_j) \delta(E_i - E_j) (T_{ji}^* - T_{ij}) = \\ & = \frac{1}{\epsilon^2} \int d\vec{k}_1 d\vec{k}_2 \sum_{m=1}^3 c_m \delta(\vec{P}_i - \vec{k}_1 - \vec{k}_2) \delta(E_i - E_j) \tau_{mi}^* \delta(\vec{P}_j - \vec{k}_1 - \vec{k}_2) \delta(E_j - E_m) \tau_{mj}. \end{aligned}$$

Substituting the partial wave decomposition of the τ 's and performing the $\vec{k}_1$ and $\vec{k}_2$ integrations we obtain

$$\begin{aligned} & \sum (2l+1) P_l(\cos\theta) [(\eta_l^{ji})^* + \eta_l^{ij}] = \\ & = \sum_{m=1}^3 c_m \sum (2l+1) P_l(\cos\theta) (\eta_l^{mi})^* \eta_l^{mj} \end{aligned}$$

or, for each l , in 3×3 matrix form

$$\eta_l^+ + \eta_l = \eta_l^+ C \eta_l$$

with

$$C = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.3)$$

If we define

$$S_l = 1 - \sqrt{C} \eta_l \sqrt{C} = \begin{pmatrix} 1 - \frac{1}{2} \eta_l^{11}, & -\frac{1}{2} \eta_l^{12}, & -\frac{1}{\sqrt{2}} \eta_l^{13} \\ -\frac{1}{2} \eta_l^{21}, & 1 - \frac{1}{2} \eta_l^{22}, & -\frac{1}{\sqrt{2}} \eta_l^{23} \\ -\frac{1}{\sqrt{2}} \eta_l^{31}, & -\frac{1}{\sqrt{2}} \eta_l^{32}, & 1 - \eta_l^{33} \end{pmatrix} \quad (2.4)$$

we find that unitarity of the S matrix is equivalent with S_1 being unitary for all l . For even l , all η_{11}^{ij} can be different from zero. For odd l , on the contrary, symmetry of all amplitudes but $\tau_{33}(s,t)$ implies that S_1 reduces to

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - \eta_{11}^{33} \end{pmatrix} \quad l \text{ odd.} \quad (2.5)$$

Unitarity of S_1 then requires

$$\eta_{11}^{33} = 1 - e^{2i\varphi_1}, \quad l \text{ odd, } \varphi_1 \text{ real.} \quad (2.6)$$

We now consider the consequences of crossing symmetry. A and B being supposed self-conjugate, each crossing gives through analytic continuation through the upper half energy plane a relation between two amplitudes $\tau_{ij}(s,t)$. We find, e.g.,

$$\tau_{12}(s,t) = \tau_{33}^*(t,s) \quad \tau_{21}(s,t) = \tau_{33}^*(t,s) \quad (2.7)$$

which implies the symmetry property

$$\tau_{12}(s,t) = \tau_{21}(s,t).$$

One also proves in this way

$$\tau_{13}(s,t) = \tau_{31}(s,t) \quad \tau_{23}(s,t) = \tau_{32}(s,t).$$

This implies that the matrix S_1 is symmetric for each l ⁶⁾.

The further consequences of crossing symmetry are of an asymptotic nature for $s \rightarrow \infty$. We make the basic assumption that the elastic scattering amplitudes have the following asymptotic behaviour at $s \rightarrow \infty$

i) for t fixed (forward scattering)

$$s^{-1} \tau_{ii}(s, t) \rightarrow ia_i(t), \quad i = 1, 2, 3, \quad (2.8)$$

the $a_i(t)$ being different from zero at least in an interval $t_0 < t \leq 0$, and tending to zero for $t \rightarrow -\infty$;

ii) for u fixed (backward scattering)

$$s^{-1} \tau_{33}(s, u) \rightarrow ib(u). \quad (2.9)$$

For the time being, $b(u)$ may or may not vanish. We suppose, however, $b(u) \rightarrow 0$ for $u \rightarrow -\infty$. We shall see later on that $b(u)$ is fully determined by the $a_i(t)$.

The foregoing assumptions are the simplest ones leading to constant cross-sections. Because of forward-backward symmetry which holds for amplitudes τ_{11} and τ_{22} (one gets from i) for u fixed

$$s^{-1} \tau_{11}(s, u) \xrightarrow{s \rightarrow \infty} ia_1(u), \quad s^{-1} \tau_{22}(s, u) \xrightarrow{s \rightarrow \infty} ia_2(u). \quad (2.10)$$

Expressing the conditions (2.8), (2.9), (2.10) in terms of partial wave amplitudes one obtains

$$\begin{aligned} \eta_{11}^{ii}(k) &\equiv \eta^{ii}(\varrho) && \text{for } i = 1, 2, 3 \quad \text{and } l \text{ even} \\ \eta_{11}^{33}(k) &\equiv \zeta(\varrho) && \text{for } l \text{ odd} \end{aligned} \quad (2.11)$$

where $\varrho = \frac{1}{k}$ is the impact parameter. The functions $\eta^{ii}(\varrho)$, $\zeta(\varrho)$ do not vanish identically.

We make the assumption that at very high energy the off-diagonal partial wave amplitudes $\eta_l^{ij}(k)$, $i \neq j$, taken for even l , are also continuous functions of ϑ and k . It then follows from unitarity and from the condition (2.11) that they do not depend on k : $\eta_l^{ij}(k) \equiv \eta_l^{ij}(\vartheta)$, $i \neq j$. Consequently for fixed t the off-diagonal elements of $\mathcal{T}$ have a similar asymptotic behaviour

$$s^{-1} \mathcal{T}_{ij}(s, t) \xrightarrow{s \rightarrow \infty} i a_{ij}(t) \quad \begin{array}{l} i \neq j \\ i, j = 1, 2, 3 \end{array} \quad (2.12)$$

where the $a_{ij}(t)$ may or may not vanish. Forward-backward symmetry tells us then that for fixed u we have

$$s^{-1} \mathcal{T}_{ij}(s, u) \xrightarrow{s \rightarrow \infty} i a_{ij}(u) \quad \begin{array}{l} i \neq j \\ i, j = 1, 2, 3 \end{array} \quad (2.12')$$

The further consequences of crossing symmetry can now be obtained by the method of Ref. ⁷⁾. Since the particles A and B are self-conjugate, the conditions (2.8) imply for large s that the $a_i(t)$ are real ⁷⁾:

$$a_i(t) = a_i^*(t) \quad i = 1, 2, 3. \quad (2.13)$$

By straightforward generalization of the same method one obtains

$$a_{13}^*(t) = a_{13}(t) \quad a_{23}^*(t) = a_{23}(t) \quad (2.14)$$

$$a_{12}^*(u) = b(u). \quad (2.15)$$

Conditions (2.13) for $i = 1, 2$, and (2.14) combined with forward-backward symmetry, imply reality conditions for the following partial wave amplitudes :

$$\eta_1^{11*} = \eta_1^{11} \quad \eta_1^{22*} = \eta_1^{22} \quad \eta_1^{13*} = \eta_1^{13} \quad \eta_1^{23*} = \eta_1^{23} \quad (2.16)$$

To exploit the consequences of the third condition (2.13) and of (2.15) is a more difficult task. The result is for even l

$$\begin{aligned} \operatorname{Im} S_1^{12} &= -\operatorname{Im} S_1^{33} = +\sin 2\varphi_1 \\ 2\operatorname{Re} S_1^{12} &= \operatorname{Re} S_1^{33} - \cos 2\varphi_1 \end{aligned} \quad (2.17)$$

where φ_1 is the angle defined in (2.6) for odd l , and interpolated by continuity to even l . The interpolation is done by expressing φ_1 as a function $\varphi(\rho)$ of the impact parameter, see (2.11). The proof of (2.17) is given in the Appendix.

In the next Section we give the most general matrix S_1 satisfying the relations derived here.

12.

3. THE GENERAL FORM OF THE S AND K MATRICES

From the results of Section 2 it follows that for even l the matrix S_l must satisfy the following conditions

- i) S_l is unitary,
- ii) S_l is symmetric,
- iii) S_l is three dimensional and has the form

$$S_l = \begin{pmatrix} r_1 & c_1 & r_2 \\ c_1 & r_3 & r_4 \\ r_2 & r_4 & c_2 \end{pmatrix} \quad \begin{array}{l} r_i - \text{real} \\ c_i - \text{complex} \end{array} \quad (3.1)$$

iv) condition (2.17),

v) S_l is continuously connected with the identity matrix (to which it tends for $l \rightarrow \infty$).

We shall derive the most general form of S_l satisfying these conditions. This will be done by way of the K matrix which will also be used later on to formulate the dynamical contents of the model. Let us first recall the definition of K in terms of the S matrix, and express its partial wave decomposition in terms of S_l . One has by definition

$$S = \frac{I + iK}{I - iK} \quad (3.2)$$

or inversely by means of (2.1)

$$2K = T - iKT. \quad (3.3)$$

The matrix elements of K are written

$$K_{ij} = \frac{k_{ij}}{\xi} \int (\vec{p}_i - \vec{p}_j) \delta(E_j - E_i)$$

with

$$k_{ij}(s,t) = \frac{\sqrt{s}}{4\pi k} \sum_{l=0}^{\infty} (2l+1) k_l^{ij} P_l(\cos\theta). \quad (3.4)$$

In the same way as was done with the unitarity condition for the S matrix, Eq. (3.3) can now be reduced to

$$-2ik_1 = \eta_1 - ik_1 C \eta_1 \quad (3.5)$$

where k_1 , η_1 and C are three-dimensional matrices. Solving (3.5) and using S_1 instead of η_1 by means of (2.4) gives

$$k_1 = i \frac{1}{\sqrt{C}} \frac{1 - S_1}{1 + S_1} \frac{1}{\sqrt{C}}$$

or

$$K_1 = i \frac{1 - S_1}{1 + S_1}$$

with

$$K_1 = \sqrt{C} k_1 \sqrt{C}.$$

We proceed now to find the most general matrix S_1 which obeys the conditions i) - v). We write $S_1 = A_1 + iB_1$, with both A_1 and B_1 real and symmetric. Then from (2.16), (2.17), B_1 takes the form

$$B_1 = \begin{pmatrix} 0 & \sin 2\varphi_1 & 0 \\ \sin 2\varphi_1 & 0 & 0 \\ 0 & 0 & -\sin 2\varphi_1 \end{pmatrix}. \quad (3.6)$$

Because S_1 is unitary, A_1 and B_1 commute. There exists therefore a real orthogonal transformation which diagonalizes both A_1 and B_1 . After this transformation S_1 and B_1 become, respectively

$$S'_1 = \begin{pmatrix} e^{2i\delta_1} & 0 & 0 \\ 0 & e^{2i\delta_2} & 0 \\ 0 & 0 & e^{2i\delta_3} \end{pmatrix} \quad B'_1 = \begin{pmatrix} \sin 2\delta_1 & 0 & 0 \\ 0 & \sin 2\delta_2 & 0 \\ 0 & 0 & \sin 2\delta_3 \end{pmatrix}$$

the $\delta_i = \delta_i^1$ being the eigenphases. The form of K_1 after the transformation is

$$K'_1 = \begin{pmatrix} \operatorname{tg} \delta_1 & 0 & 0 \\ 0 & \operatorname{tg} \delta_2 & 0 \\ 0 & 0 & \operatorname{tg} \delta_3 \end{pmatrix}.$$

From (3.6) we see immediately that the eigenvalues of B_1 are

$$\sin 2\delta_{1,2,3} = \sin 2\varphi_1, \quad -\sin 2\varphi_1, \quad -\sin 2\varphi_1.$$

On the other hand,

$$\sin 2\delta_i = 2\cos^2 \delta_i \operatorname{tg} \delta_i$$

and we can write

$$\begin{aligned} 2\cos^2 \delta_i = 1 + \cos 2\delta_i &= 1 \pm (1 - \sin^2 2\delta_i)^{\frac{1}{2}} = 1 \pm (1 - \sin^2 2\varphi_1)^{\frac{1}{2}} \\ &= 1 \pm \cos 2\varphi_1. \end{aligned}$$

Since we want continuity in l with the conditions $\delta_i \rightarrow 0$ and $\varphi_1 \rightarrow 0$ for $l \rightarrow \infty$, the upper sign must be chosen, so that

$$\sin 2\delta_i = 2\cos^2 \varphi_1 \operatorname{tg} \delta_i, \quad i = 1, 2, 3.$$

This gives the following relation between the matrices B'_1 and K'_1

$$B'_1 = 2\cos^2 \varphi_1 K'_1.$$

The same relation must hold also for the matrices K_1 and B_1 . Taking this into account we find from (3.6)

$$K_1 = \begin{pmatrix} 0 & a_1 & 0 \\ a_1 & 0 & 0 \\ 0 & 0 & -a_1 \end{pmatrix} \quad \text{with} \quad a_1 = \operatorname{tg} \varphi_1. \quad (3.7)$$

The full expression of S_1 then takes the form

$$S_1 = \begin{pmatrix} 1 - \frac{2a_1^2}{1+a_1^2}, & i \frac{2a_1}{1+a_1^2}, & 0 \\ i \frac{2a_1}{1+a_1^2}, & 1 - \frac{2a_1^2}{1+a_1^2}, & 0 \\ 0, & 0, & 1 - \frac{2a_1^2}{1+a_1^2} - i \frac{2a_1}{1+a_1^2} \end{pmatrix}. \quad (3.8)$$

One easily verifies that the conditions i) - v) are fulfilled. The foregoing analysis applies to even l . For odd l , S_1 is directly given in terms of φ_1 by (2.5) and (2.6). The K matrix for odd l takes the form

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_1 \end{pmatrix} . \quad (3.7')$$

Comparison of the formulae (2.5), (2.6) and (3.8) with (2.4) gives for each l all amplitudes η_1^{ij} in terms of only one real parameter a_1 , or one angle $\varphi_1 = \arctg a_1$, which are supposed to be continuous functions of the impact parameter l/k and to approach zero for $l \rightarrow \infty$.

We close this Section by collecting the formulae for the cross-sections in the limit of very high energy. In the limit of very high energies the differential cross-sections are given by

$$\frac{d\sigma_{ij}}{dt} = 16\pi^3 \frac{|\tau_{ij}|^2}{s^2} \quad (3.9)$$

and the optical theorem for the total cross-section with ingoing channel i reads

$$\sigma_i^T = 16\pi^2 \frac{\text{Im } \tau_{ii}(\theta=0)}{s} .$$

After substitution of the partial wave expansions these formulae become

$$\frac{d\sigma_{ij}}{dt} = \frac{\pi}{4k^4} \left| \sum_{l=0}^{\infty} (2l+1) \eta_1^{ij} P_l(\cos\theta) \right|^2 \quad (3.10)$$

and

$$\sigma_i^T = \frac{2\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \eta_1^{ii} . \quad (3.11)$$

The sums extend over all l values, even and odd, but odd l contribute only to $d\sigma_{33}/dt$ and σ_3^T . It should be remembered that in (3.10) the range of θ is from 0 to π for $j=3$ and from 0 to $\pi/2$ for $j=1, 2$. Then (3.11) can be obtained by multiplying (3.10) by $2k^2 d(\cos\theta)$, integrating over the appropriate range and summing over l .

At high energy the summations over l occurring in (3.10) and (3.11) can, to a good approximation, be replaced by an integration over the dimensionless continuous variable

$$x = \frac{1}{kr} = \frac{\rho}{r}$$

where ρ is the impact parameter and r is some length, to be chosen later. Using the well-known formula

$$\lim_{l \rightarrow \infty} P_l\left(1 - \frac{z^2}{2l^2}\right) = J_0(z)$$

we obtain for the differential cross-sections in the forward peak for $i, j = 1, 2$

$$\frac{d\sigma_{ij}}{dt} = \frac{\pi r^4}{4} \left| \int_0^\infty x \eta^{ij}(x) J_0(\sqrt{B}x) dx \right|^2 \quad \text{with} \quad B = -r^2 t. \quad (3.12)$$

Because of forward-backward symmetry the backward scattering is described by the same formula with the substitution $t \rightarrow u$. The differential cross-section in the forward peak for $3 \rightarrow 3$ channel is given by

$$\frac{d\sigma_{33}}{dt} = \pi r^4 \left| \int_0^\infty x \operatorname{Re} \eta^{33}(x) J_0(x\sqrt{B}) dx \right|^2. \quad (3.13)$$

The corresponding formula for the backward peak is

$$\frac{d\sigma_{33}}{du} = \pi r^4 \left| \int_0^\infty x \operatorname{Im} \bar{\eta}^{33}(x) J_0(x\sqrt{\bar{B}}) dx \right|^2 \quad (3.14)$$

with

$$\bar{\eta}^{33}(x) = (-1)^1 \eta_{33}^1(k) \quad \text{and} \quad \bar{B} = -r^2 u. \quad (3.15)$$

The total cross-sections become for $i = 1, 2$

$$\sigma_i^T = 2\pi r^2 \int_0^\infty x \eta^{ii}(x) dx \quad (3.16)$$

and for $i = 3$

$$\sigma_3^T = 4\pi r^2 \int_0^\infty x \operatorname{Re} \eta^{33}(x) dx. \quad (3.17)$$

We shall take for r a value independent of the incident energy. It follows from this and from the results of the previous Section that $\eta^{ij}(x)$ ($i, j = 1, 2$), $\operatorname{Re} \eta^{33}(x)$ and $\operatorname{Im} \bar{\eta}^{33}(x)$ are functions of x alone and not also of k . The same then holds for the K matrix element $a_1 = a(x)$. The values of r and $a(x)$ will be chosen in the next Section on the basis of a dynamical model.

4. THE MODEL FOR THE K MATRIX

The formulae (3.7) and (3.7') for the K matrix suggest that the only "direct" transitions occurring in our model are the transitions $1 \leftrightarrow 2$ and $3 \leftrightarrow 3$ backward, all other transitions being "indirect" in the sense that they go through intermediate states which conserve momentum and energy. Consequently, it is natural to formulate the dynamics of our model in terms of the K matrix rather than of the S matrix⁸⁾. From the previous considerations, especially Eqs. (3.7), (3.7') and (3.4), we already know that for high energy

$$\begin{aligned} k_{12}(s,u) = k_{21}(s,u) = -k_{33}(s,u) = s k(u) \quad \text{for } u \text{ fixed} \\ k_{12}(s,t) = k_{21}(s,t) = s k(t) \quad k_{33}(s,t) = 0 \quad \text{for } t \text{ fixed.} \end{aligned} \quad (4.1)$$

We now assume that the t dependence is given by the diagrams in Fig. 1 in which the object exchanged is taken to have a given mass m and no spin nor charge, and in which no t dependence is considered for the vertices. In other words, our ansatz is

$$k(t) \propto (m^2 - t)^{-1}. \quad (4.2)$$

We shall give later arguments to the effect that our main conclusions do not depend on the details of this t dependence. At c.m. energies much larger than the masses of A and B we have

$$m^2 - t = m^2 + 2k^2 (1 - \cos\theta)$$

which gives for the partial wave decomposition of the propagator

$$\frac{1}{m^2 - t} = \frac{2}{s} \sum (2l+1) F_l(s) P_l(\cos\theta) \quad (4.3)$$

with

$$F_1(s) = Q_1 \left(1 + \frac{m^2}{2k^2} \right)$$

Q_1 is a Legendre function of the second kind. Using the relation

$$\lim_{l \rightarrow \infty} Q_1 \left(1 + \frac{x^2}{2l^2} \right) = K_0(x) \quad (4.4)$$

where $K_0(x)$ is the Macdonald function of order 0⁹⁾, we find for large l

$$F_1(s) = K_0(x).$$

We have put $x = l/kr$, the length r being defined as m^{-1} . It is easily seen that r^2 is the logarithmic derivative of (4.3) with respect to t at $t=0$.

From the foregoing it will now be clear that in this one particle exchange model the parameters a_1 must be proportional to $K_0(x)$. More specifically we take

$$a_1 = a(x) = D K_0(x). \quad (4.5)$$

The dimensionless constant D gives the strength of the interaction. The set of a_1 's corresponding to $D=1$ is normalized in the way prescribed by (1.4) and (1.5) since one has $r = \bar{r}$ and, the sum extending to even l only,

$$\frac{1}{(kr)^2} \sum_{l=0,2,\dots}^{\infty} (2l+1) K_0\left(\frac{l}{kr}\right) = \int_0^{\infty} x K_0(x) dx = 1.$$

Using the formulae (3.12) - (3.17) we can calculate all differential and total cross-sections in terms of the two parameters D and m which we have at our disposal. We now present the results of these calculations, mainly for channels 1 and 2. Channel 3 will be mentioned at the end.

The elastic differential cross-section in channels 1 and 2 is

$$\frac{d\sigma_{11}}{dt} = \frac{d\sigma_{22}}{dt} = \pi r^4 |F(B)|^2$$

with

$$B = -r^2 t, \quad F(B) = \int_0^\infty x \frac{2 D^2 K_0^2(x)}{1 + D^2 K_0^2(x)} J_0(\sqrt{B}x) dx. \quad (4.6)$$

The total cross-section becomes

$$\sigma_1^T = \sigma_2^T = 4\pi r^2 F(0).$$

Before discussing intermediate values of D , we consider the limiting cases of weak and strong coupling.

- 1) Weak coupling, $D \rightarrow 0$. In the denominator of (4.3) the function $D^2 K_0^2(x)$ may then be neglected and the remaining integral can be performed analytically. The result so obtained is

$$\frac{d\sigma_{11}}{dt} = \frac{d\sigma_{22}}{dt} = \frac{4\pi D^4}{t(t-4m^2)} \log^2 \frac{\sqrt{4m^2-t} + \sqrt{-t}}{\sqrt{4m^2-t} - \sqrt{-t}} \quad (4.7)$$

It essentially corresponds to lowest order shadow scattering of a one meson exchange model of inelastic collisions. It was derived before by Salzman¹⁰⁾ and is indicated by $D = 0$ in Fig. 2 where the abscissa B is defined as above $B = -\frac{t}{m^2} = -r^2 t$.

2) Strong coupling, $D \rightarrow \infty$. In this case one must introduce the parameter $r \log D = R$ and one obtains

$$\frac{d\sigma_{11}}{dt} = \frac{d\sigma_{22}}{dt} = 4\pi R^4 \left\{ \frac{J_1(R\sqrt{-t})}{R\sqrt{-t}} \right\}^2 \quad (4.8)$$

which has the same t dependence as the differential cross-section for scattering by a black sphere¹¹⁾. It is represented by the curve $D = \infty$ in Fig. 2 where the abscissa is now $B'' = -\frac{3}{4} R^2 t$ and is so chosen as to give the same logarithmic slope at $t = 0$ as the $D = 0$ curve.

None of these two limiting cases comes close to the experimental form of the function $|F(B)|^2$, which appears to be proportional to e^{at+bt^2} over a range of t as large as $-10 \leq at < 0$, with b/a^2 positive and of the order of a few percent. Neglecting b this form could be obtained by replacing the function

$$\lambda^2(x) = \frac{D^2 K_0^2(x)}{1 + D^2 K_0^2(x)} \quad (4.9)$$

which occurs in (4.6) by $\exp\left(-\frac{r^2 x^2}{2a}\right)$. We have plotted the function $\lambda^2(x)$, given by (4.9) for different values of D and it appeared that we can approximate the exponential form rather well for D around 2. After this preliminary work we made an accurate numerical calculation of $|F(B)|^2$ for $D = 1.5$ and $D = 1.8$. The results are plotted in Fig. 2, the abscissa being again $B = -t/m^2$. The most striking effect is that the diffraction peak, when plotted on a logarithmic scale, is very straight over many orders of magnitude, with a small upward curvature. To get an idea of how well this calculated $d\sigma_{11}/dt$ could be approximated by a function of the form

$$\left(\frac{d\sigma_{11}}{dt}\right)_{t=0} e^{at+bt^2} \quad (4.10)$$

we made the following analysis. A number of points on the calculated curve were considered as experimental points and were assigned the same experimental inaccuracy as in Ref. 2). We made a best fit to these points with a function of the form (4.10) and determined a/σ_1^T , b/a^2 , χ^2 and the corresponding probability P . The results are collected in the Table. They are independent of the value of r . Considering that the experimentalist's practice is to reject a fit only if $P < 5\%$ our result is quite good. The ratio b/a^2 has a correct sign and is of the right magnitude. The value which we obtain for a/σ_1^T is two times smaller than for proton-proton scattering and four times smaller than for pion-proton scattering. In the Table we also give values of $m = r^{-1}$ in pion masses and of t_0 in $(\text{GeV}/c)^2$. r^{-1} is determined by putting for a the value $4\text{mb} = 10(\text{GeV})^{-2}$ which is close to the experimentally determined values for pp and πp diffraction peaks. The definition of t_0 is

$$\left(\frac{d\sigma_{11}}{dt}\right)_{t=t_0} = \left(\frac{d\sigma_{11}}{dt}\right)_{t=0} \cdot 10^{-3.5}$$

t_0 depends of course on r .

	a/σ_1^T	b/a^2	χ^2	P	$r^{-1} = m$	t_0
$D = 1.5$	0.05	0.031	11.25	0.20	$1.7 m_\pi$	1.16 $(\text{GeV}/c)^2$
$D = 1.8$	0.045	0.025	4.71	0.50	$1.9 m_\pi$	1.0 $(\text{GeV}/c)^2$

T A B L E

Let us now consider the inelastic differential cross-section given by the formula

$$\frac{d\sigma_{12}}{dt} = \pi r^4 |H(B)|^2$$

with

$$H(B) = \int_0^{\infty} x \frac{2D K_0(x)}{1 + D^2 K_0^2(x)} J_0(\sqrt{B}x) dx. \quad (4.11)$$

We calculated $d\sigma_{12}/dt$ for the same values of D as in the analysis of the elastic scattering $d\sigma_{11}/dt$ and $d\sigma_{22}/dt$. In the limiting case of weak coupling ($D \rightarrow 0$) we have

$$\frac{d\sigma_{12}}{dt} = \frac{4\pi D^2}{(m^2 - t)^2}.$$

The logarithmic plot of this function is given in Fig. 3.

In the limit $D \rightarrow \infty$ we clearly obtain $d\sigma_{12}/dt = 0$. We also performed the numerical calculation of $d\sigma_{12}/dt$ for the values $D = 1.5$ and $D = 1.8$. The corresponding curves are given in Fig. 3. One can see from these plots that

- i) $d\sigma_{12}/dt$ is a much steeper function of t than the elastic differential cross-section $d\sigma_{11}/dt$ (for the same value of D). This agrees with a conclusion reached by Koba⁵⁾;
- ii) for r as given in the Table $d\sigma_{12}/dt$ becomes zero at about $0.2 (\text{GeV}/c)^2$ and has a maximum at about $0.5 (\text{GeV}/c)^2$. The magnitude of this maximum is 10^{-3} times $(d\sigma_{12}/dt)_{t=0}$.

The last reaction we should consider is $A + B \rightarrow A + B$. In this case the forward and the backward scattering are different and one should consider them separately. From the formulae (2.4), (2.5), (2.6), (3.8) one can derive the equalities

$$\left(\frac{d\sigma_{33}}{dt} \right)_{\text{forward}} = \frac{d\sigma_{11}}{dt} \quad (4.12)$$

$$\left(\frac{d\sigma_{33}}{du} \right)_{\text{backward}} = \frac{d\sigma_{12}}{du} \quad (4.13)$$

from which we see that the foregoing analysis of $d\sigma_{11}/dt$ and $d\sigma_{12}/dt$ applies also to $(d\sigma_{33}/dt)_{\text{forward}}$ and $(d\sigma_{33}/du)_{\text{backward}}$, respectively.

5. TWO OTHER MODELS

In order to see how sensitive our results are to a change in the dynamical model assumed for the K matrix, we considered two other choices for the function $a_1 = D a_1^{(0)}$. First we took

$$a_1^{(0)} = a^{(0)}(x) = \frac{3}{2} e^{-x}. \quad (5.1)$$

Again $x = l/kR$ and r is a free parameter. This function is also normalized so that

$$\frac{1}{(kr)^2} \sum_{l \text{ even}} (2l+1) a_1^{(0)} = \int_0^\infty x e^{-x} dx = 1 \quad (5.2)$$

where $\bar{r}^2 = \frac{3}{2} r^2$ is the logarithmic derivative of $K(s, t)$ with respect to t at $t = 0$. For this function a_1 we calculated the elastic scattering $d\sigma_{11}/dt = d\sigma_{22}/dt$. In Fig. 4 we give the curves for two values of D . The general features are the same as those of Fig. 2.

The same conclusion could be drawn for the second model which we considered and which was defined by

$$a_1^{(0)} = a^{(0)}(x) = \begin{cases} \infty & \text{for } l \leq kR \\ K_0(x) & \text{for } l > kR. \end{cases} \quad (5.3)$$

This model is obtained from the one particle exchange model by leaving a_1 for large l values unchanged and replacing it for impact parameters below R by the value it would have for infinitely strong interaction. The $a_1^{(0)}$ of (5.3) can, of course, not be normalized. This, however, only means that we cannot compare the values of D with those of the previous models. The elastic scattering $d\sigma_{11}/dt$ is plotted in Fig. 4 for $R = 0.2 r$ and two values of D . The abscissa B in Fig. 4 is always $B = -r^2 t$.

In both examples one can also show that for $D \rightarrow \infty$ the elastic angular distributions become of the black sphere type ¹¹⁾. All general features encountered in the previous Section are therefore maintained, and one can expect this to be the case whenever $a_1 = a(x)$ is a smoothly decreasing function of x which for $x \rightarrow \infty$ tends to zero faster than any power of $1/x$. In particular the experimental angular distributions can be reproduced for moderately strong values of the coupling constant D ($D \simeq 2$ in all examples considered).

6. CONCLUSIONS

Our conclusions can be summarized in the following points.

- i) We proved that there exists a three-channel S matrix which obeys the requirements of crossing symmetry and unitarity and gives at high energies differential cross-sections which become constant when expressed in momentum transfer. It is clearly the simplest non-trivial S matrix for which this is the case. The corresponding K matrix takes a very simple form.
- ii) We calculated the differential elastic cross-sections over a range where they change by more than four orders of magnitude. The resulting curves are given in Fig. 2 for moderately strong coupling. These curves can be approximated very well by the experimental shape (1.1). This was achieved without use of any form factors and results entirely from the damping imposed by unitarity when the forces described by the K matrix are strong. The angular distribution obtained for very strong coupling is of the black sphere type.
- iii) For the inelastic scattering we calculated the distribution of momentum transfer. It is steeper than the elastic differential cross-section. The ratio of the numbers of asymmetric to symmetric jets, i.e., of $A + A \rightarrow A + B$ to $A + A \rightarrow B + B$ collisions, was found to approach zero at high energy.
- iv) We checked whether the results are sensitive to the form of the interaction. For this purpose we considered three different forms for the K matrix and found that the results are essentially the same.
- v) Our results tend to show that, in models where inelastic collisions are of two-body type, the observed shape of the elastic diffraction peak can be qualitatively explained as a consequence of a damping due to unitarity. It

does not depend critically on the specific choice of the interaction but requires a moderately strong coupling. We also know that the same shape follows when inelastic collisions produce a large number of uncorrelated particles³⁾. These two extreme types of inelastic collisions may well combine in some unknown way in the actual physical situation producing once more the same shape for the diffraction peak.

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A P P E N D I X

We try to find the consequences of the conditions $a_3^*(t) = a_3(t)$ and (2.15) for partial wave amplitudes. Let us write the amplitude for the $3 \rightarrow 3$ channel in the form

$$\tau_{33}(s, t) = \tau_{33}^F(s, t) + \tau_{33}^B(s, t) \quad (\text{A.1})$$

where $\tau_{33}^F(s, t)$ and $\tau_{33}^B(s, t)$ describe the forward and backward scattering, and satisfy for large s (2.8) and (2.9), respectively. We introduce the corresponding partial wave expansions

$$\tau_{33}^{F,B}(s, t) = \frac{i \sqrt{s}}{4\pi k} \sum_{l=0}^{\infty} (2l+1) \eta_l^{F,B}(s) P_l(\cos\theta). \quad (\text{A.2})$$

From (2.8), (2.9) it follows that η_l^F and

$$\bar{\eta}_l^B = (-1)^l \eta_l^B \quad (\text{A.3})$$

are continuous functions of the impact parameter $\varrho = 1/k$, and that $\eta_l^F = \eta_l^F(\varrho)$ does not vanish identically. The definitions (A.1) and (A.2) give the relation

$$\eta_{33} = \eta_l^F + \eta_l^B \quad (\text{A.4})$$

from which we deduce for odd l , using the formula (2.6)

$$\bar{\eta}_l^B = \eta_l^F - 1 + e^{2i\varphi_l}. \quad (\text{A.5})$$

The reality condition $a_3^*(t) = a_3(t)$ gives

$$\eta_1^{*F} = \eta_1^F.$$

To see what are the consequences of (2.15) let us observe that this condition is equivalent to

$$-\lim_{s \rightarrow \infty} \frac{\tau_{12}^{*}(s,u)}{s} = \lim_{s \rightarrow \infty} \frac{\tau_{33}^B(s,u)}{s}, \quad u \text{ fixed.} \quad (\text{A.7})$$

Remembering that τ_{12} has forward-backward symmetry whereas τ_{33}^B is supposed to vanish in the forward hemisphere, we get

$$\eta_1^{*12} = 2\bar{\eta}_1^B = 2(\eta_1^F)^{-1+e^{2i\varphi_1}} \quad (\text{A.8})$$

where (A.5) has been used.

From the formulae (A.4), (A.5), (A.6), (A.8) and the definition (2.4) of S_1 one can easily derive the conditions (2.17).

REFERENCES AND FOOTNOTES

- 1) A series of recent experiments have shown the ratio $\alpha = \text{Re}T/\text{Im}T$, T forward scattering amplitude, to be $-0,2$ to $-0,3$ at laboratory energies around 10 GeV (πp and pp) and 20 GeV (pp); for a review see S.J. Lindenbaum's report at the International Conference on High-Energy Physics, Dubna (USSR), August 1964. The forward dispersion relations predict a slow approach of α to zero at higher energies; see, e.g., P. Söding, *Physics Lett.* **8**, 285 (1964) for pp and G. Hohler, G. Ebel and J. Giesecke, *Zeitschr. f. Physik* **180**, 430 (1964) for πp .
- 2) Among the many experimental papers on this subject, we quote the following which contains information on a great variety of processes: K.J. Foley, S.J. Lindenbaum, W.A. Love, S. Ozaki, J.J. Russel and L.C.L. Yuan, *Phys. Rev. Letters* **11**, 425 and 503 (1963).
- 3) L. Van Hove, *Rev. Mod. Phys.* **36**, 655 (1964). As kindly pointed out by Dr. R.F. Peierls, the right-hand side of Eq. (4.12) of this paper should read 0,185 instead of 0,213. Also a_1 in Eq. (4.16) should be changed from 0,86 to 0,885. The uncorrelated model presented in this paper has been applied to pion-proton and proton-proton scattering by W.N. Cottingham and R.F. Peierls, *The Impact Parameter Expansion of High Energy Elastic Scattering Amplitudes*, Brookhaven National Laboratory preprint (August 1964).
- 4) A. Białas, *Nuovo Cimento* **33**, 972 (1964).
- 5) A similar model of high-energy collisions has been investigated recently by L. Chang and Z. Koba, to be published. See also Z. Koba, *Proceedings of the International Conference on High-Energy Physics, Dubna (1964)*, to be published. This model is more realistic than ours, but it does not use crossing symmetry and makes no predictions on the shape of the angular distribution for elastic scattering.

- 6) This symmetry, which usually follows from invariance under time reversal, is here a consequence of crossing symmetry.
- 7) L. Van Hove, Physics Letters 7, 76 (1963). For an extension of this theorem to more general asymptotic behaviours, see Section 2 of Ref. ³⁾.
- 8) A similar idea was exploited by D.S. Chernavskii, JETP 45, 1558 (1963), English translation: Soviet Physics JETP 18, 1072 (1964).
- 9) The definition of the function $K_0(x)$ and the formula (4.4) can be found in G.N. Watson, Theory of Bessel Functions, Cambridge University Press 1952, p. 78 and p. 156, respectively.
- 10) F. Salzman, Phys. Rev. 131, 1345 (1963).
- 11) The relation (4.5) also holds if we replace $K_0(x)$ by any function which for $x \rightarrow \infty$ approaches zero faster than any power of $1/x$.
- 12) It is worth noting that the coefficients a and b calculated in this way differ appreciably from those obtained by means of the Taylor expansion of the function $\ln d\sigma_{11}/dt$ around $t = 0$.

FIGURE CAPTIONS

Figure 1 The diagrams contributing to the K matrix.

Figure 2 Elastic differential cross-sections in the one particle exchange model

$$\frac{d\sigma_{11}}{dt} = \frac{d\sigma_{22}}{dt} = \left(\frac{d\sigma_{33}}{dt}\right)_{\text{forward}}$$

The ordinate is $\ln(d\sigma/dt)/(d\sigma/dt)_{t=0}$. For the abscissa, see the text.

Figure 3 Inelastic and backward $3 \rightarrow 3$ differential cross-sections in the one particle exchange model. The ordinate is the natural logarithm of the differential cross-section normalized to 1 at $t = 0$. The abscissa is B for all curves.

Figure 4 Elastic differential cross-sections in the models described in Section 5. Continuous curves - the model with the function e^{-x} . Dotted curves - the modified one particle exchange model.

FIG. 1

FIG. 2

FIG. 3

FIG. 4